Applied Problems in Probability Theory

E. Wentzel and L. Ovcharov

Mir Publishers Moscow

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Probability Theory

E. С. Вентцель, Л. А Овчаров

Прикладные задачи по теории вероятностей

Москва · Радио и Связь ·

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E. Wentzel and L. Ovcharov

Translated from Russian
by Irene Aleksanova

Mir Publishers Moscow

First published 1986
Revised from the 1983 Russian edition

На английском языке

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AUTHORS' PREFACE

This book is based on many years of experience of teaching probability theory and its applications at higher educational establishments. It contains many of the problems we ourselves encountered in our research and consultative work. The problems are related to a variety of fields including electrical engineering, radio engineering, data transmission, computers, information systems, reliability of technical devices, preventive maintenance and repair, accuracy of apparatus, consumer service, transport, and the health service.

The text is divided into eleven chapters; each of which begins with a short theoretical introduction which is followed by relevant formulas.

The problems differ both in the fields of application and in difficulty. At the beginning of each chapter the reader will find comparatively simple problems whose purpose is to help the reader grasp the fundamental concepts and acquire and consolidate the experience of applying probabilistic methods. Then follow more complicated applied problems, which can be solved only after the requisite theoretical knowledge has been acquired and the necessary techniques mastered.

We have avoided the standard typical problems which can be solved mechanically. Many problems may prove difficult for both beginners and experienced readers alike (the problems we believe most difficult are marked by an asterisk). In the interest of the reader most of the problems have both answers and detailed solutions, and they are given immediately after the problem rather than at the end of the book; we wrote the book for a laborious and thoughful reader who will first try to find his own answer. This structure is very convenient and has justified itself in another book we have written, *The Theory of Probability* (Nauka, Moscow, 1973), which has been republished many times both at home and abroad (some problems in that edition have been repeated in this book).

We believe that statements and detailed solutions of nontrivial problems which demonstrate certain of the techniques of problem solving are particularly interesting. Our aim is not just to solve a problem but to use the simplest and most general technique. Some problems have been given several different solutions. In many cases a method of solution used has a general nature and can be applied in several fields. We have paid special attention to numerical characteristics which makes it possible to solve a number of problems with exceptional simplicity. The applied problems using the theory of Markov stochastic processes has been given the greatest consideration.

The brief theoretical sections which open each chapter do not usually

repeat what existing textbooks on probability theory present, but have been based on new methods.

Thus this book is, in a certain sense, intermediate between a collection of problems and a textbook on the theory of probabilities. It should be useful to a wide variety of readers such as students and lecturers at higher schools, engineers and research workers who require a probabilistic approach to their work. Note that the detailed solutions and the consideration given to problem solving techniques make this book suitable for independent study.

We wish to express our gratitude to B. V. Gnedenko, Academician of the Ukrainian Academy of Sciences, who read the manuscript very attentively and made a number of valuable remarks, and also to V. S. Pugachev, Academician of the USSR Academy of Sciences, whom we consulted frequently when we worked on the book and whose methodical principles and notation we substantially followed.

PREFACE TO THE ENGLISH EDITION

We wish to express our satisfaction at having the opportunity to bring our techniques of solving applied problems in probability theory to the notice of the English reader.

We wrote this book so that it could be used both as a study aid in probability theory and as a collection of problems, of which are about 700.

The theoretical part at the beginning of each chapter and the methodical instructions for solving the various problems make it possible to use the book as a study aid. The solutions of many of the problems in the book are important both from an educational point of view and because they can be used when investigating various applied engineering problems.

The methodology and the notation in this book correspond, in the main, to those used in V. S. Pugachev's book [6] which was recently published in Great Britain.

When the book was being prepared for translation into English, a number of corrections and additions were made which improved the content of the book. During this preparatory work, Assistant Professor Danilov made an essential contribution, for which we want to express our gratitude.

E. Wentzel, L. Ovcharov

CONTENTS

CHAPTER 1

Fundamental Concepts of Probability Theory. Direct Calculation of Probability in an Urn Model

1.0. The theory of probability is the mathematical study of random phenomena. The concept of a *random event* (or simply event) is one of the principal concepts in probability theory. An *event* is the outcome of an experiment (or a trial). Six dots appearing on the top face of a die, the failure of a device during its service life, and a distortion in a message transmitted over a communication channel are all examples of events. Each event has an associated *quantity* which characterizes how likely its occurrence is; this is called the *probability of the event*.

There are several approaches to the concept of probability. The "classical" approach is to calculate the number of favourable outcomes of a trial and divide it by the total number of possible outcomes [see formula (1.0.6) below]. The frequency or statistical approach is based on the concept of the frequency of an event in a long series of trials. The *frequency* of an event in a series of N trials is the ratio of the number of trials in which it occurs to the total number of trials. There are random events for which a *stability of the frequencies* is observed; with an increase in the number N of independent trials, the frequency of the event stabilizes, and tends to a certain constant quantity, which is called the *probability of an event*.

The modern construction of probability theory is based on an axiomatic approach using the fundamental concepts of set theory. This approach to probability theory is known as the set-theoretical approach.

Here are the fundamental concepts of set theory.

A *set* is a collection of objects each of which is called an *element of the set*. A set of students who study at a given school, the set of natural numbers which do not exceed 100, a set of points on a plane lying within or on a circle with a unit radius and centre at the origin are all examples of sets.

There are several ways of designating sets. It can be denoted either by one capital letter or by the enumeration of its elements given in braces or by indicating (also in braces) a rule which associates an element with a set. For example, the set M of natural numbers from 1 to 100 can be written

$$M = \{1, 2, \ldots, 100\} = \{i \text{ is an integer}; 1 \leqslant i \leqslant 100\}.$$

The set C of points on a plane which lie within or on a circle with centre at the origin can be written in the form $C = \{x^2 + y^2 \leqslant R^2\}$, where x and y are the Cartesian coordinates of the point and R is the radius of the circle.

Depending on how many elements it has, a set may be *finite* or *infinite*. The set $M = \{1, 2, \ldots, 100\}$ is finite and contains 100 elements (in a particular case a finite set can consist of only one element). The set of all natural numbers $N = \{1, 2, \ldots, n, \ldots\}$ is infinite; the set of even numbers $N_2 = \{2, 4, \ldots, 2n, \ldots\}$ is also infinite. An infinite set is said to be *countable* if all of its terms can be enumerated (both of the infinite sets N and N_2 given above are countable). The set C of points within or on a circle of radius $R > 0$

$$C = \{x^2 + y^2 \leqslant R\} \tag{1.0.1}$$

is infinite and uncountable (its elements cannot be enumerated).

Two sets A and B *coincide* (or are equivalent) if they consist of the same elements (the coincidence of sets is expressed thus: $A = B$). For example, the set of roots of the equation $x^2 - 5x + 4 = 0$ coincides with the set $\{1, 4\}$ (and also with the set $\{4, 1\}$).

The notation $a \in A$ means that an object a is an element of a set A; or, in other words, *a belongs to A*. The notation $a \notin A$ means that an object a is not an element of a set A.

An *empty set* is a set with no elements. It is designated $\varnothing$. Example: the set of points on a plane whose coordinates (x, y) satisfy the inequality $x^2 + y^2 \leqslant -1$ is an empty set: $\{x^2 + y^2 \leqslant -1\} = \varnothing$. All empty sets are equivalent.

A set B is said to be a *subset* of a set A if all the elements of B also belong to A. The notation is $B \subseteq A$ (or $A \supseteq B$). Examples: $\{1, 2, \ldots, 100\} \subseteq \{1, 2, \ldots, 1000\}$; $\{1, 2, \ldots, 100\} \subseteq \{1, 2, \ldots, 100\}$; $\{x^2 + y^2 \leqslant 1\} \subseteq \{x^2 + y^2 \leqslant 2\}$.

An empty set is a subset of any set A: $\varnothing \subseteq A$.

We can use a geometrical interpretation to depict the inclusion of sets; the points on the plane are elements of the set (see Fig. 1.0.1, where B is a subset of A).

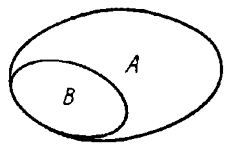

Fig. 1.0.1

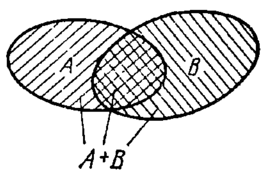

Fig. 1.0.2

The *union* (logical sum) of the sets A and B is a set $C = A + B$ which consists of all the elements of A and all those of B (including those which belong to both A and B). In short, a union of two sets is a collection of elements belonging to *at least one* of them. Examples: $\{1, 2, \ldots, 100\} + \{50, 51, \ldots, 200\} = \{1, 2, \ldots, 200\}$, $\{1, 2, \ldots 100\} + \{1, 2, \ldots, 1000\} = \{1, 2, \ldots, 1000\}$, $\{1, 2, \ldots, 100\} + \varnothing = \{1, 2, \ldots, 100\}$. The union of the two sets A and B is shown in Fig. 1.0.2; the shaded area is $A + B$.

We can similarly define a union of any number of sets: $A_1 + A_2 + \ldots + A_n = \sum_{i=1}^{n} A_i$ is a set consisting of all the elements which belong to at least one of the sets $A_1, \ldots, A_n$. We can also consider a union of an infinite (countable) number of sets $\sum_{i=1}^{\infty} A_i = A_1 + A_2 + \ldots + A_n + \ldots$. Example: $\{1, 2\} + \{2, 3\} + \{3, 4\} + \ldots + \{n, n-1\} + \ldots = \{1, 2, 3, \ldots, n, \ldots\}$.

The *intersection* (logical product) of two sets A and B is the set $D = A \cdot B$ that consists of the elements which belong to both A and B. Examples: $\{1, 2, \ldots, 100\} \times \{50, 51, \ldots, 200\} = \{50, 51, \ldots, 100\}$, $\{1, 2, \ldots 100\} \cdot \{1, 2, \ldots, 1000\} = \{1, 2, \ldots, 100\}$, $\{1, 2, \ldots, 100\} \cdot \varnothing = \varnothing$. An intersection of two sets A and B is shown in Fig. 1.0.3.

We can similarly define the intersection of any number of sets. The set $A_1 \cdot A_2 \ldots A_n = \prod_{i=1}^{n} A_i$ consists of all the elements which belong to all the sets $A_1, A_2, \ldots A_n$ simultaneously. The definition can be extended to an infinite (countable) number of sets: $\prod_{i=1}^{\infty} A_i$ is a set consisting of elements belonging to all the sets $A_1, A_2, \ldots, A_n, \ldots$ simultaneously.

Two sets A and B are said to be *disjoint* (nonintersecting) if their intersection is an empty set: $A \cdot B = \varnothing$, i.e. no element belongs to both A and B (Fig. 1.0.4). Figure 1.0.5 illustrates several disjoint sets.

As in the notation of an ordinary multiplication, the $\cdot$ sign is usually omitted.

It is sufficient to have this elementary knowledge of set theory in order to use the set-theoretical construction of probability theory.

Assume that an experiment (trial) is conducted whose result is not known beforehand, i.e. is accidental. Let us consider the set Ω of all possible outcomes of the

experiment: each of its elements $\omega \in \Omega$ (each outcome) is known as an *elementary event* and the whole set Ω as the *space of elementary events* or the *sample space*. Any subset of the set Ω is known as an *event* (or *random event*); and any event A is a subset of the set Ω, viz. $A \subseteq \Omega$.

Example: an experiment involves tossing a die; the space of elementary events $\Omega = \{1, 2, 3, 4, 5, 6\}$; an event A is an even score; $A = \{2, 4, 6\}$; $A \subseteq \Omega$. In particular, we can consider the event Ω (since every set is a subset of itself); it is said to be a *certain event* (it must occur in every experiment). We can add the empty set $\varnothing$ to the whole space Ω of elementary events; this set is also an event and is said to be an *impossible event* (it cannot occur as a result of an experiment). An example

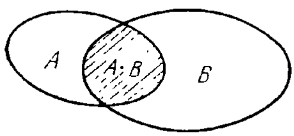

Fig. 1.0.3

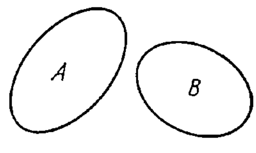

Fig. 1.0.4

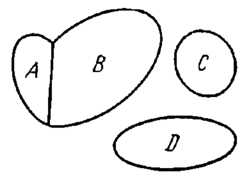

Fig. 1.0.5

of a certain event: {a score not exceeding 6 when a die is tossed}; an example of an impossible event: {a score of 7 dots on a face of a die}.

Note that there are different ways of defining an elementary event in the same experiment; say, in a random throw of a point on a plane, the position of the point can be defined both by a pair of Cartesian coordinates (x, y) and by a pair of polar coordinates (ρ, φ).

Two mutually disjoint events A and B (such that $AB = \varnothing$) are said to be *incompatible*; the occurrence of one precludes the occurrence of the other. Several events $A_1, A_2, \ldots, A_n$ are said to be *pairwise incompatible* (or simply incompatible), or *two-by-two mutually exclusive events*, if the occurrence of one of them precludes the occurrence of each of the others.

We say that several events $A_1, A_2, \ldots, A_n$ *form a complete group* if $\sum_{i=1}^{n} A_i = \Omega$, i.e. if their sum is a certain event (in other words, if at least one of them is certain to occur as a result of an experiment). Example: an experiment consists in tossing a die, the events $A = \{1, 2\}$, $B = \{2, 3, 4\}$, and $C = \{4, 5, 6\}$ form a complete group; $A + B + C = \{1, 2, 3, 4, 5, 6\} = \Omega$.

We now introduce axioms of probability theory. Assume that every event is associated with a *number* called its *probability*. The probability of an event A is designated as $\mathrm{P}(A)$*). We require that the probabilities of the events should satisfy the following axioms.

I. *The probability of an event A falls between zero and unity*

$$0 \leqslant \mathrm{P}(A) \leqslant 1. \tag{1.0.2}$$

II. Probability addition rule: *if A and B are mutually exclusive, then*

$$\mathrm{P}(A + B) = \mathrm{P}(A) + \mathrm{P}(B). \tag{1.0.3}$$

Axiom (1.0.3) can be immediately generalized to any finite number of events: *if $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then*

$$\mathrm{P}\Big(\sum_{i=1}^{n} A_i\Big) = \sum_{i=1}^{n} \mathrm{P}(A_i). \tag{1.0.4}$$

*) If an event (a set) is denoted by a verbal description of its properties, or by a formula of type (1.0.1), or by an enumeration of the elements of the set rather than by a letter, we do not use parentheses but use braces when designating the probability, e.g. $\mathrm{P}\{x^2 + y^2 < 2\}$.

III. Probability addition rule for an infinite sequence of events: *if* $A_1, A_2, \ldots, A_n$ *are mutually exclusive events, then*

$$P\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{1.0.5}$$

The axioms of probability theory can be used to calculate the probability of any events (the subsets of Ω) from the probabilities of the elementary events (if there is a finite or countable number of them). It is not necessary to consider here the ways of determining the probability of the elementary events. In practice they are found either from a consideration of the symmetry of the experiment (for a symmetric die, for instance, it is natural to assume the appearance of each face to be equipossible), or on the basis of experimental data (frequencies).

If the possible outcomes of an experiment have symmetry, then the probabilities can be directly calculated from the so-called urn model (*model of events*). This technique is based on the assumption that the elementary events are equipossible. Several events $A_1, A_2, \ldots, A_n$ are said to be *equipossible* if they have the same probability by virtue of the symmetry of the conditions of the experiment relative to those events: $P(A_1) = P(A_2) = \ldots = P(A_n)$.

If, in an experiment, we can represent the sample space Ω as a complete group of disjoint and equipossible events $\omega_1, \omega_2, \ldots, \omega_n$, then the events are called *cases* (*chances*) and the experiment is said to reduce to the urn model.

A case ω_i is said to be *favourable* to an event A if it is an element of the set A: $\omega_i \in A$.

Since the cases $\omega_1, \omega_2, \ldots, \omega_n$ form a complete group of events, it follows that

$$\sum_{i=1}^{n} \omega_i = \Omega.$$

Since the elementary events $\omega_1, \omega_2, \ldots, \omega_n$ are incompatible, it follows, from the probability addition rule, that

$$P\left(\sum_{i=1}^{n} \omega_i\right) = P(\Omega) = \sum_{i=1}^{n} P(\omega_i) = 1.$$

Since the elementary events $\omega_1, \omega_2, \ldots, \omega_n$ are equipossible, their probability is the same and is equal to $1/n$:

$$P(\omega_1) = P(\omega_2) = \ldots = P(\omega_n) = 1/n.$$

This formula yields a so-called *classical formula* for the probability of an event: *if an experiment reduces to an urn model, then the probability of event A in that experiment can be calculated by the formula*

$$P_A(A) = m_A/n, \tag{1.0.6}$$

where m_A is the number of cases favourable to the event A, and n is the total number of cases.

Formula (1.0.6), which was once accepted as the definition of probability, is now, with the modern axiomatic approach, a corollary of the probability addition rule.

Let us consider an example. Three white and four black balls are thoroughly stirred in an urn and a ball is drawn at random. Construct the sample space for this experiment and find the probability of the event A = {a white ball is drawn}. We now label the balls 1 to 7 inclusive: the first three balls are white and the last four are black. Hence

$$\Omega = \{1, 2, 3, 4, 5, 6, 7\}; \quad A = \{1, 2, 3\}.$$

Since the experimental conditions are symmetric with respect to all the balls (a ball is drawn at random), the elementary events are equipossible. Since they are

incompatible and form a complete group, the probability of event A can be found by formula (1.0.6): $\mathrm{P}(A) = m_A/n = 3/7$.

When the experiment is symmetric with respect to the possibility of outcomes, formula (1.0.6) makes it possible to calculate the probabilities of events directly from the conditions of the experiment.

A "geometrical" approach to the calculation of the probabilities of events is a natural generalization and extension of a direct calculation of probabilities in the urn model. It is employed when the sample space Ω includes an uncountable set of elementary events $\omega \in \Omega$ and by symmetry none of them is more likely than the

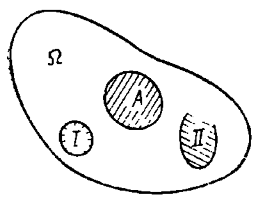

Fig. 1.0.6

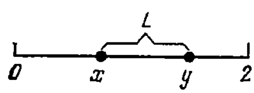

Fig. 1.0.7

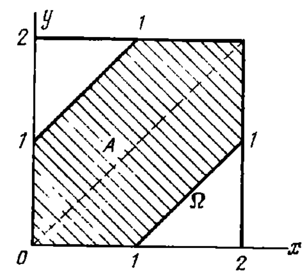

Fig. 1.0.8

others as concerns the possibility of occurrence*). Assume that the sample space Ω is a domain on a plane (Fig. 1.0.6) and that the elementary events ω are points within the domain. If the experiment has a symmetry of possible outcomes (say, a "point" object is thrown at random in the interior of the domain), then all the elementary events are "equal in rights" and it is natural to assume that the probabilities that the elementary event ω will fall in domains I and II of the same size S are equal and the probability of any event $A \subseteq \Omega$ is equal to the ratio of the area S_A of domain A to the area of the whole domain Ω:

$$\mathrm{P}(A) = S_A/S_\Omega. \tag{1.0.7}$$

Formula (1.0.7) is a generalization of the classical formula (1.0.6) to an uncountable set of elementary events. The symmetry of the experimental conditions with respect to its elementary outcomes ω is usually formulated using the words "at random". In essence this is equivalent to the random choice of a ball in an urn (see above). In textbooks the probabilities calculated by this technique are often called "geometrical probabilities".

Assume, for instance, that two points with abscissas x and y are put at random on the interval from 0 to 2 (Fig. 1.0.7). Find the probability that the distance L between them is less than unity. The elementary event ω is characterized by a pair of coordinates (x, y). The space of elementary events is a square with side 2 on the x, y plane (Fig. 1.0.8). We then have $L = |y - x|$ and so the event $A = \{|y - x| < 1\}$ is associated with the domain A which is hatched in Fig. 1.0.8.

$$\mathrm{P}(A) = \mathrm{P}\{|y - x| < 1\} = S_A/S_\Omega = 3/4.$$

If the space of elementary events is not plane but three-dimensional, then we must replace the areas S_A and S_Ω in formula (1.0.7) by the volumes V_A and V_Ω, and for a one-dimensional space, by the lengths L_A and L_Ω of the corresponding segments of a straight line.

*) We do not say that the elementary events ω are "equipossible"; we shall ascertain in Chapter 5 that the probability of each of them is equal to zero.

Problems and Exercises

1.1. Find out whether the events indicated in each example form a complete group of events for the given experiment (answer yes or no).

(1) An experiment involves tossing a coin; the events are

$$A_1 = \{\text{heads}\};\ A_2 = \{\text{tails}\}.$$

(2) An experiment involves tossing two coins; the events are

$$B_1 = \{\text{two heads}\};\ B_2 = \{\text{two tails}\}.$$

(3) An experiment involves throwing two dice; the events are

$$C_1 = \{\text{each die comes up 6}\}:$$
$$C_2 = \{\text{none of the dice comes up 6}\};$$
$$C_3 = \{\text{one die comes up 6 and the other does not}\}.$$

(4) Two signals are sent over a communication channel; the events are

$$D_1 = \{\text{at least one signal is not distorted}\};$$
$$D_2 = \{\text{at least one signal is distorted}\}.$$

(5) Three messages are sent over a communication channel; the events are

E_1 = {the three messages are transmitted without an error};

E_2 = {the three messages are transmitted with errors};

E_3 = {two messages are transmitted with errors and one without an error}.

Answer. (1) yes, (2) no, (3) yes, (4) yes, (5) no.

1.2. Regarding each group of events say whether they are incompatible in the given experiment (yes, no).

(1) An experiment involves tossing a coin; the events are

$$A_1 = \{\text{a head}\};\ A_2 = \{\text{a tail}\}.$$

(2) An experiment involves tossing two coins; the events are

$$B_1 = \{\text{the first coin comes up heads}\};$$
$$B_2 = \{\text{the second coin comes up heads}\}.$$

(3) Two shots are fired at a target; the events are

$$C_0 = \{\text{no hits}\};\ C_1 = \{\text{one hit}\};\ C_2 = \{\text{two hits}\}.$$

(4) The same as above; the events are

$$D_1 = \{\text{one hit}\};\quad D_2 = \{\text{one miss}\}.$$

(5) Two cards are selected from a pack; the events are

E_1 = {both cards are from black suits};
E_2 = {there is a queen of clubs among the selected cards};
E_3 = {there is an ace of spades among the selected cards}.

(6) Three messages are transmitted by radio; the events are

F_1 = {there is an error in the first message};
F_2 = {there is an error in the second message};
F_3 = {the first message contains an error and the second does not}.

Answer. (1) yes, (2) no, (3) yes, (4) no, (5) no, (6) no.

1.3. Regarding each group of events say whether they are equipossible in the given experiment (answer yes or no).

(1) An experiment involves tossing a coin; the events are

A_1 = {it comes up heads}; A_2 = {it comes up tails}.

(2) An experiment involves tossing an unfair (concave) coin; the events are as above, A_1 and A_2.

(3) A shot is fired at a target; the events are

B_1 = {a hit}; B_2 = {a miss}.

(4) An experiment involves tossing two coins; the events are

C_1 = {two heads}; C_2 = {two tails}; C_3 = {one head and one tail}.

(5) A card is selected from a pack at random; the events are

D_1 = {a heart}; D_2 = {a diamond};
D_3 = {a club}; D_4 = {a spade}.

(6) An experiment involves throwing a die; the events are

E_1 = {the score is no less than 3};
E_2 = {the score is no more than 3}

(7) Three messages of the same length are sent over a communication channel under the same conditions; the events are

F_1 = {an error in the first message};
F_2 = {an error in the second message};
F_3 = {an error in the third message}.

Answer. (1) yes, (2) no, (3) no in a general case, (4) no, (5) yes, (6) no, (7) yes.

1.4. Regarding each of the following group of events say whether they form a complete group, whether they are incompatible, whether they are equipossible, or whether they form a group of cases.

(1) An experiment involves tossing a (fair) coin; the events are

$$A_1 = \{\text{it comes up heads}\}; \quad A_2 = \{\text{it comes up tails}\}.$$

(2) An experiment involves tossing two coins; the events are

$$B_1 = \{\text{two heads}\}; \quad B_2 = \{\text{two tails}\};$$
$$B_3 = \{\text{one head and one tail}\}.$$

(3) An experiment involves throwing a die; the events are

$$C_1 = \{1 \text{ or } 2\}, \quad C_2 = \{2 \text{ or } 3\}, \quad C_3 = \{3 \text{ or } 4\},$$
$$C_4 = \{4 \text{ or } 5\}, \quad C_5 = \{5 \text{ or } 6\}.$$

(4) One card is selected from a pack of 36 at random; the events are

$$D_1 = \{\text{an ace}\}; \quad D_2 = \{\text{a king}\}; \quad D_3 = \{\text{a queen}\};$$
$$D_4 = \{\text{a knave}\}; \quad D_5 = \{\text{a ten}\}; \quad D_6 = \{\text{a nine}\};$$
$$D_7 = \{\text{an eight}\}; \quad D_8 = \{\text{a seven}\}; \quad D_9 = \{\text{a six}\}.$$

(5) A shot is fired at a target; the events are

$$E_1 = \{\text{a hit}\}; \quad E_2 = \{\text{a miss}\}.$$

(6) Three messages of the same length are sent under the same conditions; the events are

$$F_1 = \{\text{the first message is distorted}\};$$
$$F_2 = \{\text{the second message is distorted}\};$$
$$F_3 = \{\text{the third message is distorted}\}.$$

(7) Two devices are operated for a time τ; the events are

$$G_0 = \{\text{none of the devices failed}\};$$
$$G_1 = \{\text{one device failed and the other did not}\};$$
$$G_2 = \{\text{both devices failed}\}.$$

Answer. (1) yes, yes, yes, yes; (2) yes, yes, no, no; (3) yes, no, yes, no; (4) yes, yes, yes, yes; (5) yes, yes, no, no; (6) no, no, yes, no; (7) yes, yes, no, no.

1.5. A coin is tossed until two heads or two tails are obtained in succession. Construct the sample space for the experiment and isolate the subset corresponding to the event A = {no more than three tossings will be necessary}. Can the probability of the event be found as the ratio of the number of elementary events favourable to A to the total number of elementary events, and if not why?

Solution. The elementary events which form the set Ω (the heads are denoted by the letter "h" and the tails by "t") are: $\omega_1 = \{h, h\}$, $\omega_2 =$

{t, t}, ω_3 = {h, t, t}, ω_4 = {t, h, h}; ω_5 = {h, t, h, h}, ω_6 = {t, h, t, t} . . ., the number of events is infinite but countable: $\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$. The subset of elementary events favourable to A is $A = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

It is impossible to find here the probability of the event A as the ratio of the number of elementary events favourable to A to the total number of elementary events since the elementary events $\omega_1, \omega_2, \omega_3, \ldots$ are not equipossible: each pair is less probable than its predecassor [see problem 2.18 to find P (A)].

1.6. A polyhedron with k faces ($k > 3$) labelled 1, 2, . . ., k is thrown onto a plane at random and it falls one or another face down. Construct the sample space for the experiment and isolate the subset corresponding to the event A = {the polyhedron falls the face not exceeding the number $k/2$ down}.

Solution. The space Ω consists of k elementary events: $\Omega = \{1, 2, \ldots, k\}$, where the numbers correspond to the number of faces. The subset A consists of elementary events $A = \{1, 2, \ldots, [k/2]\}$, where $[k/2]$ is an integral part of $k/2$.

1.7. Under the conditions of the preceding problem, the polyhedron is regular; the possible number of faces are $k = 4$ (a tetrahedron); $k = 6$ (a cube); $k = 8$ (an octahedron); $k = 12$ (a dodecahedron); $k = 20$ (an icosahedron). Find the probability of the event A for each polyhedron.

Solution. For a regular (symmetric) polyhedron the appearance of each face is equipossible and, therefore, we can calculate P (A) by formula (1.0.6). Since the number of faces in each regular polyhedron is even, it follows that $[k/2] = k/2$ and so for each of them P $(A) = 1/2$.

1.8. There are a white and b black balls in an urn. A ball is drawn from the urn at random. Find the probability that the ball is white.

Answer. $a/(a + b)$.

1.9. There are a white and b black balls in an urn. A ball is drawn from the urn and put aside. The ball is white. Then one more ball is drawn. Find the probability that the ball is also white.

Answer. $(a - 1)/(a + b - 1)$.

1.10. There are a white and b black balls in an urn. A ball is drawn from the urn and put aside without noticing its colour. Then one more ball is drawn. It is white. Find the probability that the first ball, which was put aside, is also white.

Answer. $(a - 1)/(a + b - 1)$.

1.11. An urn contains a white and b black balls. All the balls except for one ball are drawn from it. Find the probability that the last ball remaining in the urn is white.

Answer. $a/(a + b)$.

1.12. An urn contains a white and b black balls. All the balls are drawn from it in succession. Find the probability that the second drawn ball is white.

Answer. $a/(a + b)$.

1.13. There are a white and b black balls in an urn ($a \geqslant 2$). Two balls are drawn together. Find the probability that both balls are white.

Solution. An event A = {two white balls}. The total number of outcomes

$$n = C_{a+b}^2 = (a + b)\ (a + b - 1)/(1 \cdot 2),$$

where $C_k^m = \frac{k!}{m!\,(k-m)!}$ is the number of combinations of k elements taken m at a time.

The number of favourable outcomes

$$m_A = C_a^2 = a\ (a - 1)/(1 \cdot 2).$$

The probability of event A

$$\mathrm{P}\ (A) = m_A/n = (a - 1)/[(a + b)\ (a + b - 1)].$$

1.14. There are a white and b black balls in an urn ($a \geqslant 2$, $b \geqslant 3$). Five balls are drawn together. Find the probability p that two of them are white and three are black.

Solution. The total number of cases

$$n = C_{a+b}^5 = \frac{(a+b)\ (a+b-1)\ (a+b-2)\ (a+b-3)\ (a+b-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}.$$

The number of favourable outcomes

$$m = C_a^2 C_b^3 = \frac{a\ (a-1)}{1 \cdot 2}\ \frac{b\ (b-1)\ (b-2)}{1 \cdot 2 \cdot 3},$$

$$p = \frac{m}{n} = \frac{10a\ (a-1)\ b\ (b-1)\ (b-2)}{(a+b)\ (a+b-1)\ (a+b-2)\ (a+b-3)\ (a+b-4)}.$$

1.15. A batch consists of k articles, l of which are faulty. A total of r articles are taken from the batch for inspection. Find the probability p that exactly s of the selected articles are faulty.

Answer. $p = C_l^s C_{k-l}^{r-s} / C_k^r$.

1.16. A die is thrown once. Find the probabilities of the following events:

A = {the score is even};

B = {the score is no less than 5};

C = {the score is no more than 5}.

Answer. P (A) = 1/2; P (B) = 1/3; P (C) = 5/6.

1.17. There are a white and b black balls in an urn ($a \geqslant 2$, $b \geqslant 2$). Two balls are drawn at once. Which of the following events is more likely:

A = {the balls are of the same colour};

B = {the balls are different in colour}?

Solution.

$$\mathrm{P}(A)=\frac{C_a^2+C_b^2}{C_{a+b}^2}=\frac{a(a-1)+b(b-1)}{(a+b)(a+b-1)},$$

$$\mathrm{P}(B)=\frac{C_a^1C_b^1}{C_{a+b}^2}\frac{2ab}{(a+b)(a+b-1)}.$$

Comparing the numerators of the fractions, we find that

$$\mathrm{P}(A)<\mathrm{P}(B) \quad \text{for} \quad a(a-1)+b(b-1)<2ab,$$

i.e. $(a-b)^2<a+b$;

$$\mathrm{P}(A)=\mathrm{P}(B) \text{ for } (a-b)^2=a+b,$$
$$\mathrm{P}(A)>\mathrm{P}(B) \text{ for } (a-b)^2>a+b.$$

1.18. A box contains n enumerated articles. All the articles are taken out one by one at random. Find the probability that the numbers of the selected balls are successive: 1, 2, . . ., n.

Answer. $1/n!$

1.19. The same box, as in the preceding problem, but each article is drawn and its number is written down, after which it is replaced and the articles are stirred. Find the probability that a natural sequence of numbers will be written: 1, 2, . . ., n.

Answer. $1/n^n$.

1.20. Eighteen teams participate in a basketball championship, out of which two groups, each consisting of 9 teams, are formed at random. Five of the teams are first class. Find the probabilities of the following events:

A = {all the first class teams get into the same group};

B = {two first class teams get into one group and three into the other}.

Answer.

$$\mathrm{P}(A)=\frac{2C_5^5C_{13}^4}{C_{18}^9}=\frac{1}{34},\quad \mathrm{P}(B)=\frac{C_5^2C_{13}^7+C_5^3C_{13}^6}{C_{18}^9}=\frac{12}{17}.$$

1.21. A certain Petrov buys a bingo ticket and marks 6 of the 49 numbers. Then the six winning numbers, out of the 49, are announced. Find the probabilities of the following events:

A_3 = {he guessed 3 out of 6 winning numbers};

A_4 = {he guessed 4 out of 6 winning numbers};

A_5 = {he guessed 5 out of 6 winning numbers};

A_6 = {he guessed all the winning numbers}.

Solution. The problem is equivalent to drawing 6 balls from an urn in which there are 6 white balls (winning numbers) and $49-6=43$

black balls (not winning).

$$\mathrm{P}(A_3)=\frac{C_6^3C_{43}^3}{C_{49}^6}\approx 0.01765;\quad \mathrm{P}(A_4)=\frac{C_6^4C_{43}^2}{C_{49}^6}\approx 0.000969,$$

$$\mathrm{P}(A_5)=\frac{C_6^5C_{43}^1}{C_{49}^6}\approx 0.00001845;\quad \mathrm{P}(A_6)=\frac{1}{C_{49}^6}\approx 0.715\cdot 10^{-7}.$$

1.22. Nine cards are labelled 0, 1, 2, 3, 4, 5, 6, 7, 8. Two cards are drawn at random and put on a table in a successive order, and then the resulting number is read, say, 07 (seven), 14 (fourteen) and so on. Find the probability that the number is even.

Solution. The evenness of a number is defined by its last digit which must be even (zero is also an even number). The required probability is the probability that one of the numbers 0, 2, 4, 6, 8 will be the second to appear, i.e. 5/9.

1.23. Five cards are labelled 1, 2, 3, 4, 5. Two cards are drawn one after another. Find the probability that the number on the second card is larger than that on the first.

Solution. The experiment has two possible outcomes:

A = {the second number is larger than the first};

B = {the second number is smaller than the first}.

Since the conditions of the experiment are symmetric with respect to A and B, we have $\mathrm{P}(A) = \mathrm{P}(B) = 1/2$.

1.24. A box contains articles of the same type manufactured at different factories; a of them are manufactured at factory I, b at factory II and c at factory III. All the articles are taken from the box one after another and the factories at which they were manufactured are written down. Find the probability that an article manufactured at factory I will appear before that manufactured at factory II.

Solution. It is irrelevant whether any of the articles are from factory III. The probability we seek is the probability that an article manufactured at factory I is the first to be drawn, where a articles are from factory I and b articles are from factory II, i.e., $a/(a+b)$.

1.25. There are two boxes containing standard elements for replacement. The first box contains a sound elements and b faulty ones, and the second box contains c sound and d faulty elements. One element is drawn at random from each box. Find the probability that both elements are sound.

Solution. Each element from the first box can be combined with each element from the second box; the number of cases $n = (a+b)(c+d)$. The number of favourable cases $m = ac$ and the probability of the event is $ac/[(a+b)(c+d)]$.

1.26. Under the conditions of problem 1.25 find the probability that the two elements are different in quality.

Answer. $(ad+bc)/[(a+b)(c+d)]$.

1.27. Under the same conditions find the probability that both elements are faulty.

Answer. $bd/[(a+b)(c+d)]$.

1.28. A box contains k articles of the same type labelled 1, 2, . . ., k. A total of l articles are taken from the box in succession at random, the number of each article taken out is written down and the article is replaced. Find the probability p that all the numbers written down are different.

Solution. The number of cases $n = k^l$. The number of favourable cases is equal to the number of arrangements of k elements taken l at a time, i.e. $m = k(k-1) \dots (k-l+1)$. The probability of the event

$$p = \frac{m}{n} = \frac{k(k-1)\dots(k-l+1)}{k^l} = \frac{k!}{k^l(k-l)!}$$

1.29. A child who cannot read plays with blocks labelled with the letters of the alphabet. He takes five blocks from which the word "table" is formed. Then he scatters them and puts them side by side in an arbitrary fashion. Find the probability p that he will again form the same word.

Answer. $p = 1/5! = 1/120$.

1.30. The same question for the word "papaya", a tropical American tree.

Solution. The number of outcomes $n = 6!$; however the number of favourable outcomes is now not unity as in problem 1.29 but $m = 3!2!$ since the repeating letters "a" and "p" can be put in any order: $p = 3!2!/6! = 1/60$.

1.31. Several cards are selected from a complete pack of 52 cards. How many cards must we select to state, with a probability greater than 0.50, that cards of the same suit will be among them?

Solution. We designate A = {the presence of at least two cards of the same suit among the selected k cards}.

For $k = 2$ we have $n = C_{52}^2$, $m_A = C_{13}^2 \cdot 4$; $\mathrm{P}(A) = 0.235 < 0.50$.

For $k = 3$ we have $n = C_{52}^3$, $m_A = C_{13}^3 \cdot 4 + C_{13}^2 C_{39}^1 \cdot 4$; $\mathrm{P}(A) = 0.602 > 0.50$.

Thus we have to select $k \geqslant 3$ cards.

1.32. N people take seats at random at a round table ($N > 2$). Find the probability p that two fixed people A and B will sit side-by-side.

Solution. The total number of cases $n = N!$ Let us calculate the number of favourable cases m. There are two ways of seating two people A and B side-by-side and the other people can be seated in $(N-2)!$ ways; then $m = 2N(N-2)!$ and $p = 2N(N-2)!/N! = 2/(N-1)$. There is a simpler way of solving the problem: let A sit where he pleases. Now there are $N - 1$ seats B can take, two of which are favourable; hence $p = 2/(N-1)$.

1.33. The same problem for a rectangular table with N people sitting arbitrarily along one side of the table.

Solution. $n = N!$; the favourable cases divide into two groups: (1) A sits at an end; (2) A does not sit at an end. The number in the first $m_1 = 2(N-2)!$ and the number in the second $m_2 = 2(N-2) \times (N-2)!$; $p = (m_1 + m_2)/n = 2(N-1)(N-2)!/N! = 2/N$.

1.34. There are M operators and N enumerated devices which they can service. Each operator chooses a device at random and with equal probability but under the condition that none of the devices can be serviced by more than one operator. Find the probability that the devices labelled 1, 2, . . ., M will be chosen.

Solution. The number of ways of distributing M operators over N devices is equal to the number of permutations of N elements taken M at a time; viz. $n = N(N-1)\ldots(N-M+1)$. By the hypothesis all these ways are equipossible, i.e. they form a group of cases. The number of favourable cases (when only the first M devices are serviced) is $m = M!$ Hence the required probability

$$p = \frac{M!}{N(N-1)\ldots(N-M+1)} = \frac{1}{C_N^M}.$$

1.35. There are K standard elements for replacement in a box among which K_1 elements are of the 1st type, . . ., K_i elements of the ith type, . . ., K_m elements are of the mth type; $\sum_{i=1}^{m} K_i = K$; k elements are taken from the box at random. Find the probability that there will be k_1 elements of the 1st type, . . ., k_i elements of the ith type, . . ., k_m elements of the mth type among them.

Solution. The total number of cases n is equal to the number of ways in which k elements can be selected from K elements: $n = C_K^k$. The number of favourable cases

$$m = C_{K_1}^{k_1} C_{K_2}^{k_2} \ldots C_{K_m}^{k_m} = \prod_{i=1}^{m} C_{K_i}^{k_i},$$

since k_1 elements of the 1st type can be chosen in $C_{K_1}^{k_1}$ ways, . . ., k_m elements of the mth type can be chosen in $C_{K_m}^{k_m}$ ways and various combinations of them can be formed. The required probability

$$p = \prod_{i=1}^{m} C_{K_i}^{k_i} / C_K^k.$$

1.36. A local post office is to send four telegrams; there are four communication channels in all. The telegrams are distributed at random over the communication channels; each telegram is sent over any channel with equal probability. Find the probability of the event A = {three telegrams are sent over one channel, one telegram over another channel and two channels remain empty}.

Solution. The total number of cases $n = 4^4$. The number of ways to choose a channel over which three telegrams are sent $C_4^1 = 4$; the number of ways to choose a channel over which one telegram is sent $C_3^1 = 3$. The number of ways to choose three telegrams out of four to send them over one channel $C_4^3 = 4$. The total number of favourable cases $m_A = 4\cdot3\cdot4$.

$$\mathrm{P}(A) = m_A/n = 4\cdot3\cdot4/4^4 = 3/16.$$

1.37. M telegrams are distributed at random over N communication channels ($N > M$). Find the probability of an event

A = {no more than one telegram will be sent over each channel}.

Solution. The total number of cases is N^M. The number of ways to choose M channels out of N in order to send one telegram over each of them is C_N^M. The number of ways to choose one telegram out of M and to send it over the first of the channels is $C_M^1 = M$. The number of ways to choose the second telegram out of the remaining $M - 1$ is $C_{M-1}^1 = M - 1$, and so on. The total number of favourable cases $m_A = M(M-1) \ldots 1 = M!$.

$$\mathrm{P}(A) = C_N^M \cdot M!/N^M.$$

1.38*. A local post office is to send M talegrams and to distribute them at random over N communication channels. The channels are enumerated. Find the probability that exactly k_1 telegrams will be sent over the 1st channel, k_2 telegrams over the 2nd channel, and so on, k_N telegrams over the Nth channel, with $\sum_{i=1}^{N} k_i = M$.

Solution. The number of cases $n = N^M$. Let us find the number of favourable cases m. The number of ways to choose k_1 telegrams out of M is $C_M^{k_1}$; the number of ways to choose k_2 telegrams out of the remaining $M-k_1$ is $C_{M-k_1}^{k_2}$, and so on. The number of ways to choose k_N telegrams out of $M-(k_1+\ldots+k_{N-1}) = k_N$ is $C_{k_N}^{k_N} = 1$. These numbers must be multiplied together.

$$m = C_M^{k_1} C_{M-k_1}^{k_2} \ldots C_{M-(k_1+\ldots+k_{N-1})}^{k_{N-1}} \cdot 1$$

$$= \frac{M!}{k!\,(M-k_1)!}\,\frac{(M-k_1)!}{k_2!\,[M-(k_1+k_2)]!} \cdots \frac{[M-(k_1+k_2+\ldots+k_{N-2})]!}{k_{N-1}!\,k_N!}$$

$$= \frac{M!}{k_1!\,k_2!\ldots k_N!} = \frac{M!}{\prod_{i=1}^{N} k_i!};$$

$$\mathrm{P}(A) = m/n = M!/\left(N^M \prod_{i=1}^{N} k_i!\right).$$

1.39*. Under the conditions of problem 1.37 find the probability that no telegrams will be sent over l_0 of N channels, one telegram will be sent over l_1 channels, and so on; and all M telegrams will be sent over l_M channels:

$$l_0 + l_1 + \ldots + l_M = N;\ 0 \cdot l_0 + 1 \cdot l_1 + \ldots + M l_M = M.$$

Solution. The total number of cases is $n = N^M$. To find the number of favourable cases m, we must multiply the number of ways in which the channels can be chosen by the number of ways in which the tele-

grams can be chosen. The number of ways to choose the channels is

$$N!/l_0!l_1! \dots l_M!) = N!/\prod_{k=0}^{M} l_k!.$$

Let us find the number of ways in which the telegrams can be chosen. They fall into a number of groups: the initial group (0 telegrams) is empty; the first group contains l_1 telegrams; in general kth group contains kl_k telegrams ($k = 1, 2, \dots, M$). The number of ways to choose the groups of telegrams is

$$\frac{M!}{(1\cdot l_1)!\,(2l_2)!\,(3l_3)! \dots (Ml_M)!} = \frac{M!}{\prod_{k=1}^{M} (kl_k)!}. \tag{1.39.1}$$

Let us now find the number of ways to choose the telegrams from the kth group so that k telegrams are sent over each channel. This number of ways is

$$(kl_k)!/\underbrace{(k!k! \dots k!)}_{l_k \text{ times}} = (kl_k)!/(k!)^{l_k},$$

and the number of ways to choose all the telegrams for all the groups is equal to the product of these numbers for different k:

$$\prod_{k=1}^{M} \frac{(kl_k)!}{(k!)^{l_k}}. \tag{1.39.2}$$

Multiplying (1.39.1) and (1.39.2), we get the number of ways in which the telegrams can be chosen:

$$\frac{M!}{\prod_{k=1}^{M} (kl_k)!} \cdot \frac{\prod_{k=1}^{M} (kl_k)!}{\prod_{k=1}^{M} (k!)^{l_k}} = \frac{M!}{\prod_{k=1}^{M} (k!)^{l_k}}.$$

Multiplying this number by the number of ways in which the channel can be chosen, we find the number of favourable cases

$$m = \frac{N!}{\prod_{i=0}^{M} l_i!} \cdot \frac{M!}{\prod_{k=1}^{M} (k!)^{l_k}}.$$

Hence the probability of the event we are interested in:

$$\mathrm{P}(A) = \frac{N!M!}{N^M \prod_{i=0}^{M} l_i! \prod_{k=1}^{M} (k!)^{l_k}}. \tag{1.39.3}$$

1.40. There is a point image of an object M on a circular radar screen (Fig. 1.40) which is positioned at random in the circle, no domain within the circle being more probable (the image of the object is "thrown at random" onto the screen). We consider an event A consisting in the distance ρ from the point M to the centre of the screen being smaller than $r/2$, i.e. $A = \{\rho < r/2\}$. Find the probability of the event.

Solution. The sample space Ω is the interior of the circle of radius r. The domain A is hatched in Fig. 1.40.

$$P(A) = \frac{S_A}{S_\Omega} = \frac{\pi r^2/4}{\pi r^2} = \frac{1}{4}.$$

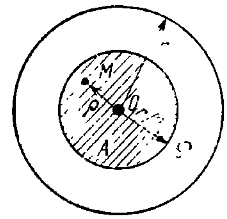

Fig. 1.40

1.41. We have a magnetic tape 200 m long with messages written on two tracks. The message on the first track is $l_1 = 30$ m long and that on the other track is $l_2 = 50$ m long; the location of the records is unknown. Because of a damage, we had to remove a section of the tape $l_0 = 10$ m long which begins 80 m from the start. Find the probabilities of the following events:

A = {neither record is damaged};
B = {the first record is damaged and the second is not};
C = {the second record is damaged and the first is not};
D = {both records are damaged}.

Solution. Since the location of neither record is known, we infer that each may begin anywhere as long as the whole record lies on the tape. We designate the abscissa of the beginning of the first record as x and that of the second record as y. The sample space is a rectangle of length $L - l_1 = 170$ m and of height $L - l_2 = 150$ m (Fig. 1.41). In the figure the different kinds of hatching designate the domains corresponding to damages to the first and second records, and the letters A, B, C, D designate the domains corresponding to the events A, B, C, D (every domain, except for D, consists of separate parts).

$$S_\Omega = 170 \times 150 = 25\,500 \text{ m}^2,$$
$$S_A = 50 \times 60 + 80 \times 60 + 30 \times 50 + 80 \times 30 = 11\,700 \text{ m}^2,$$
$$S_B = 40 \times 60 + 40 \times 30 = 3600 \text{ m}^2,$$
$$S_C = 50 \times 60 + 80 \times 60 = 7800 \text{ m}^2,$$
$$S_D = 60 \times 40 = 2400 \text{ m}^2.$$
$$P(A) = 0.459, \quad P(B) = 0.141,$$
$$P(C) = 0.306, \quad P(D) = 0.094.$$

1.42. Two signals $\tau < 1/2$ long are transmitted by radio for a time interval $(0, 1)$, each of them beginning at any moment of the interval

$(0,\ 1-\tau)$ and with equal probability. If the signals overlap even partially, they both become distorted and cannot be received. Find the probability that the signals will be received without distortion.

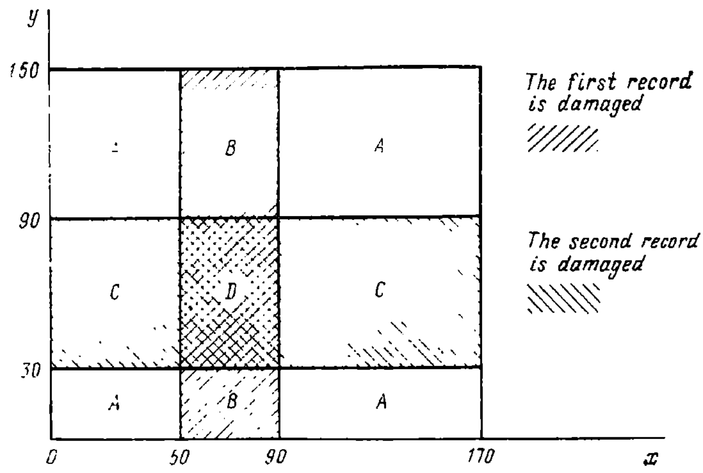

Fig. 1.41

Solution. We denote the moment when the first signal begins by x and the moment when the second signal begins by y. The sample space is shown in Fig. 1.42. The hatched domains A correspond to the event

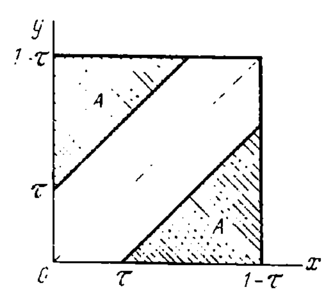

Fig. 1.42

$$A = \{\text{the signals are not distorted}\} = \{|x-y| > \tau\}.$$

$$P(A) = S_A/S_\Omega = (1-2\tau)^2/(1-\tau)^2.$$

1.43. There are two parallel telephone lines of length l (Fig. 1.43*a*) $d < l$ apart. It is known that there is a break in each of them (the location of each break is unknown). Find the probability that the distance R between the breaks is not larger than a $(d < a < \sqrt{l^2+d^2})$.

Solution. We denote the abscissa of the first break by x and that of the second by y; $R = \sqrt{|x-y|^2+d^2}$. The event we speak of is $A = \{|x-y|^2 + d^2 \leqslant a^2\} = \{|x-y| \leqslant \sqrt{a^2-d^2}\}$. The sample space is a square with side l, $S_\Omega = l^2$. The domain A is hatched in Fig. 1.43*b*.

$$S_A = 2l\sqrt{a^2-d^2} - a^2 + d^2,$$

$$P(A) = \frac{2}{l}\sqrt{a^2-d^2} - \frac{a^2-d^2}{l^2}.$$

1.44. A bar of unit length is broken into three parts x, y, z (Fig. 1.44*a*). Find the probability that a triangle can be formed from the resulting parts.

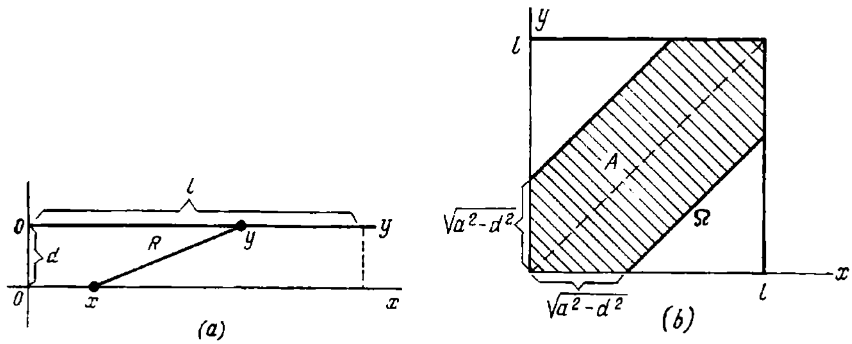

Fig. 1.43

Solution. The elementary event ω is characterized by two parameters, x and y [since $z = 1 - (x + y)$]. We depict the event by a point on the x, y plane (Fig. 1.44*b*). The conditions $x > 0$, $y > 0$, $x + y < 1$ are imposed on the quantities x and y; the sample space is the interior of a right triangle with unit legs, i.e. $S_\Omega = 1/2$. The condition A requiring that a triangle could be formed from the segments x, y, $1 -$

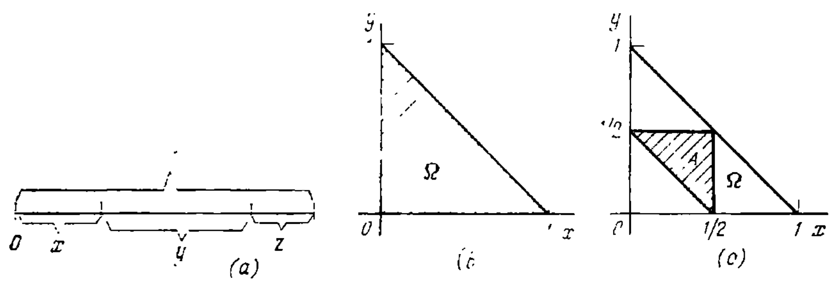

Fig. 1.44

$(x + y)$ reduces to the following two conditions: (1) the sum of any two sides is larger than the third side; (2) the difference between any two sides is smaller than the third side. This condition is associated with the triangular domain A (Fig. 144*c*) with area $S_A = (1/2)\ (1/4) = 1/8$; $P(A) = S_A/S_\Omega = 1/4$.

1.45. *Buffon's needle problem.* A plane is ruled with parallel straight lines L distant from each other (Fig. 1.45*a*). A needle (a line segment) of length $l < L$ is thrown at random on the plane. Find the probability that it will hit one of the lines.

Solution. We characterize the outcome of the experiment (the position of the needle) by two numbers: the abscissa x of the centre of the

needle with respect to the nearest line on the left and by the angle φ the needle makes with the direction of the lines (Fig. 1.45*b*). The fact that the needle is thrown on the plane at random means that all the values of x and φ are equipossible. We can evidently limit (without losing generality) the possible values of x to the interval from 0 to $L/2$ and those of φ to the interval from 0 to $\pi/2$, and consider the possibility of the needle hitting a line for only one of the lines (the nearest left

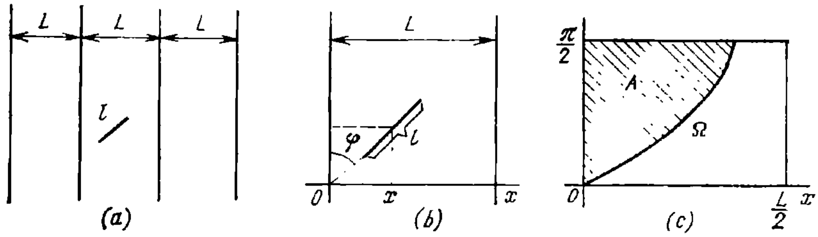

Fig. 1.45

line). The sample space Ω is a rectangle with sides $L/2$ and $\pi/2$ (Fig. 1.45*c*); $S_\Omega = L\pi/4$. The needle will hit the line if the abscissa x of its centre is smaller than $(l/2)\sin\varphi$; the event we are interested in is $A = \{x < (l/2)\sin\varphi\}$. The domain A is hatched in Fig. 1.45*c*; its area is

$$S_A = \int_0^{\pi/2} \frac{l}{2}\sin\varphi\, d\varphi = \frac{l}{2};$$

$$\mathrm{P}(A) = \frac{S_A}{S_\Omega} = \frac{2l}{\pi L}.$$

Remark. This formula was obtained by Buffon in the 18th century and was repeatedly verified experimentally, for which purpose the frequency of hits was calculated for a long series of throws. The formula was even used to make an approximate calculation of the number π and satisfactory results were obtained.

CHAPTER 2

Algebra of Events. Rules for Adding and Multiplying Probabilities

2.0. The theory of probability is based on indirect rather than direct methods of calculating probabilities when the probability of an event is expressed in terms of the probabilities of other related events. It is necessary, first of all, to know how to express an event in terms of other events using the so-called *algebra of events*. We now introduce the concepts of "a sum of events" and "a product of events" and operations on them. Note that these concepts are only introduced for events which are subsets of the same space of elementary events Ω.

Since in our set-theoretical representation events are *sets*, the actions performed on them (addition and multiplication) are defined as those corresponding to set operations (see Chapter 1).

Nevertheless, we shall repeat certain definitions and give them a geometric interpretation. We shall depict the space of elementary events as a rectangle and the

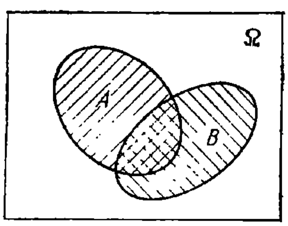

Fig. 2.0.1

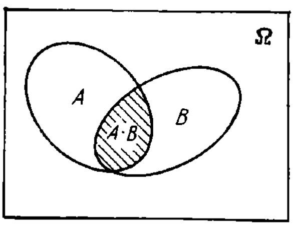

Fig. 2.0.2

events as parts of the rectangle (Fig. 2.0.1). The *sum $A + B$ of two events A and B* is an event consisting in the occurrence of at least one of those events (the hatched domain in Fig. 2.0.1). The sum of two events is evidently a union (sum) of the respective sets. Similarly, the *sum of several events* $A_1, A_2, \ldots, A_n$ is the event $\sum_{i=1}^{n} A_i = A_1 + A_2 + \ldots + A_n$ consisting in an occurrence of at least one of them. We can also form the union or sum of an infinite (countable) number of events: $\sum_{i=1}^{\infty} A_i = A_1 + A_2 + \ldots + A_n + \ldots$.

The *product $A \cdot B$ of two events A and B* is the event consisting in a simultaneous realization of the two events (the intersection of the sets, Fig. 2.0.2). The *product of several events* $A_1, A_2, \ldots, A_n$ is an event $\prod_{i=1}^{n} A_i = A_1 A_2 \ldots A_n$ consisting in a simultaneous realization of all the events (an intersection of the respective sets). We can also multiply an infinite (countable) number of events: $\prod_{i=1}^{\infty} A_i = A_1 A_2 \ldots A_n \ldots$.

It follows from the definitions of a sum and a product of events that

$$A + A = A, \quad A \cdot A = A,$$
$$A + \Omega = \Omega, \quad A \cdot \Omega = A,$$
$$A + \varnothing = A, \quad A \cdot \varnothing = \varnothing.$$

If $A \subseteq B$, then $A + B = B$, $A \cdot B = A$.

The operations of addition and multiplication of events have some of the properties of ordinary addition and multiplication.

1. Commutativity:

$$A + B = B + A, \quad A \cdot B = B \cdot A.$$

2. Associativity:

$$(A + B) + C = A + (B + C), \quad (AB)C = A(BC).$$

3. Distributivity:

$$A\,(B + C) = AB + AC.$$

All these properties follow from the fact that events are sets (see Chapter 1).

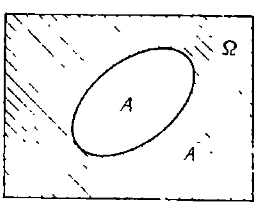

Fig. 2.0.3

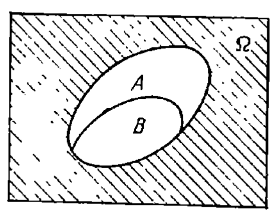

Fig. 2.0.4

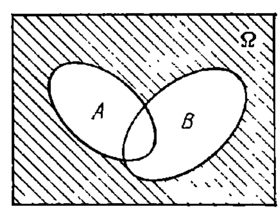

Fig. 2.0.5

An "*opposite*" or *complementary* event of A is the event $\bar{A}$ of nonoccurrence of the event A : $\bar{A} = \{$A has not occurred$\}$. A complementary event is represented in Fig. 2.0.3. The domain $\bar{A}$ is the complement of A with respect to the complete space Ω.

It follows from the definition of a complementary event that

$$(\bar{\bar{A}}) = A, \quad \bar{\Omega} = \varnothing, \quad \bar{\varnothing} = \Omega.$$

It is easy to verify (Fig. 2.0.4) that if $B \subseteq A$, then $\bar{A} \subseteq \bar{B}$. It is as easy to verify (Fig. 2.0.5) the following properties of complementary events:

$$\overline{A + B} = \bar{A} \cdot \bar{B}; \quad \overline{A \cdot B} = \bar{A} + \bar{B}.$$

The rules of the event algebra make it possible to combine various simple events and form, in that way, other, more complex, events.

The probabilities of compound events can be calculated from the probabilities of simpler events, using two basic rules (addition and multiplication) of probability theory. These rules are often called the fundamental theorems of probability theory, but in reality they are theorems only for the urn model and are introduced axiomatically for experiments which do not reduce to the urn model.

1. The addition rule for probabilities. *The probability of the sum of two mutually exclusive events is equal to the sum of the probabilities of those events*, i.e. if $A \cdot B = \varnothing$, then

$$\mathrm{P}\,(A + B) = \mathrm{P}\,(A) + \mathrm{P}\,(B). \tag{2.0.1}$$

This rule was introduced axiomatically in Chapter 1 (axiom II).

The addition rule of probabilities can be easily generalized to an arbitrary number n of incompatible events: if $A_i \cdot A_j = \varnothing$ for $i \neq j$, then

$$\mathrm{P}\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathrm{P}(A_i). \tag{2.0.2}$$

The addition rule can be as easily generalized to the case of an infinite (countable) number of events: if $A_i \cdot A_j = \varnothing$ for $i \neq j$, then

$$\mathrm{P}\left(\sum_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{P}(A_i). \tag{2.0.3}$$

(see axiom III in Chapter 1).

It follows from the addition rule that *if $A_1, A_2, \ldots, A_n$ are disjoint events that form a complete group, then the sum of their probabilities is equal to unity*, i.e. if

$$\sum_{i=1}^{n} A_i = \Omega, \quad A_i \cdot A_j = \varnothing \quad \text{for} \quad i \neq j,$$

then

$$\sum_{i=1}^{n} \mathrm{P}(A_i) = 1. \tag{2.0.4}$$

In particular, since two opposite events A and $\bar{A}$ are mutually exclusive and form a complete group, then *the sum of their probabilities is equal to unity*:

$$\mathrm{P}(A) + \mathrm{P}(\bar{A}) = 1. \tag{2.0.5}$$

To formulate the multiplication rule for probabilities, it is necessary to introduce the concept of conditional probability. The *conditional probability of an event A, relative to the hypothesis that an event B occurs* [designated as $\mathrm{P}(A \mid B)$] is the probability of the event A calculated on the hypothesis that the event B has occurred.

Advancing a hypothesis that the event B has occurred is equivalent to changing the conditions of the experiment, i.e. retaining only those elementary events which are favourable to the event B and removing all the other events. It follows that instead of the space of elementary events Ω we have a new space Ω_B corresponding to the event B (Fig. 2.0.6). The domain AB, which corresponds to the intersection of A and B, is favourable to the event A on the hypothesis that the event B is realized.

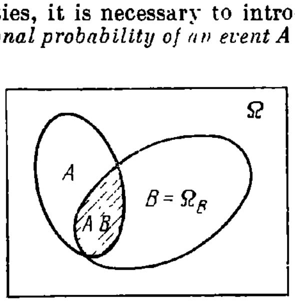

Fig. 2.0.6

2. The multiplication rule for probabilities. *The probability of the product of two events A and B is equal to the probability of one of them* (say, A) *multiplied by the conditional probability of the other, provided that the first event has occurred*:

$$\mathrm{P}(AB) = \mathrm{P}(A) \cdot \mathrm{P}(B \mid A), \tag{2.0.6}$$

or, if we take B as the first event,

$$\mathrm{P}(AB) = \mathrm{P}(B) \cdot \mathrm{P}(A \mid B). \tag{2.0.7}$$

The multiplication rule for probabilities can be generalized to an arbitrary number of events:

$$\mathrm{P}(A_1 A_2 \ldots A_n) = \mathrm{P}(A_1) \cdot \mathrm{P}(A_2 \mid A_1) \cdot \mathrm{P}(A_3 \mid A_1 A_2) \ldots \mathrm{P}(A_n \mid A_1 A_2 \ldots A_{n-1}). \tag{2.0.8}$$

Formula (2.0.6) yields the following expression for a conditional probability:

$$\mathrm{P}\,(B \mid A) = \mathrm{P}\,(AB)/\mathrm{P}\,(A), \tag{2.0.9}$$

i.e. *the conditional probability of one event, on the hypothesis that the other event occurs, is equal to the probability of the product of the two events divided by the probability of the event which is assumed to be realized.*

By analogy with formula (2.0.9) we can write

$$\mathrm{P}\,(A \mid B) = \mathrm{P}\,(AB)/\mathrm{P}\,(B). \tag{2.0.10}$$

Two events A and B are said to be *independent* if the occurrence of one of them does not affect the probability of the occurrence of the other:

$$\mathrm{P}\,(A \mid B) = \mathrm{P}\,(A), \tag{2.0.11}$$

or, which is the same thing,

$$\mathrm{P}\,(B \mid A) = \mathrm{P}\,(B). \tag{2.0.12}$$

For two independent events the multiplication rule for probabilities assumes the form

$$\mathrm{P}\,(AB) = \mathrm{P}\,(A) \cdot \mathrm{P}\,(B), \tag{2.0.13}$$

i.e. *the probability of the product of two independent events is equal to the product of the probabilities of those events.*

Several events are said to be *independent in their totality* (or simply independent) if the occurrence of any number of them does not affect the probabilities of the other events. For several independent events the multiplication rule (2.0.8) assumes the form

$$\mathrm{P}\,(A_1A_2 \ldots A_n) = \mathrm{P}\,(A_1) \cdot \mathrm{P}\,(A_2) \ldots \mathrm{P}\,(A_n), \tag{2.0.14}$$

i.e. *the probability of the product of independent events is equal to the product of their probabilities.*

Several trials are said to be *independent* if the probability of the outcome of each of them does not depend on the outcomes of the other trials.

The following theorem on a repetition of a trial is a corollary of the addition and multiplication rules for probabilities. *If n independent trials are performed in each of which an event A occurs with probability p, then the probability of the event A occurring exactly m times in the given experiment is expressed by the formula*

$$\mathrm{P}_{m,n} = C_n^m p^m (1-p)^{n-m}, \tag{2.0.15}$$

or, designating $1 - p = q$,

$$\mathrm{P}_{m,\,n} = C_n^m p^m q^{n-m}. \tag{2.0.16}$$

The probability of the event A occurring not less than m times in a series of n independent trials is expressed by the formula

$$R_{m,\,n} = \sum_{k=m}^{n} C_n^k p^k q^{n-k}, \tag{2.0.17}$$

or by the formula

$$R_{m,\,n} = 1 - \sum_{k=0}^{m-1} C_n^k p^k q^{n-k} \tag{2.0.18}$$

From the two formulas (2.0.17) and (2.0.18) we choose the one which contains a smaller number of terms.

Problems and Exercises

2.1. Can the sum of two events A and B coincide with their product?

Answer. It can if the event A is equivalent to the event B ($A \subseteq B$ and $B \subseteq A$). For example, if a message transmitted over a communication channel can only be distorted by a noise on the time interval occupied by the message and is surely distorted when there is a noise, then the events

A = {the message is distorted},

B = {there is a noise on the time interval occupied by the message} are equivalent: $A = B$; $A + B = A = B$; $AB = A = B$.

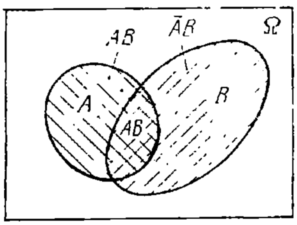

Fig. 2.2

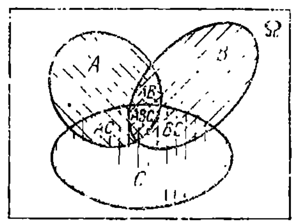

Fig. 2.3

2.2. Prove that if two events A and B are compatible, then

$$P(A + B) = P(A) + P(B) - P(AB). \tag{2.2.1}$$

Solution. Let us represent the event $A + B$ as a sum of three incompatible events: $A\bar{B}$ (A and not B); $\bar{A}B$ (B and not A) and AB (both A and B) (see Fig. 2.2):

$$A + B = A\bar{B} + \bar{A}B + AB.$$

We find the expressions of the events A and B:

$$A = AB + A\bar{B}; \quad B = \bar{A}B + AB.$$

Using the addition rule for probabilities, we find:

$$P(A + B) = P(A\bar{B}) + P(\bar{A}B) + P(AB), \tag{2.2.2}$$

$$P(A) = P(A\bar{B}) + P(AB),$$

$$P(B) = P(\bar{A}B) + P(AB).$$

Adding the two last expressions together, we get

$$P(A) + P(B) = P(A\bar{B}) + P(\bar{A}B) + 2P(AB).$$

Subtracting (2.2.2) from this equation, we get

$$P(A) + P(B) = P(A + B) + P(AB),$$

whence follows (2.2.1). From (2.2.1) it follows that

$$\mathrm{P}(A+B) \leqslant \mathrm{P}(A)+\mathrm{P}(B)$$

always and the equality sign can be put only for incompatible events.

2.3. Write the expression for the probability of the sum of three joint events.

Solution. From Fig. 2.3 we obtain

$$\mathrm{P}(A+B+C)=\mathrm{P}(A)+\mathrm{P}(B)+\mathrm{P}(C)-\mathrm{P}(AB)-\mathrm{P}(AC) \\ -\mathrm{P}(BC)+\mathrm{P}(ABC).$$

2.4. An experiment consists in tossing two coins. The following events are considered:

A = {the first coin comes up heads}, B = {the first coin comes up tails}, C = {the second coin comes up heads};

D = {the second coin comes up tails}; E = {at least one head};

F = {at least one tail}; G = {one head and one tail};

H = {no heads}; K = {two heads}.

Determine the events to which the following events are equivalent: (1) $A+C$, (2) AC, (3) EF, (4) $G+E$, (5) GE, (6) BD, (7) $E+K$.

Answer. (1) $A+C=E$, (2) $AC=K$, (3) $EF=G$, (4) $G+E=E$, (5) $GE=G$, (6) $BD=H$, (7) $E+K=E$.

2.5. Three messages are sent in succession over a communication channel, each of which may either be sent correctly or be distorted. We consider the following events:

A_i = {the ith message was sent correctly};

$\bar{A}_i$ = {the ith message was distorted} ($i=1, 2, 3$).

Express the following events as sums, products and sums of products of the events A_i and $\bar{A}_i$:

A = {the three messages were sent correctly};

B = {the three messages were distorted};

C = {at least one message was sent correctly};

D = {at least one message was distorted};

E = {not less than two messages were sent correctly};

F = {not more than one message was sent correctly};

G = {the first message sent correctly was the third in the sequence}.

Answer.

$A = A_1A_2A_3, \quad B = \bar{A}_1\bar{A}_2\bar{A}_3,$

$C = A_1A_2A_3 + A_1A_2\bar{A}_3 + A_1\bar{A}_2A_3 + \bar{A}_1A_2A_3 + A_1\bar{A}_2\bar{A}_3 + \bar{A}_1\bar{A}_2A_3 + \bar{A}_1A_2\bar{A}_3.$

$D = \bar{A}_1\bar{A}_2\bar{A}_3 + \bar{A}_1\bar{A}_2A_3 + \bar{A}_1A_2\bar{A}_3 + A_1\bar{A}_2\bar{A}_3 + \bar{A}_1A_2A_3 + A_1\bar{A}_2A_3 + A_1A_2\bar{A}_3,$

$E = A_1A_2A_3 + A_1A_2\bar{A}_3 + A_1\bar{A}_2A_3 + \bar{A}_1A_2A_3,$

$F = \bar{A}_1\bar{A}_2\bar{A}_3 + A_1\bar{A}_2\bar{A}_3 + \bar{A}_1A_2\bar{A}_3 + \bar{A}_1\bar{A}_2A_3,$

$G = \bar{A}_1\bar{A}_2A_3.$

2.6. A group of four homogeneous objects is being tracked. Each of the objects can be either detected or not detected. We consider the following events:

A = {exactly one of the four objects is detected};
B = {at least one object is detected};
C = {not less than two objects are detected};
D = {exactly two objects are detected};
E = {exactly three objects are detected};
F = {all four objects are detected}.

Find what the following events consist in: (1) $A + B$; (2) AB; (3) $B + C$; (4) BC; (5) $D + E + F$; (6) BF. Are the events BF and CF the same? Are the events BC and D the same?

Answer. (1) $A + B = B$; (2) $AB = A$; (3) $B + C = B$; (4) $BC = C$; (5) $D + E + F = C$; (6) $BF = F$. The events BF and CF are the same; BC and D are not the same.

2.7. Given below are experiments and the events that can occur in them. Indicate their complementary events.

(1) Two messages are sent over a communication channel: an event A = {both messages are sent correctly}.

(2) A ball is drawn from an urn which contains two white, three black and four red balls; an event B = {a white ball is drawn}.

(3) Five messages are sent; an event C = {not less than three messages are sent correctly}.

(4) n shots are fired at a target; an event D = {at least one hit is registered}.

(5) A preventive inspection of an instrument is made, the instrument consisting of k units, each of which can be either put in order at once or sent for repair. An event E = {none of the units is sent for repair}.

(6) Two people play chess; an event F = {the whites win}.

Answer. (1) $\bar{A}$ = {at least one message is distorted}; (2) $\bar{B}$ = {a black or a red ball is drawn}; (3) $\bar{C}$ = {not more than two messages are sent

correctly}. (4) $\bar{D}$ = {no hits are registered}; (5) $\bar{E}$ = {at least one unit must be repaired}; (6) $\bar{F}$ = {the blacks win or the game is drawn}.

2.8. An event B is a special case of an event $A : B \subseteq A$, i.e. it follows from the occurrence of the event B that the event A has occurred. Does it follow from $\bar{B}$ that $\bar{A}$ has occurred?

Answer. No, it does not. For instance, an experiment may consist in sending two messages; an event A = {at least one message is distorted} and an event B = {both messages are distorted}. If an event $\bar{B}$ has occurred, $\bar{B}$ = {less than two messages are distorted}, it does not follow that neither of the messages is distorted (event $\bar{A}$). On the contrary, $\bar{B}$ follows from $\bar{A}$ ($\bar{A} \subseteq \bar{B}$).

Figure 2.8 shows events A and B, $B \subseteq A$, and the complementary events $\bar{A}$ (vertical hatching) and $\bar{B}$ (horizontal hatching). It can be immediately seen that $\bar{A} \subseteq \bar{B}$.

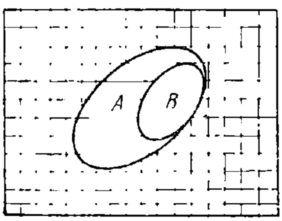

Fig. 2.8

2.9. If an event B is a special case of an event A ($B \subseteq A$), are the events dependent?

Answer. They are dependent if $P(A) \neq 1$ since $P(A/B) = 1$.

2.10. Are the following events dependent: (1) mutually disjoint events; (2) events which form a complete group; (3) equipossible events?

Answer. (1) Dependent, since the occurrence of any of them turns into zero the probabilities of all the others happening; (2) dependent, since the nonoccurrence of all but one, turns into unity the probability of the last one; (3) may be either dependent or independent.

2.11. An experiment consists in a successive tossing of two coins. We consider the following events:

A = {the first coin comes up heads}; D = {at least one head comes up}; E = {at least one tail comes up}; F = {the second coin comes up heads}. Determine whether the following pairs of events are dependent. (1) A and E; (2) A and F; (3) D and E; (4) D and F. Determine the conditional and unconditional probabilities of the events in each pair.

Answer.

(1) $P(E) = 3/4$, $P(E \mid A) = 1/2$, the events are dependent;
(2) $P(A) = 1/2$, $P(A \mid F) = 1/2$, the events are independent;
(3) $P(D) = 3/4$, $P(D \mid E) = 2/3$, the events are dependent;
(4) $P(D) = 3/4$, $P(D \mid F) = 1$, the events are dependent.

2.12. An urn contains a white and b black balls. Two balls are drawn either simultaneously or one after another. Find the probability that both balls are white*).

*) This problem, like a number of other problems in this chapter, can be also solved by a direct calculation of the number of cases. The point here is to solve them by using the addition and multiplication rules.

Answer. By the multiplication rule for probabilities

$$\mathrm{P}\{\text{both balls are white}\}=\mathrm{P}\{\mathrm{ww}\}=\frac{a}{a+b}\,\frac{a-1}{a+b-1}.$$

2.13. There are a white and b black balls in an urn. A ball is drawn, its colour is noted and the ball is replaced. Then one more ball is drawn. Find the probability that both drawn balls are white.

Answer. $[a/(a+b)]^2$.

2.14. There are a white and b black balls in an urn. Two balls are drawn at once. Find the probability that the balls are different in colour.

Solution. The event can be in two incompatible variants: {wb} or {bw} by the addition and multiplication rules

$$\mathrm{P}\{\mathrm{wb}+\mathrm{bw}\}=\frac{a}{a+b}\,\frac{b}{a+b-1}+\frac{b}{a+b}\,\frac{a}{a+b-1}=\frac{2ab}{(a+b)(a+b-1)}.$$

2.15. The problem is the same as 2.14, but the balls are drawn one after another and the first ball is replaced.

Answer. $2ab/(a+b)^2$.

2.16. There are a white and b black balls in an urn. All the balls are drawn from the urn at random, one after another. Find the probability that a white ball will be the second in the sequence.

Solution. We can find the probability of the event directly (see problem 1.12). The same result can be obtained by the addition and multiplication rules, i.e.

$$\mathrm{P}\{\mathrm{ww}+\mathrm{bw}\}=\frac{a}{a+b}\,\frac{a-1}{a+b-1}+\frac{b}{a+b}\,\frac{a}{a+b-1}=\frac{a}{a+b}.$$

2.17. There are a white, b black and c red balls in an urn. Three of them are drawn at random. Find the probability that at least two of them are of the same colour.

Solution. To find the probability of an event consisting in at least two balls being the same colour, we pass to the complementary event $\bar{A}$ = {all the balls are different in colour}.

$$\mathrm{P}(\bar{A})=\mathrm{P}\{\underbrace{\mathrm{wbr}+\mathrm{wrb}+\mathrm{rbw}+\dots}_{6\text{ combinations}}\}$$
$$=6\,\frac{a}{a+b+c}\,\frac{b}{a+b+c-1}\,\frac{c}{a+b+c-2}.$$

Hence

$$\mathrm{P}(A)=1-\mathrm{P}(\bar{A})=1-\frac{6abc}{(a+b+c)(a+b+c-1)(a+b+c-2)}.$$

2.18. A coin is tossed until two heads or two tails appear in succession (see problem 1.5). Find the probability of the event A = {not more than three tosses are needed}.

Solution. Denoting heads by "h" and tails by "t", we write A as $A = \{hh\} + \{tt\} + \{htt\} + \{thh\}$. By the addition and multiplication rules

$$P(A) = P\{hh\} + P\{tt\} + P\{htt\} + P\{thh\}$$
$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4}.$$

2.19. There are nine instruments of the same type on a shelf. At the beginning of an experiment they are all new (never have been used). Three instruments are taken at random for a temporary use; after being used, they are replaced. If we look at the instruments, we cannot distinguish between the new instruments and the used ones. Find the probability of the event

A = {after the instruments are chosen and used for three times no new instruments are left}.

Solution. Event A may occur in only one way: the first, the second and the third time new instruments are taken from the shelf. That all new instruments are taken the first time is a certainty and, therefore

$$P(A) = 1\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{9}\cdot\frac{2}{8}\cdot\frac{1}{7} \approx 0.0028.$$

2.20. The hypothesis of problem 2.19 is changed as follows: $N = lM$ instruments are on the shelf and M instruments are taken at random for a temporary use. Find the probability p that no new instruments remain on the shelf after the procedure is repeated l times.

Answer.

$$p = \frac{(N-M)!}{[N(N-1)\ldots(N-M+1)]^{l-1}}.$$

2.21. There are n new instruments on a shelf, k of them are taken at random and used for some time ($k \leqslant n/2$). Then the instruments are replaced and another k instruments are chosen at random. Find the probability p that the second set of k instruments are all new.

Answer.

$$p = \frac{n-k}{n}\,\frac{n-k-1}{n-1}\ldots\frac{n-2k+1}{n-k+1} = \frac{[(n-k)!]^2}{n!\,(n-2k)!}.$$

2.22. An instrument consists of four units, which can fail, independently of one another, during the time τ the instrument is used. The reliability (the probability of failure-free performance) of the ith unit is p_i; the probability of failure $q_i = 1 - p_i$ ($i = 1, 2, 3, 4$). Find the probabilities of the following events:

A = {failure-free operation of all the units};
B = {the first unit fails and the others do not};
C = {one of the units fails and the others do not};
D = {exactly two units out of the four fail};
E = {no less than two units fail};
F = {at least one unit fails}.

Answer.

$$\mathrm{P}(A) = p_1p_2p_3p_4, \quad \mathrm{P}(B) = q_1p_2p_3p_4, \quad \mathrm{P}(C) = q_1p_2p_3p_4$$
$$+ p_1q_2p_3p_4 + p_1p_2q_3p_4 + p_1p_2p_3q_4;$$
$$\mathrm{P}(D) = q_1q_2p_3p_4 + p_1p_2q_3q_4 + p_1q_2p_3q_4 + q_1p_2q_3p_4$$
$$+ p_1q_2q_3p_4 + q_1p_2p_3q_4, \quad \mathrm{P}(E) = \mathrm{P}(D)$$
$$+ q_1q_2q_3p_4 + q_1q_2p_3q_4 + q_1p_2q_3q_4 + p_1q_2q_3q_4$$
$$+ q_1q_2q_3q_4; \quad \mathrm{P}(F) = 1 - \mathrm{P}(A).$$

2.23. A train consisting of k carriages, which are intended for different destinations, arrives at a marshalling yard; k_1 carriages are intended for destination A_1, k_2 carriages for destination A_2 and k_3 carriages for destination A_3 ($k_1 + k_2 + k_3 = k$). The carriages are randomly scattered along the train, each place being equally likely for each carriage. Find the probability that all the carriages for each destination are side-by-side.

Answer. There can be six variants of the event B we speak of (the number of permutations of three elements $P_3 = 6$). We seek the probability of one of these variants.

C = {the first places are occupied by k_1 carriages intended for A_1; they are followed by k_2 carriages intended for A_2 and then by k_3 carriages intended for A_3}.

$$\mathrm{P}(C) = \frac{k_1}{k}\,\frac{k_1-1}{k-1}\cdots\frac{1}{k-k_1+1}\,\frac{k_2}{k-k_1}\,\frac{k_2-1}{k-k_1-1}$$
$$\cdots\frac{1}{k-k_1-k_2+1}\,\frac{k_3}{k-k_1-k_2}\,\frac{k_3-1}{k-k_1-k_2-1}\cdots 1,$$
$$\mathrm{P}(B) = 6\mathrm{P}(C).$$

2.24. A box contains standard articles, a of which are from factory I and b from factory II. We select $2k$ articles at random ($2k < a$, $2k < b$). Find the probability that there are more articles from factory I than from factory II among the selected articles.

Solution. It is easier to solve the problem by combining the method of direct calculation of probabilities with the addition rule. The event A = {there are more articles from factory I than from factory II} can be represented as the sum

$$A = A_{k+1} + A_{k+2} + \ldots + A_{2k} = \sum_{i=k+1}^{2k} A_i,$$

where A_i = {i articles from factory I are selected}.

We find by a direct calculation that

$$\mathrm{P}(A_i) = \frac{C_a^i C_b^{2k-i}}{C_{a+b}^{2k}}, \quad \text{whence} \quad \mathrm{P}(A) = \sum_{i=k+1}^{2k} \frac{C_a^i C_b^{2k-i}}{C_{a+b}^{2k}}.$$

2.25. In a batch of N articles M articles are faulty. We take n articles from the batch for inspection. If more than m articles in the batch are faulty, we reject the whole batch. Find the probability that the batch will be rejected.

Solution. The event A = {the batch is rejected} can be represented as a sum

$$A = A_{m+1} + A_{m+2} + \ldots + A_n = \sum_{j=m+1}^{n} A_i,$$

where A_i = {there are i faulty articles among those being inspected}.

$$\mathrm{P}(A_i) = \frac{C_M^i C_{N-M}^{n-i}}{C_N^n}, \quad \mathrm{P}(A) = \sum_{i=m+1}^{n} \frac{C_M^i C_{N-M}^{n-i}}{C_N^n}.$$

2.26. A box contains homogeneous articles of different qualities; a articles are of the best quality, b articles are of the first quality and c articles are of the second quality ($a \geqslant 4$, $b \geqslant 4$, $c \geqslant 4$). Four articles are taken simultaneously and at random from the box without regard for their quality. The following events are considered:

A = {at least one of the chosen articles is of the best quality};

B = {at least one of the chosen articles is of the second quality}.

Find the probability of the event $C = A + B$.

Solution. Passing to the complementary event $\bar{C}$ = {there is neither A nor B} = {all the articles are of the first quality}, we have

$$\mathrm{P}(\bar{C}) = \frac{b}{a+b+c} \frac{b-1}{a+b+c-1} \frac{b-2}{a+b+c-2} \frac{b-3}{a+b+c-3},$$

whence it follows that $\mathrm{P}(C) = 1 - \mathrm{P}(\bar{C})$.

2.27. During one surveillance cycle of a radar unit, tracking a target, the target is detected with probability p. The detection of the target in each cycle is independent of other cycles. Find the probability that the target will be detected in n cycles.

Answer. $1 - (1 - p)^n$.

2.28. There are m radar units, each of which detects a target during one surveillance cycle with probability p (independently of other cycles and other units). During time T each unit makes n cycles. Find the probability of the following events:

A = {the target is detected at least by one unit};

B = {the target is detected by all units}.

Answer.

$$\mathrm{P}(A) = 1 - (1 - p)^{mn}; \quad \mathrm{P}(B) = [1 - (1 - p)^n]^m.$$

2.29. There is a group of k targets, each of which, independently of the other targets, can be detected by a radar unit with probability p.

Each of m radar units tracks the targets independently of other units. Find the probability that not all the targets in the group will be detected.

Solution. We pass to a complementary event $\bar{A}$ = {all the targets will be detected}:

$$P(\bar{A}) = [1 - (1-p)^m]^k; \quad P(A) = 1 - [1 - (1-p)^m]^k.$$

2.30. A total of k workers take part in succession in the manufacture of an article; when an article is transferred to the next worker, its quality is not inspected. The first worker may spoil the article with probability p_1, the second worker with probability p_2, and so on. Find the probability that a faulty article will be manufactured.

Answer.

$$1 - \prod_{i=1}^{k} (1 - p_i).$$

2.31. In a certain lottery n tickets were sold, l of which were winning tickets. A certain Petrov buys k tickets. Find the probability that he will receive at least one prize.

Answer.

$$1 - \frac{n-l}{n} \frac{n-l-1}{n-1} \cdots \frac{n-l-k+1}{n-k+1} = 1 - \frac{(n-l)!\,(n-k)!}{n!\,(n-l-k)!}.$$

2.32. Two balls are distributed at random and independently of one another among four cells which lie in a straight line. Each ball can fall in each cell with the same probability of 1/4. Find the probability that the balls will fall in adjacent cells.

Solution. We divide the event A = {the balls fall in adjacent cells} into a sum of as many variants as there are pairs of adjacent cells we can form; we get $A = A_1 + A_2 + A_3$, where

A_1 = {the balls fall in the first and the second cell};

A_2 = {the balls fall in the second and the third cell};

A_3 = {the balls fall in the third and the fourth cell}.

The probability of each variant is the same and equal to

$$\frac{1}{4} \cdot \frac{1}{4} \cdot 2 = \frac{1}{8}, \quad P(A) = \frac{3}{8}.$$

2.33. k balls are distributed at random and independently of one another among n cells which lie in a straight line ($k < n$). Find the probability that they will occupy k adjacent cells.

Solution. We can choose k adjacent cells out of n in $n - k + 1$ ways. The probability that the k balls will fall in each group of adjacent cells is equal to $(1/n)^k k!$ (since they can be distributed among the cells in $k!$ ways). The probability of the event A = {the balls fall in k adjacent cells} is $P(A) = (1/n)^k\, k!\, (n - k + 1)$.

2.34. One day a post office received 20 telegrams intended for four different addresses (five for each address). Four telegrams are chosen at random. Find the probabilities of the following events:

A = {all the telegrams are intended for different addresses};

B = {all the telegrams are intended for the same address}.

Solution. For the event A to occur, the first telegram may be intended for any address; the address of the second telegram must differ from that of the first one, that of the third must differ from the addresses of the first two, and that of the fourth must differ from the addresses of the first three. By the multiplication rule for probabilities we have

$$\mathrm{P}(A) = 1 \cdot \frac{15}{19} \cdot \frac{10}{18} \cdot \frac{5}{17} \approx 0.130.$$

Similarly,

$$\mathrm{P}(B) = 1 \cdot \frac{4}{19} \cdot \frac{3}{18} \cdot \frac{2}{17} \approx 0.00413.$$

2.35. A computer consists of n units. The reliability (failure-free performance) of the first unit for the time T is p_1, that of the second unit is p_2 and so on. The units may fail independently of one another. When any unit fails, the computer fails. Find the probability that the computer will fail during the time T.

Answer.

$$1 - \prod_{i=1}^{n} p_i.$$

2.36. When the ignition is turned on, the engine picks up with probability p. Find: (1) the probability that the engine picks up when the ignition is turned on for the second time; (2) the probability that it is necessary to switch on the ignition no more than two times for the engine to begin working.

Answer. (1) $(1 - p)\,p$; (2) $1 - (1 - p)^2 = (2 - p)\,p$.

2.37. Three messages are sent over a communication channel, each of which may be transmitted with different accuracy. The transmission of one message can lead to one of the following events:

A_1 = {the message is transmitted in a correct form};

A_2 = {the message is partially distorted};

A_3 = {the message is completely distorted}.

The probabilities of the events A_1, A_2 and A_3 are known to equal p_1, p_2 and p_3 $(p_1 + p_2 + p_3 = 1)$. Considering that messages may be distorted or transmitted correctly independently of one another, find the probabilities of the following events:

A = {all three messages are transmitted in a correct form};

B = {at least one of the messages is completely distorted};

C = {no less than two messages are completely or partially distorted}.

Answer.

$$P(A) = p_1^3, \quad P(B) = 1 - (p_1 + p_2)^3,$$
$$P(C) = 3(p_2 + p_3)^2 p_1 + (p_2 + p_3)^3.$$

2.38. Two radar units are scanning a region of space in which a target is moving for a time τ. During that time the first unit makes $2n_1$ surveillance cycles and the second unit $2n_2$ cycles. During one surveillance cycle the first unit detects the target (independently of the other units) with probability p_1, the second unit with probability p_2. Find the probabilities of the following events:

A = {the target is detected in time τ by at least one of the units};

B = {the target is detected by the first unit and is not detected by the second unit};

C = {the target is not detected during the first half of the time τ and is detected during the second half of the time}.

Solution. $P(A) = 1 - P(\bar{A})$.

$$\bar{A} = \{\text{the object is not detected by either unit}\};$$
$$P(\bar{A}) = (1 - p_1)^{2n_1}(1 - p_2)^{2n_2}, \quad P(A) = 1 - (1 - p_1)^{2n_1}(1 - p_2)^{2n_2},$$
$$P(B) = [1 - (1 - p_1)^{2n_1}](1 - p_2)^{2n_2},$$
$$P(C) = (1 - p_1)^{n_1}(1 - p_2)^{n_2}[1 - (1 - p_1)^{n_1}(1 - p_2)^{n_2}].$$

2.39. During one surveillance cycle a radar installation can detect a target with probability p. How many surveillance cycles are needed for the target to be detected with a probability no less than $\mathcal{P}$?

Solution. We denote the unknown number of cycles by N. The condition $1 - (1 - p)^N \geqslant \mathcal{P}$ must be fulfilled, whence it follows that $(1 - p)^N \leqslant 1 - \mathcal{P}$. Taking logarithms, we have

$$N \log(1 - p) \leqslant \log(1 - \mathcal{P}),$$
$$N \geqslant \log(1 - \mathcal{P})/\log(1 - p). \tag{2.39}$$

2.40. A message being sent over a communication channel consists of n signs (symbols). During the transmission each sign is distorted (independently of the other signs) with probability p. For the sake of reliability, each message is repeated k times. Find the probability that at least one of the messages being transmitted will not be distorted in any sign.

Solution. The probability that one separate message will not be distorted is equal to $(1 - p)^n$; the probability that at least one out of k messages will not be distorted is

$$P(A) = 1 - [1 - (1 - p)^n]^k.$$

2.41. Under the conditions of Problem 2.40 how many times must a message be repeated for the probability that at least one message will not be distorted to be no less than $\mathscr{P}$?

Answer. By formula (2.39) we have $N \geqslant \log(1 - \mathscr{P}) / \log[1 - (1-p)^n]$.

2.42. A significant message is sent simultaneously over n communication channels and is repeated k times over each channel for the sake of reliability. During one transmission a message (independently of the other messages) is distorted with probability p. Each communication channel (independently of the other channels) is "blocked up" with noise with probability Q; a "blocked-up" channel cannot transmit any messages. Find the probability of the event

A = {a message is transmitted in a correct form at least once}.

Solution. We introduce the event

B — {a message is transmitted over one communication channel without distortion at least once}.

For the event B to occur, first the channel must not be "blocked up" with noise and, second, at least one message sent over it must not be distorted:

$$\mathrm{P}(B) = (1 - Q)(1 - p^k).$$

The probability of the event A, which is that the event B occurs at least over one channel, is

$$\mathrm{P}(A) = 1 - [1 - \mathrm{P}(B)]^n = 1 - [1 - (1 - Q)(1 - p^k)]^n.$$

2.43. An air battle between two aircraft, a fighter and a bomber, is going on. The fighter is the first to fire. It fires once at the bomber and brings it down with probability p_1. If the bomber is not brought down, it fires once at the fighter and brings it down with probability p_2. If the fighter is not brought down, it fires at the bomber again and brings it down with probability p_3. Find the probabilities of the following outcomes:

A = {the bomber is brought down};

B = {the fighter is brought down};

C = {at least one of the aircraft is brought down}.

Answer.

$$\mathrm{P}(A) = p_1 + (1 - p_1)(1 - p_2)p_3, \quad \mathrm{P}(B) = (1 - p_1)p_2,$$
$$\mathrm{P}(C) = \mathrm{P}(A) + \mathrm{P}(B).$$

2.44. An air battle between a bomber and two attacking fighters is going on. The bomber is the first to fire; it fires once at each fighter and brings it down with probability p_1. If a fighter is not brought down, it fires at the bomber irrespective of the destiny of the other fighter and

brings it down with probability p_2. Find the probabilities of the following outcomes:

A = {the bomber is brought down};
B = {both fighters are brought down};
C = {at least one fighter is brought down};
D = {at least one aircraft is brought down};
E = {exactly one fighter is brought down};
F = {exactly one aircraft is brought down}.

Solution. The probability of one of the fighters bringing down the bomber is $(1 - p_1) p_2$; the probability that neither of them brings down the bomber is $[(1 - (1 - p_1) p_2]^2$, whence

$$P(A) = 1 - [1-(1 - p_1) p_2]^2, \quad P(B) = p_1^2,$$
$$P(C) = 1 - (1 - p_1)^2, \quad P(D) = 1 - (1 - p_1)^2 (1 - p_2)^2,$$
$$P(E) = 2p_1 (1 - p_1).$$

The event F can be represented in the form $F = F_1 + F_2 + F_3$, where

F_1 = {the bomber is brought down and both fighters are safe};

F_2 = {the first fighter is brought down and the second fighter and the bomber are safe};

F_3 = {the second fighter is brought down and the first fighter and the bomber are safe}.

$$P(F_1) = (1 - p_1)^2 [1 - (1 - p_2)^2],$$
$$P(F_2) = P(F_3) = p_1 (1 - p_1) (1 - p_2),$$
$$P(F) = (1 - p_1)^2 [1 - (1 - p_2)^2] + 2p_1 (1 - p_1) (1 - p_2).$$

2.45. The conditions and the questions are the same as in the preceding problem, with the only difference that the fighters attack only in pairs: if one of them is brought down, then the second one disengages.

Answer.

$$P(A) = (1 - p_1)^2 [1 - (1 - p_2)^2], \quad P(B) = p_1^2,$$
$$P(C) = 1 - (1 - p_1)^2, \quad P(D) = 1 - (1 - p_1)^2 (1 - p_2)^2,$$
$$P(E) = 2p_1 (1 - p_1), \quad P(F) = (1 - p_1)^2 [1 - (1 - p_2)^2] + 2p_1 (1 - p_1).$$

2.46. There are a white and b black balls in an urn. Two players draw one ball each in turn, replacing it each time and stirring the balls. The player who is the first to draw a white ball wins the game. Find the probability $\mathscr{P}_1$ of the first player (the one who begins the game) being the winner.

Solution. The first player can win either in the first or in the third selection (in the latter case the first two times black balls must be drawn and the third time a white ball must be drawn) and so on.

$$\mathcal{P}_1=\frac{a}{a+b}+\left(\frac{b}{a+b}\right)^2\frac{a}{a+b}+\ldots+\left(\frac{b}{a+b}\right)^{2k}\frac{a}{a+b}+\ldots$$
$$=\frac{a}{a+b}\sum_{k=0}^{\infty}\left(\frac{b}{a+b}\right)^{2k}=\frac{a}{a+b}\,\frac{1}{1-\left(\frac{b}{a+b}\right)^2}=\frac{a+b}{a+2b}$$

(it is evident that $\mathcal{P}_1>1/2$ for any a and b).

2.47. There are two white and three black balls in an urn. Two players each draw a ball in turn without replacing it. The first player who is the first to draw a white ball wins the game. Find the probability $\mathcal{P}_1$ that the first player wins the game.

Solution.

$$\mathcal{P}_1=\frac{2}{5}+\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}=\frac{3}{5}.$$

2.48. An instrument is being tested. Upon each trial the instrument fails with probability p. After the first failure the instrument is repaired, after the second failure it is considered to be unfit for operation. Find the probability that the instrument is rejected exactly in the kth trial.

Solution. For the given event to occur, it is necessary, first, that the instrument should fail in the kth trial, the probability of that event being p. In addition, it is necessary that in the preceding $k-1$ trials the instrument should fail exactly once; the probability of that event being $(k-1)\,p\,(1-p)^{k-2}$. The required probability is equal to $(k-1)\,p^2\,(1-p)^{k-2}$.

2.49. Missiles are fired at a target. The probability of each missile hitting the target is p; the hits are independent of one another. Each missile which reaches the target brings it down with probability p_1. The missiles are fired until the target is brought down or the missile reserve is exhausted. The reserve consists of n missiles ($n>2$). Find the probability that some missiles will remain in the reserve.

Solution. We pass to the complementary event $\bar{A}$ = {the whole reserve is exhausted}. For $\bar{A}$ to occur, the first $n-1$ missiles must not bring the target down:

$$\mathrm{P}(\bar{A})=(1-pp_1)^{n-1},\qquad \mathrm{P}(A)=1-(1-pp_1)^{n-1}.$$

2.50. Under the conditions of the preceding problem find the probability that no less than two missiles remain in the reserve after the target is brought down.

Solution. The complementary event $\bar{A}$ = {less than two missiles remain in the reserve} is equivalent to the first $n-2$ missiles not bringing down the target:

$$\mathrm{P}(\bar{A})=(1-pp_1)^{n-2},\quad \mathrm{P}(A)=1-(1-pp_1)^{n-2}.$$

2.51. Under the conditions of problem 2.49 find the probability that no more than two missiles will be fired.

Solution. For no more than two missiles to be sent, the target must be brought down after the first two shots: the probability of this event is equal to $1 - (1 - pp_1)^2$.

2.52. A radar unit tracks k targets. During a surveillance period the ith target may be lost with probability p_i ($i = 1, 2, \ldots, k$). Find the probabilities of the following events

$$A = \{\text{none of the targets is lost}\};$$
$$B = \{\text{no less than one target is lost}\};$$
$$C = \{\text{no more than one target is lost}\}.$$

Answer.

$$\mathrm{P}(A) = \prod_{i=1}^{k} (1 - p_i), \quad \mathrm{P}(B) = 1 - \prod_{i=1}^{k} (1 - p_i),$$

$$\mathrm{P}(C) = \prod_{i=1}^{k} (1 - p_i) + p_1 (1 - p_2) \ldots (1 - p_k) + (1 - p_1) p_2 (1 - p_3)$$
$$\ldots (1 - p_k) + \ldots + (1 - p_1)(1 - p_2) \ldots (1 - p_{k-1}) p_k.$$

The last probability can be written in the form

$$\mathrm{P}(C) = \prod_{i=1}^{k} (1 - p_i) + \sum_{j=1}^{k} \frac{p_j}{1 - p_j} \prod_{i=1}^{k} (1 - p_i).$$

2.53. An instrument consisting of k units is in use for a period t During that time, the first unit may fail with probability q_1, the second with probability q_2 and so on. A repairman is summoned to check the instrument. He checks each unit and either locates and clears the fault, if one exists, with probability p or passes it as faultless with probability $q = 1 - p$. Find the probability of the event A = {at least one unit remains faulty after the inspections}.

Solution. The probability that the ith unit remains faulty is equal to the probability that it became faulty during the time t multiplied by the probability that the repairman failed to locate the fault: $q_i q$. The probability that this event will occur in at least one unit is

$$\mathrm{P}(A) = 1 - \prod_{i=1}^{k} (1 - q_i q).$$

2.54. A new condition is added to those of Problem 2.53: after time t a repairman is not found with probability Q and the instrument is used without inspection. Find the probability of the event A = {the instrument is used with at least one faulty unit}.

Answer.

$$(1 - Q)\left[1 - \prod_{i=1}^{k} (1 - q_i q)\right] + Q\left[1 - \prod_{i=1}^{k} (1 - q_i)\right].$$

2.55. A message is sent which consists of n binary symbols "0" and "1". Each symbol is distorted with a small probability p (is changed to the opposite). To be on the safe side, the message is repeated twice; the information is considered to be correct if both messages coincide. Find the probability that both messages are faulty in spite of their coincidence.

Solution. A = {the messages coincide but are distorted} = {the errors are in the same places}. We introduce an event B = {the messages are similar}. $B = A + C$, where C = {the messages are similar and correct}.

$$\mathrm{P}(A) = \mathrm{P}(B) - \mathrm{P}(C).$$

The event $B = \prod_{i=1}^{n} B_i$, where B_i = {the symbols which occupy the ith place in the two messages are either both true or both false}.

$$\mathrm{P}(B_i) = p^2 + (1-p^2), \quad \mathrm{P}(B) = [p^2 + (1-p)^2]^n;$$

$$\mathrm{P}(C) = [(1-p)^2]^n = (1-p)^{2n}. \quad \mathrm{P}(A) = [p^2 + (1-p)^2]^n - (1-p)^{2n}.$$

2.56. A train consists of n carriages, each of which may have a defect with probability p. All the carriages are inspected, independently of one another, by two inspectors; the first detects defects (if any) with probability p_1, and the second with probability p_2. If none of the carriages is found to have a defect, the train departs. Find the probability of the event

A = {a train departs with at least one defective carriage}.

Solution. We consider one isolated carriage and the event B = {the carriage has an undetected defect}. $\mathrm{P}(B) = p(1-p_1)(1-p_2)$. Hence

$$\mathrm{P}(A) = 1 - [1 - \mathrm{P}(B)]^n = 1 - [1 - p(1-p_1)(1-p_2)]^n.$$

2.57. A computer with a suspected flaw is tested in order to localize the flaw, for which purpose n tests are carried out in succession: T_1, T_2, . . ., T_n. As soon as the flaw is detected, the tests are terminated. The probability of localizing the flaw in the first test is p_1; the conditional probability of localizing the flaw in the second test (provided that it was not localized in the first test) is p_2; the conditional probability of localizing the flaw in the ith test (provided that it was not localized in the first $i-1$ tests) is p_i ($i = 1, 2, \ldots, n$). Find the probabilities of the following events:

A = {no less than three tests were carried out};

B = {no more than three tests were carried out};

C = {the flaw was localized exactly in the fourth test};

D = {the flaw was not localized in n tests};

E = {all n tests were carried out}.

Solution. An event $\bar{A}$ = {less than three tests were carried out} = $F_1 + F_2$, where

F_1 = {only one test was carried out, the flaw was localized};

F_2 = {the first test did not produce any results, the flaw was localized in the second test}.

$$P(F_1) = p_1, \quad P(F_2) = (1-p_1)p_2, \quad P(\bar{A}) = p_1 + (1-p_1)p_2,$$
$$P(A) = 1 - [p_1 + (1-p_1)p_2].$$

An event B = {one, two or three tests were carried out} = $F_1 + F_2 + F_3$, where F_3 = {the flaw was localized in the third test}.

$$P(F_3) = (1-p_1)(1-p_2)p_3, \quad P(B) = p_1 + (1-p_1)p_2$$
$$+ (1-p_1)(1-p_2)p_3, \quad P(C) = (1-p_1)(1-p_2)(1-p_3)p_4,$$
$$P(D) = (1-p_1)(1-p_2)\ldots(1-p_n)$$
$$= \prod_{i=1}^{n}(1-p_i), \quad P(E) = \prod_{i=1}^{n-1}(1-p_i).$$

2.58. At a railway station a passenger leaves his luggage in a locker which is opened by dialling a three-digit code (say, 253, 009, 325, etc.). The passenger chooses the code, closes the locker and leaves for the town. A strange man, who does not know the code, tries to open the locker by dialling three digits at random. Find the probabilities of the following events:

A = {the locker opens on the first trial};

B = {the locker opens after k trials}.

Solution. $P(A) = 0.1 \cdot 0.1 \cdot 0.1 = 0.001$. If k trials are made, then it is natural to assume that the unsuccessful combinations are not repeated. We pass from B to a complementary event $\bar{B}$ = {all k trials are unsuccessful}. Then

$$P(B) = 1 - P(\bar{B}) = 1 - \frac{999}{1000}\cdot\frac{998}{999}\cdots\frac{1000-k+1}{1000-k+2}\cdot\frac{1000-k}{1000-k+1}$$
$$= 1 - \frac{1000-k}{1000} = \frac{k}{1000}. \qquad (2.58)$$

It stands to reason that formula (2.58) is meaningful only for $k < 1000$; for $k \geqslant 1000$ the probability of finding the correct combination $P(B) = 1$.

2.59. In the town of Tbilisi the three most widely used languages are Georgian, Armenian and Russian. A group of m people is chosen, of whom m_1 people speak only Georgian, m_2 people speak only Armenian, and m_3 people speak only Russian; m_{12} people speak Georgian and Armenian; m_{13} people speak Georgian and Russian; m_{23} people speak Armenian and Russian; m_{123} people speak all the three languages;

$m_1 + m_2 + m_3 + m_{12} + m_{13} + m_{23} + m_{123} = m$. Two people are chosen from the group at random. What is the probability p that they can use one of the three languages to talk to each other without an interpreter?

Solution. We enumerate all the seven groups of people and write in front of each group the languages which all the members of the group speak, denoting them by the letters G, A and R.

(I) G (m_1 people); (II) A (m_2 people); (III) R (m_3 people); (VI) G, A (m_{12} people); (V) G, R (m_{13} people); (VI) A, R (m_{23} people); (VII) G, A, R (m_{123} people).

For two people to talk to each other, they must belong to a pair of groups which have a language in common. It is simpler in this case to find the probability that two people cannot talk to each other. They must then belong to one of the following pairs: (I, II); (I, III); (I, VI); (II,III); (II, V); (III, IV). The probabilities that one of the two selected people belongs to one group and the other to another group are

$$P(I, II) = \frac{2m_1m_2}{m(m-1)},$$

$$P(I, III) = \frac{2m_1m_3}{m(m-1)}, \quad P(I, VI) = \frac{2m_1m_{23}}{m(m-1)},$$

$$P(II, III) = \frac{2m_2m_3}{m(m-1)}, \quad P(II, V) = \frac{2m_2m_{13}}{m(m-1)},$$

$$P(III, IV) = \frac{2m_3m_{12}}{m(m-1)}.$$

Adding these probabilities and subtracting the resulting sum from unity, we find the required probability p:

$$p = 1 - \frac{2}{m(m-1)}(m_1m_2 + m_1m_3 + m_1m_{23} + m_2m_3 + m_2m_{13} + m_3m_{12}).$$

2.60. A factory manufactures a certain type of article. Each article may have a defect, the probability of which is p. A manufactured article is checked successively by k inspectors; the ith inspector detects a defect (if any) with probability p_i ($i = 1, 2, \ldots, k$). If a defect is detected, the article is rejected. Find the probabilities of the following events:

A = {an article is rejected};

B = {an article is rejected by the second inspector and not by the first};

C = {an article is rejected by all k inspectors}.

Answer.

$$P(A) = p\left[1 - \prod_{i=1}^{k}(1-p_i)\right],$$

$$P(B) = p(1-p_1)p_2, \quad P(C) = p\prod_{i=1}^{k}p_i.$$

2.61. A factory manufactures a certain type of article. Each article may have a defect with probability p. An article is checked by one inspector who detects a defect with probability p_1. If he does not detect a defect, he lets the article pass. In addition, the inspector may make a mistake and reject a flawless article, the probability of which event is α. Find the probabilities of the following events:

A = {an article is rejected};

B = {an article is rejected by mistake};

C = {an article with a defect is passed to a lot of finished products}.

Answer.

$$\mathrm{P}(A) = pp_1 + (1-p)\alpha, \quad \mathrm{P}(B) = (1-p)\alpha,$$
$$\mathrm{P}(C) = p(1-p_1).$$

2.62. The conditions are the same as in the preceding problem, but each article is checked by two inspectors. The probabilities that the first and the second inspector will reject a defective article are p_1 and p_2 respectively; the probabilities of rejecting a flawless article by mistake are α_1 and α_2 respectively. If at least one inspector rejects an article, it goes to waste. Find the probabilities of the same events.

Answer.

$$\mathrm{P}(A) = p[1-(1-p_1)(1-p_2)] + (1-p)[1-(1-\alpha_1)(1-\alpha_2)],$$
$$\mathrm{P}(B) = (1-p)[1-(1-\alpha_1)(1-\alpha_2)],$$
$$\mathrm{P}(C) = p(1-p_1)(1-p_2).$$

2.63. An instrument consists of n units (Fig. 2.63), and a failure of any unit leads to a failure of the instrument as a whole. The units may

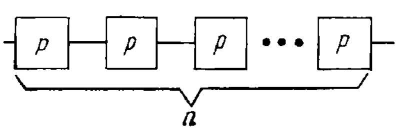

Fig. 2.63

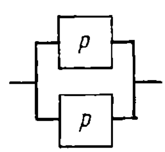

Fig. 2.64

fail independently. The reliability (the probability of failure-free performance) of each unit is p. Find the reliability P of the whole instrument. What must be the reliability p_1 of each unit to ensure the reliability P_1 of the instrument?

Remark. From now on elements without which a system cannot operate are represented as links connected in series; elements which repeat each other are connected in parallel. The reliability of each element is written in a respective square.

Answer. $P = p^n$, $p_1 = \sqrt[n]{P_1}$.

2.64. To increase the reliability of an instrument, it is repeated by another instrument of the same kind (Fig. 2.64); the reliability (the probability of failure-free performance) of each instrument is p. When the first instrument fails, the second instrument is instantaneously switched on (the reliability of the switching device is equal to unity). Find the reliability P of a system of two instruments which repeat each other.

Solution. For the system to fail, the two instruments must fail simultaneously; the reliability of the system $P = 1 - (1 - p)^2$.

2.65. The same problem, but the reliability of the switching device Sw is p_1 (Fig. 2.65).

Answer. The reliability of the system $P = 1 - (1 - p)(1 - p_1p)$.

2.66. To increase the reliability of an instrument it is repeated by $(n - 1)$ instruments of the same kind (Fig. 2.66); the reliability of

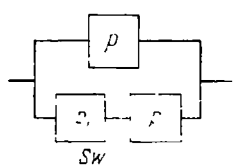

Fig. 2.65

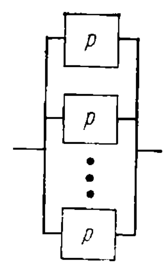

Fig. 2.66

each instrument is p. Find the reliability P of the system. How many instruments must be used to increase the reliability to the assigned value P_1?

Answer. $P = 1 - (1 - p)^n$; $n \geqslant \log(1 - P_1)/\log(1 - p)$.

2.67. The same problem but a device with reliability p_1 is used to switch on every stand-by instrument (Fig. 2.67).

Answer.

$$P = 1 - (1 - p)(1 - p_1p)^{n-1}, \quad n \geqslant \frac{\log(1 - P_1) - \log(1 - p)}{\log(1 - p_1p)} + 1.$$

2.68*. A system consists of n units, the reliability of each of which is p. A failure of at least one unit leads to the failure of the whole system. To increase the reliability of the system, its operation is repeated, for which purpose another n units are allotted. The switching devices are completely reliable. Determine which of the following repeating techniques is the more reliable; (a) each unit is repeated (Fig. 2.68a), (b) the whole system is repeated (Fig. 2.68b).

Solution. The reliability of the system repeated by technique (a) is $p_a = [1 - (1 - p)^2]^n$ and that of the system repeated by technique (b) is $p_b = 1 - (1 - p^n)^2$. Let us show that $p_a > p_b$ for any $n > 1$ and

$0 < p < 1$. Since

$$p_a = [1-(1-p)^2]^n = [1-1+2p-p^2]^n = p^n(2-p)^n,$$
$$p_b = 1-(1-p^n)^2 = 1-1+2p^n-p^{2n} = p^n(2-p^n),$$

it is sufficient to prove that $(2-p)^n > 2-p^n$. We set $q = 1-p$ $(q > 0)$ and the inequality assumes the form $(2-1+q)^n > 2-$

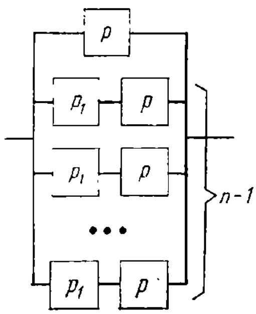

Fig. 2.67

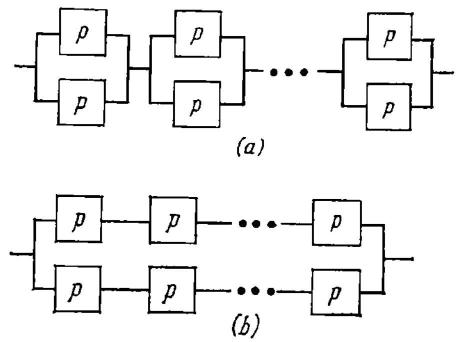

Fig. 2.68

$1-q)^n$, or $(1+q)^n + (1-q)^n > 2$. Applying the binomial theorem, we note that all the negative terms are eliminated:

$$\begin{aligned}(1+q)^n+(1-q)^n &= 1+nq+\frac{n(n-1)}{2}q^2+\ldots\\ &+1-nq+\frac{n(n-1)}{2}q^2-\ldots\\ &= 2+n(n-1)q^2+\ldots > 2,\end{aligned}$$

and this proves the required inequality.

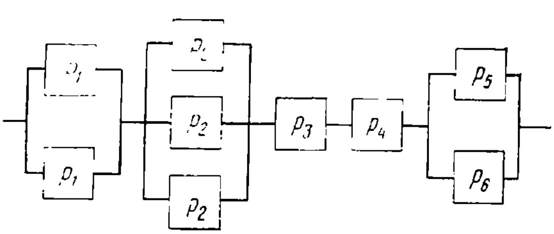

Fig. 2.69

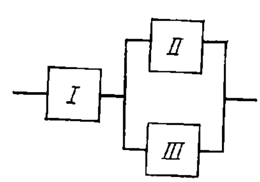

Fig. 2.70

2.69. In a technical system not all units are repeated but only some of them which are less reliable. The reliabilities of the units are shown in Fig. 2.69. Find the reliability P of the system.

Answer. $P = [1-(1-p_1)^2][1-(1-p_2)^3]p_3p_4[1-(1-p_5)^2]$.

2.70. An instrument consists of three units. The first unit contains n_1 elements, the second contains n_2 elements and the third contains n_3 ele-

ments. For the instrument to function, unit I is obviously necessary, the two other units II and III repeat each other (Fig. 2.70). The reliability of the elements is the same and equal to p. A failure of one element means a failure of the whole unit. The elements fail independently of one another. Find the reliability P of the instrument.

Solution. The reliability of unit I is $p_{\mathrm{I}} = p^{n_1}$; the reliability of unit II is $p_{\mathrm{II}} = p^{n_2}$; the reliability of unit III is $p_{\mathrm{III}} = p^{n_3}$; the reliability of the repeaters (II and III) is $1-(1-p^{n_2})(1-p^{n_3})$, the reliability of the instrument is

$$P = p^{n_1}[1-(1-p^{n_2})(1-p^{n_3})].$$

2.71. There is an electrical device which can fail (burn out) only at the moment when it is switched on. If the device has been switched on $k-1$ times and has not burned out, then the conditional probability of its burning out on the kth switching operation is Q_k. Find the probabilities of the following events:

A = {the device withstands no less than n switching operations};

B = (the device withstands no more than n switching operations};

C = {the device burns out exactly on the nth switching operation}.

Solution. The probability of the event A is equal to the probability that the device will not burn out on the first n switching operations:

$$\mathrm{P}(A) = \prod_{k=1}^{n}(1-Q_k).$$

To find the probability of the event B, we pass to a complementary event $\bar{B}$ = {the device will withstand more than n switching operations}. For that event to occur, it is sufficient that the device should not burn out on the first $(n+1)$ operations:

$$\mathrm{P}(\bar{B}) = \prod_{k=1}^{n+1}(1-Q_k), \quad \mathrm{P}(B) = 1-\prod_{k=1}^{n+1}(1-Q_k).$$

For the device to burn out exactly on the nth switching operation it is necessary that it should not burn out on the first $(n-1)$ operations and should burn out on the nth operation:

$$\mathrm{P}(C) = Q_n\prod_{k=1}^{n-1}(1-Q_k).$$

2.72. An instrument consists of four units; two of them (I and II) are obviously necessary for the instrument to function and the other two (III and IV) repeat each other (Fig. 2.72). The units can only fail when they are switched on. On the kth switching operation unit I fails

(independently of the other units) with probability $q_{I}(k)$, unit II with probability $q_{II}(k)$, unit III and unit IV fail with the same probability $q_{III}(k) = q_{IV}(k) = q(k)$. Find the probabilities of the events A, B, C as in Problem 2.71.

Solution. The problem reduces to the preceding problem when the conditional probability Q_k that a sound device will fail on the kth switching operation is found: $Q_k = 1 - [1 - q_{I}(k)]\,[1 - q_{II}(k)] \times [1 - (1 - q(k))^2]$, and is substituted into the solution obtained.

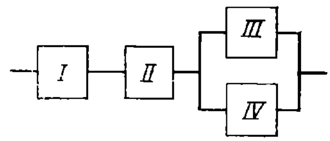

Fig. 2.72

2.73. An instrument consists of three units. When the instrument is switched on, a fault occurs in the first unit with probability p_1, in the second unit with probability p_2, and in the third unit with probability p_3. Faults occur in the units independently of one another. Each of the three units is obviously necessary for the instrument to function. For a unit to fail completely at least two faults must occur. Find the probability of an event

$$A = \{\text{the device withstands } n \text{ switching operations}\}.$$

Solution. For the event A to occur, it is necessary that all the three units should function. The probability that the first unit withstands n switching operations is equal to the probability that no more than one fault (0 or 1) occurs when it is switched on n times: $(1 + p_1)^n + np_1(1 - p_1)^{n-1}$. The probability that all the three units withstand n switching operations is

$$\mathrm{P}(A) = \prod_{i=1}^{3} [(1-p_i)^n + np_i(1-p_i)^{n-1}].$$

2.74. An instrument consists of three units, only one of which is necessary for the instrument to function, the other two repeat each other. Faults only occur in the instrument when it is functioning and can occur in any of the components constituting the units with the same probability. The first unit consists of n_1 components, the second of n_2 components, the third of n_3 components $(n_1 + n_2 + n_3 = n)$. When a fault occurs in at least one component the unit fails.

Four faults are known to have occurred in the instrument (in four different components). Find the probability that these faults prevent the instrument from functioning.

Solution. An event $A = \{\text{the instrument cannot function}\}$ is divided in two: $A = A_1 + A_2$, where

$A_1 = \{\text{the first unit fails}\}$;

$A_2 = \{\text{the first unit does not fail but the second and the third do}\}$.

For the event A_1 to occur, it is necessary that at least one of the four faults be in the first unit:

$$\mathrm{P}(A_1) = 1 - \mathrm{P}(\bar{A}_1) = 1 - \frac{n-n_1}{n}\,\frac{n-n_1-1}{n-1}\,\frac{n-n_1-2}{n-2}\,\frac{n-n_1-3}{n-3}\,.$$

To find the probability of the event A_2, the probability of the event $\bar{A}_1 = \{$the first unit does not fail$\}$ must be multiplied by the probability that the second and third units fail (with due regard for the fact that all the four faults occur in the second and third units). The last event may be in three variants: either one fault occurs in the second unit and the other three in the third unit, or, conversely, three faults occur in the second unit and one fault occurs in the third unit, or else two faults occur in the second and in the third unit each.

The probability of the first variant is

$$C_4^1\,\frac{n_2}{n_2+n_3}\,\frac{n_3}{n_2+n_3-1}\,\frac{n_3-1}{n_2+n_3-2}\,\frac{n_3-2}{n_2+n_3-3} = \mathcal{P}_1.$$

That of the second variant is

$$C_4^1\,\frac{n_3}{n_2+n_3}\,\frac{n_2}{n_2+n_3-1}\,\frac{n_2-1}{n_2+n_3-2}\,\frac{n_2-2}{n_2+n_3-3} = \mathcal{P}_2.$$

And the probability of the third variant is

$$C_4^2\,\frac{n_2}{n_2+n_3}\,\frac{n_2-1}{n_2+n_3-1}\,\frac{n_3}{n_2+n_3-2}\,\frac{n_3-1}{n_2+n_3-3} = \mathcal{P}_3.$$

Hence

$$\mathrm{P}(A_2) = [1-\mathrm{P}(A_1)]\,[\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3], \quad \mathrm{P}(A) = \mathrm{P}(A_1) = \mathrm{P}(A_2).$$

2.75. An instrument which must be very reliable is assembled from k parts $D_1, D_2, \ldots, D_k$. Before the assembly starts, each part is inspected thoroughly and, if it proves to be of very high quality, is included in the instrument and if not, is replaced by another part, which is also inspected. There is a stock of spare parts of each type: m_i parts of type D_i ($i = 1, \ldots, k$), a total of $m = \sum_{i=1}^{k} m_i$ parts. If there are not enough spare parts, the assembly is postponed. The probability that a part of type D_i proves to be of high quality is equal to p_i and does not depend on the quality of the other parts. Find the probabilities of the following events:

$A = \{$the stock of spare parts is sufficient for the assembly of the instrument$\}$;

$B = \{$all the spare parts have been used (tested and included in the instrument or only tested)$\}$;

$C = \{$using the available stock of spare parts, the fitter will assemble the instrument and at least one spare part will remain$\}$.

Solution. The event $A = \prod_{i=1}^{k} A_i$, where A_i = {at least one part of type D_i proves to be high quality}.

$$P(A_i) = 1-(1-p_i)^{m_i}, \quad P(A) = \prod_{i=1}^{k} [1-(1-p_i)^{m_i}].$$

The event $B = \prod_{i=1}^{k} B_i$, where B_i = {all the parts of type D_i have been used}. For event B_i to occur, the first $m_i - 1$ parts of type D_i must be low quality:

$$P(B_i) = (1-p_i)^{m_i-1}, \quad P(B) = \prod_{i=1}^{k} (1-p_i)^{m_i-1}.$$

To find P (C), we represent the event A as a sum of mutually exclusive events

$$A = C + F, \tag{2.75}$$

where F = {the fitter manages to assemble the instrument but no spare parts remain in stock}; $F = \prod_{i=1}^{k} F_i$, where F_i = {the first $m_i - 1$ parts of type D_i prove to be low quality and the m_ith part proves to be high quality}.

$$P(F_i) = (1-p_i)^{m_i-1} p_i, \quad P(F) = \prod_{i=1}^{k} [(1-p_i)^{m_i-1} p_i].$$

From formula (2.75) we have

$$P(C) = P(A) - P(F) = \prod_{i=1}^{k} [1-(1-p_i)^{m_i}] - \prod_{i=1}^{k} [(1-p_i)^{m_i-1} p_i].$$

2.76. A device S consists of n units, each of which may end up in one of the following states as a result of its operation for the time τ:

s_1 = {the unit is in good order};

s_2 = {the unit must be adjusted};

s_3 = {the unit must be repaired};

s_4 = {the unit is out of order}.

$$P(s_1) = p_1, \quad P(s_2) = p_2, \quad P(s_3) = p_3, \quad P(s_4) = p_4,$$
$$p_1 + p_2 + p_3 + p_4 = 1.$$

The states of separate units are independent. Find the probabilities of the following events:

A = {all the units are in good order};

B = {all the units must be adjusted};

$C =$ {one unit must be repaired and the other units must be adjusted};

$D =$ {at least one unit is completely out of order};

$E =$ {two units must be adjusted, one unit must be repaired, and the other units are in good order}.

Solution. $\mathrm{P}(A) = p_1^n$, $\mathrm{P}(B) = p_2^n$. One unit, which must be repaired, can be chosen in $C_n^1 = n$ ways:

$$\mathrm{P}(C) = np_3p_1^{n-1}, \quad \mathrm{P}(D) = 1 - (1 - p_4)^n.$$

Two units, which must be adjusted, can be chosen out of n units in $C_n^2 = \frac{n(n-1)}{2}$ ways; one unit, which must be repaired, can be chosen in $C_{n-2}^1 = n - 2$ ways:

$$\mathrm{P}(E) = \frac{n(n-1)}{2}(n-2)p_2^2p_1^{n-3}.$$

2.77. Six messages are sent over a communication channel, each of which independently of other messages may be distorted with probability $p = 0.2$. Find the probabilities of the following events:

$C =$ {exactly two messages of the six are distorted},

$D =$ {no less than two messages of the six are distorted}.

Solution. By the theorem on a repetition of trials [see formula (2.0.15)] $\mathrm{P}(C) = P_{2,6} = C_6^2 \cdot 0.2^2 \cdot 0.8^4 \approx 0.197$. By formula (2.0.18) we have

$$\mathrm{P}(D) = 1 - (\mathrm{P}_{0,6} + P_{1,6}) = 1 - (0.8^6 + 6 \cdot 0.2 \cdot 0.8^5)$$
$$= 1 - (0.262 + 0.393) = 0.345.$$

2.78. A total of n messages are sent over a communication channel. Each of them may be distorted by interference independently of the other messages with probability p. Find the probabilities of the following events:

$A =$ {exactly m of the n messages are distorted};

$B =$ {no less than m messages of the n are transmitted without distortion};

$C =$ {no more than half of the messages are distorted};

$D =$ {all the messages are transmitted without distortion};

$E =$ {no less than two messages are distorted}.

Solution. By the theorem on a repetition of trials [see formula (2.0.15)]

$$\mathrm{P}(A) = C_n^m p^m (1 - p)^{n-m}.$$

Using formula (2.0.17) and replacing p by $1-p$, we obtain

$$\mathrm{P}(B)=\sum_{k=m}^{n} C_n^k (1-p)^k p^{n-k}, \quad \mathrm{P}(C)=\sum_{k=0}^{[n/2]} C_n^k p^k (1-p)^{n-k},$$

where $[n/2]$ is an integral part of the number $n/2$.

$$\mathrm{P}(D)=(1-p)^n,$$

the same value can be obtained from formula (2.0.15).:

$$\mathrm{P}(D)=P_{0,n}=C_n^0 p^0 (1-p)^n=1\cdot 1\cdot (1-p)^n.$$

By formula (2.0.18) we find that

$$\mathrm{P}(E)=R_{2,n}=1-\sum_{k=0}^{1} C_n^k p^k (1-p)^{n-k}$$
$$=1-[(1-p)^n+np(1-p)^{n-1}]$$
$$=1-(1-p)^{n-1}[1+(n-1)p].$$

2.79. A device S consists of five units, each of which may fail during its operation with probability $p=0.4$. Units fail independently of one another. If more than three units fail, the device cannot function; if one or two units fail, the device functions but with a lower efficiency. Find the probabilities of the following events:

$A=$ {not a single unit fails in the device};

$B=$ {the device can function};

$F=$ {the device functions with a lower efficiency}.

Solution.

$$\mathrm{P}(A)=0.6^5\approx 0.0778, \quad \mathrm{P}(B)=\mathrm{P}(A)+\mathrm{P}(C)+\mathrm{P}(D),$$

where $C=$ {exactly one unit fails}; $D=$ {exactly two units fail}.

$$\mathrm{P}(C)=P_{1,5}=5\cdot 0.4\cdot 0.6^4\approx 0.259,$$
$$\mathrm{P}(D)=P_{2,5}=C_5^2\cdot 0.4^2\cdot 0.6^3\approx 0.346,$$
$$\mathrm{P}(B)=\mathrm{P}(A)+\mathrm{P}(C)+\mathrm{P}(D)\approx 0.683,$$
$$\mathrm{P}(F)=\mathrm{P}(C)+\mathrm{P}(D)\approx 0.605.$$

2.80*. There is a contest between k marksmen, each of which fires n shots at his target. The probability of hitting the target on one shot is p_i $(i=1, \ldots, k)$ for the ith marksman. The marksman who hits a target the most wins the contest. Find the probability that one and only one marksman will be the winner.

Solution. Any of the k marksmen may win. We find the probability that the ith marksman is the winner (event A_i). This event can occur in the following ways: $A_i^{(m)}=$ {the ith participant hits the target exactly m times and each of the other participants makes no more than $m-1$ hits} $(m=1, \ldots, n)$.

The probability $P_m(i)$ that the ith marksman makes m hits is equal to $C_n^m p_i^m q_i^{n-m}$, where $q_i = 1 - p_i$. We designate the probability that the jth marksman makes no more than $m - 1$ hits as $R_m(j)$:

$$R_m(j) = \sum_{s=0}^{m-1} C_n^s p_j^s q_j^{n-s} \quad (m \geqslant 1).$$

Then the probability that all the other participants, except for the ith, make no more than $m - 1$ hits, is

$$R_{m-1}(1) R_{m-1}(2) \ldots R_{m-1}(i-1) R_{m-1}(i+1) \ldots R_{m-1}(k) = \prod_{j \neq i} R_{m-1}(j).$$

Summing up the probabilities obtained for all the values of m, we get the probability that the ith participant is the only winner of the contest:

$$\mathrm{P}(A_i) = \sum_{m=1}^{n} \mathrm{P}_m(i) \prod_{j \neq i} R_{m-1}(j) \quad (i = 1, \ldots, k).$$

Summing up these probabilities for all the participants, we obtain

$$\mathrm{P}(A) = \sum_{i=1}^{k} \mathrm{P}(A_i) = \sum_{i=1}^{k} \sum_{m=1}^{n} \mathrm{P}_m(i) \prod_{j \neq i} R_{m-1}(j).$$

2.81. A family who got a new flat installed $2k$ new electric lamps. In the course of a year each lamp may burn out with probability r. Find the probability of an event A = {in the course of a year no less than half of the lamps will have to be replaced}.

Answer.

$$\mathrm{P}(A) = 1 - \sum_{m=0}^{k-1} C_{2k}^m r^m (1-r)^{2k-m^3}.$$

2.82. A factory manufactures articles, each of which, independently of the other articles, may have a defect with probability r. When an article is inspected, any defect may be detected with probability p. The checking operation consists in choosing n articles for inspection. Find the probabilities of the following events:

A = {no defects are detected in any of the articles};

B = {a defect is detected in exactly two of the n articles};

C = {a defect is detected in no less than two of the n articles}.

Solution. The probability that one article chosen at random contains a defect which is detected is pr.

$$\mathrm{P}(A) = (1 - pr)^n, \quad \mathrm{P}(B) = C_n^2 (pr)^2 (1 - pr)^{n-2},$$

$$\mathrm{P}(C) = 1 - (1 - pr)^{n-1} [(1 - pr) + npr].$$

2.83. The conditions are the same as in Problem 2.82 but now the whole batch of articles chosen for the checking operation is rejected if

no less than four articles among the chosen n have defects. Find the probability p that the whole batch of articles chosen for inspection is rejected.

Answer.

$$p=\sum_{i=4}^{n} C_n^i (pr)^i (1-pr)^{n-i}.$$

2.84. Given two instruments, the first of which consists of n_1 units and the second consists of n_2 units. Each instrument operates for a time t. In the course of that time each unit of the first instrument may, independently of the other units, fail with probability q_1 and each unit of the other instrument may fail with probability q_2. Find the probability p that during the time t m_1 units will fail in the first instrument and m_2 units will fail in the second instrument.

Answer.

$$p=C_{n_1}^{m_1} q_1^{m_1} (1-q_1)^{n_1-m_1} C_{n_2}^{m_2} q_2^{m_2} (1-q_2)^{n_2-m_2}.$$

2.85. A coin is tossed m times. Find the probability that the heads will occur no less than k times and no more than l times ($k \leqslant l \leqslant m$).

Solution.

$$p=\sum_{i=k}^{l} \mathrm{P}_{i,m}=\sum_{i=k}^{l} C_m^i (1/2)^m=(1/2)^m \sum_{i=k}^{l} C_m^i.$$

2.86. An instrument which consists of k units operates for a time t. The reliability (the probability of failure-free performance) of each unit during the time t is p. When the time t passes, the instrument stops functioning, a technician inspects it and replaces the units that failed. He needs a time τ to replace one unit. Find the probability P that in the time 2τ after the stop the instrument will be ready for work.

Solution. For the conditions of the problem to be fulfilled, it is necessary that no more than two units should fail during the time t:

$$\mathrm{P}=p^k+k(1-p)\,p^{k-1}+\frac{k(k-1)}{2}(1-p)^2 p^{k-2}.$$

2.87. What is more probable, when playing with a person of the same ability, to win (1) three games out of four or five games out of eight? (2) no less than three games out of four or at least five games out of eight?

Solution. (1) A = {winning 3 games out of 4}; B = {winning 5 games out of 8}. $\mathrm{P}(A)=C_4^3(1/2)^4=1/4$, $\mathrm{P}(B)=C_8^5(1/2)^8=7/32$; $\mathrm{P}(A)>\mathrm{P}(B)$.

(2) C = {winning no less than 3 games out of 4}; D = {winning no less than 5 games out of 8}; $\mathrm{P}(C)=C_4^3(1/2)^4+(1/2)^4=5/16$; $\mathrm{P}(D)=C_8^5(1/2)^8+C_8^6(1/2)^8+C_8^7(1/2)^8+(1/2)^8=93/256$; $\mathrm{P}(D)>\mathrm{P}(C)$.

2.88. A person belonging to a certain group of citizens may be dark-haired with probability 0.2, brown-haired with probability 0.3, fair-

haired with probability 0.4, and red-haired with probability 0.1. A group of six people is selected at random. Find the probabilities of the following events:

A = {there are no less than four fair-haired people in the chosen group};

B = {there is no less than one red-haired person in the group};

C = {there is an equal number of fair-haired and brown-haired people in the group}.

Solution.

$$\mathrm{P}(A) = 1 - [0.6^6 + 6 \times 0.4 \cdot 0.6^5 + 15 \times 0.4^2 \times 0.6^4] \approx 0.455;$$

$$\mathrm{P}(B) = 1 - (1 - 0.1)^6 \approx 0.468; \quad C = C_0 + C_1 + C_2 + C_3,$$

where

C_0 = {there are neither fair-haired nor brown-haired people in the group};

C_1 = {there is one fair-haired and one brown-haired person in the group and the other people are neither fair-haired nor brown-haired};

C_2 = {there are two fair-haired and two brown-haired people in the group and the other people are neither fair-haired nor brown-haired};

C_3 = {there are three fair-haired and three brown-haired people in the group}.

$$\mathrm{P}(C_0) = (1 - 0.7)^6 \approx 0.0007,$$

$$\mathrm{P}(C_1) = \frac{6!}{1!\,1!\,4!} 0.3 \times 0.4 (1 - 0.7)^4 \approx 0.0292,$$

$$\mathrm{P}(C_2) = \frac{6!}{2!\,2!\,2!} 0.3^2 \times 0.4^2 (1 - 0.7)^2 \approx 0.1166,$$

$$\mathrm{P}(C_3) = \frac{6!}{3!\,3!} 0.3^3 \times 0.4^3 \approx 0.0346, \quad \mathrm{P}(C) \approx 0.181.$$

2.89. N instruments operate for a period of time t. Each instrument has a reliability p and fails independently of the other instruments. Find the probability $\mathrm{P}(A)$ that the repairman who is called after the time t to repair the faulty instruments will not make the repairs in a time τ if he needs a time τ_0 to repair one faulty instrument.

Solution. The event A consists in the number of faulty instruments larger than $l = [\tau/\tau_0]$, where $[\tau/\tau_0]$ is the greatest integer included in τ/τ_0.

$$\mathrm{P}(A) = \sum_{m=l+1}^{N} C_N^m (1-p)^m p^{N-m}.$$

2.90. N faulty instruments are tested to locate a defect. In each test, independent of the other tests, a defect may be located with probabili-

ty p. When a defect has been located, the instrument is passed to a repair shop and the other instruments are tested. When the defects in all the N instruments are located, the tests are terminated. The inspection procedure only allows n tests ($n > N$). Find the probability that the defects in all the N instruments will be located.

Solution. A = {all the defects are located} = {the defects are located at least N times}.

$$\mathrm{P}(A) = \sum_{m=N}^{n} C_n^m p^m (1-p)^{n-m}.$$

2.91. The conditions are the same as in the preceding problem. Find the probability that as a result of n tests at least k instruments of the N remain with unlocated defects ($k < N$).

Solution. The problem is equivalent to finding the probability that in n tests the defects are located in no more than $N-k$ instruments.

$$\mathrm{P}(A) = \sum_{m=0}^{N-k} C_n^m p^m (1-p)^{n-m}.$$

2.92*. An instrument consists of n units. The probability that the ith unit will fail is p_i ($i = 1, 2, \ldots, n$). For the instrument to function, all the units must operate without failure. To calculate the probability R that the instrument fails, the probabilities p_i ($i = 1, 2, \ldots, n$) are approximated by their arithmetic mean:

$$\tilde{p} = \frac{1}{n} \sum_{i=1}^{n} p_i. \tag{2.92}$$

Will the approximation $\tilde{R}$ of the probability R be greater or smaller than the true R?

Solution. The exact value $R = 1 - \prod_{i=1}^{n} q_i$, where $q_i = 1 - p_i$.

The approximate value (in the mean probability $\tilde{p}$)

$$\tilde{R} = 1 - (1-\tilde{p})^n = 1 - \left[\frac{1}{n}\left(n - \sum_{i=1}^{n} p_i\right)\right]^n = 1 - \left[\frac{1}{n}\sum_{i=1}^{n} q_i\right]^n.$$

We must compare the quantities $\prod_{i=1}^{n} q_i$ and $\left[\frac{1}{n}\sum_{i=1}^{n} q_i\right]^n$. The geometric mean of unequal positive values is known to be smaller than their arithmetic mean, whence it follows that

$$\sqrt[n]{\prod_{i=1}^{n} q_i} < \frac{1}{n}\sum_{i=1}^{n} q_i, \quad \prod_{i=1}^{n} q_i < \left[\frac{1}{n}\sum_{i=1}^{n} q\right]^n,$$

and consequently $\tilde{R} < R$.

2.93. The articles a factory manufactures must each undergo four tests. An article may pass the first test safely with probability 0.9, the second test with probability 0.95, the third test with probability 0.8 and the fourth test with probability 0.85. Find the probability that the article will pass

$$A = \{\text{all the four tests}\};$$
$$B = \{\text{exactly two tests}\};$$
$$C = \{\text{at least two tests}\}.$$

Answer. $\mathrm{P}(A) \approx 0.581$, $\mathrm{P}(B) \approx 0.070$, $\mathrm{P}(C) \approx 0.994$.

2.94. A message consists of n symbols and to increase the reliability of transmission each symbol is repeated m times. A valid symbol at the receiving end is that which is repeated at least k times out of m. If, at the receiving end, a symbol is repeated less than k times, the symbol is not recognized and is considered to be distorted. The probability p of any symbol being transmitted correctly is the same and does not depend on whether the other symbols are recognized. Find the probabilities of the following events:

A = {a symbol in the message will be recognized at the receiving end};

B = {the whole message will be recognized at the receiving end};

C = {no more than l symbols will be distorted in the message}.

Solution. For a symbol to be recognized at the receiving end, it must be repeated without distortion no less than k times out of m. The probability that a symbol will be reproduced correctly j times out of m [in accordance with formula (2.0.15)] is

$$C_m^j p^j (1 - p)^{m-j}.$$

The probability that a symbol will be transmitted in a correct form no less than k times out of m can be calculated by formula (2.0.17):

$$\mathrm{P}(A) = \sum_{j=k}^{m} C_m^j (1-p)^{m-j} p^j$$

or, for a comparatively small $k < m/2$, by formula (2.0.18):

$$\mathrm{P}(A) = 1 - \sum_{j=0}^{k-1} C_m^j p^j (1-p)^{m-j}.$$

For the whole message to be recognized at the receiving end all n symbols must be reproduced correctly:

$$\mathrm{P}(B) = [\mathrm{P}(A)]^n.$$

The probability of the event C can be calculated by the formula

$$\mathrm{P}(C) = \sum_{i=0}^{l} [1 - \mathrm{P}(A)]^i [\mathrm{P}(A)]^{n-i}.$$

CHAPTER 3

The Total Probability Formula and Bayes's Theorem

3.0. If n mutually exclusive hypotheses $H_1, H_2, \ldots, H_n$ can be made concerning the staging of an experiment, and if an event A can occur only together with one of these hypotheses, then

$$P(A) = \sum_{i=1}^{n} P(H_i)\, P(A \mid H_i), \tag{3.0.1}$$

where $P(H_i)$ is the probability of the hypothesis H_i: $P(A \mid H_i)$ is the conditional probability of the event A on that assumption. Formula (3.0.1) is known as the *total probability formula.*

If, before the experiment was carried out, the probabilities of the hypotheses $H_1, H_2, \ldots, H_n$ were equal to $P(H_1), P(H_2), \ldots, P(H_n)$, and the experiment resulted in an event A, then the new (conditional) probabilities of the hypotheses can be calculated by the formula

$$P(H_i \mid A) = \frac{P(H_i)\, P(A \mid H_i)}{\sum_{i=1}^{n} P(H_i)\, P(A \mid H_i)} \quad (i = 1, 2 \ \ldots, n). \tag{3.0.2}$$

Formula (3.0.2) expresses *Bayes's theorem* or the *inverse probability theorem*, The initial probabilities (before the experiment) of the hypotheses $P(H_1)$, $P(H_2)$, $\ldots$, $P(H_n)$ are called a priori, or prior, probabilities and the post-experiment probabilities $P(H_1 \mid A)$, $P(H_2 \mid A)$, $\ldots$, $P(H_n \mid A)$ are called a posteriori, or inverse, probabilities. Bayes's theorem makes it possible to "revise" the possibilities of the hypotheses regarding the result of the experiment.

If an experiment resulting in an event A is succeeded by one more experiment as a result of which an event B may occur, then the probability (conditional) of the event B can be calculated by the total probability formula into which new probabilities of hypotheses $P(H_i \mid A)$ are substituted instead of the former ones

$$P(B \mid A) = \sum_{i=1}^{n} P(H_i \mid A)\, P(B \mid H_i A). \tag{3.0.3}$$

Formula (3.0.3) is sometimes called the "formula for the probabilities of future events".

Problems and Exercises

3.1. We have three identical urns. The first urn contains a white and b black balls, the second urn contains c white and d black balls, and the third urn contains only white balls. A ball is drawn at random from one of the urns. Find the probability that the ball is white.

Solution. Assume that an event A = {an appearance of a white ball}. We formulate the hypotheses:

$$H_1 = \{\text{the first urn is selected}\};$$

$$H_2 = \{\text{the second urn is selected}\};$$

$H_3 = \{$**the third urn is selected**$\}$.

$$\mathrm{P}(H_1) = \mathrm{P}(H_2) = \mathrm{P}(H_3) = \frac{1}{3}, \quad \mathrm{P}(A|H_1) = \frac{a}{a+b},$$

$$\mathrm{P}(A|H_2) = \frac{c}{c+d}, \quad \mathrm{P}(A|H_3) = 1.$$

By the total probability formula we have

$$\mathrm{P}(A) = \frac{1}{3}\frac{a}{a+b} + \frac{1}{3}\frac{c}{c+d} + \frac{1}{3}\cdot 1 = \frac{1}{3}\left(\frac{a}{a+b} + \frac{c}{c+d} + 1\right).$$

3.2. An instrument can operate under two kinds of conditions: normal and abnormal. Normal conditions are observed in 80 per cent of all cases, and abnormal, in 20 per cent of cases. The probability that the instrument fails during a time t in normal conditions is 0.1, and in abnormal conditions it may fail with probability 0.7. Find the total probability p that the instrument fails during the time t.

Solution. $p = 0.8 \cdot 0.1 + 0.2 \times 0.7 = 0.22$.

3.3. The articles a factory manufactures may each have a defect with probability p. There are three inspectors in the shop; each article is only checked by one inspector (by the first, the second, or the third inspector with equal probability). The probability of detecting the defect (if any) for the ith inspector is p_i $(i = 1, 2, 3)$. If an article is not rejected in the shop, it passes to the quality control department of the factory, where a defect may be detected with probability p_0. Find the probabilities of the following events:

$A = \{$**the article is rejected**$\}$;

$B = \{$**the article is rejected in the shop**$\}$;

$C = \{$**the article is rejected by the quality control department**$\}$.

Solution. Since the events B and C are mutually exclusive and $A = B + C$, it follows that $\mathrm{P}(A) = \mathrm{P}(B) + \mathrm{P}(C)$. We find $\mathrm{P}(B)$. For an article to be rejected in the shop, first, it must have a defect and, second, the defect must be detected. By the total probability formula the probability that the defect will be detected is $\frac{1}{3}(p_1 + p_2 + p_3)$; hence $\mathrm{P}(B) = \frac{1}{3}p(p_1 + p_2 + p_3)$.

Similarly, $\mathrm{P}(C) = p\left[1 - \frac{1}{3}(p_1 + p_2 + p_3)\right]p_0$, or using the notation $\frac{1}{3}(p_1 + p_2 + p_3) = \bar{p}$, we have $\mathrm{P}(B) = p\bar{p}$, $\mathrm{P}(C) = pp_0(1 - \bar{p})$, whence $\mathrm{P}(A) = p[\bar{p} + p_0(1 - \bar{p})]$.

3.4. We have two urns. The first contains a white and b black balls and the second contains c white and d black balls. A ball is drawn at random from the first urn and put into the second. Then a ball is drawn from the second urn. Find the probability that the ball is white.

Solution. An event A = {an appearance of a white ball}; the hypotheses are

$$H_1 = \{\text{a white ball has been transferred}\};$$
$$H_2 = \{\text{a black ball has been transferred}\}.$$

$$P(H_1) = \frac{a}{a+b}, \quad P(H_2) = \frac{b}{a+b},$$

$$P(A|H_1) = \frac{c+1}{c+d+1},$$

$$P(A|H_2) = \frac{c}{c+d+1},$$

$$P(A) = \frac{a}{a+b}\frac{c+1}{c+d+1} + \frac{b}{a+b}\frac{c}{c+d+1}.$$

3.5. On the hypothesis of the preceding problem three balls are drawn from the first urn and put into the second (it is assumed that $a \geqslant 3$, $b \geqslant 3$). Find the probability that a white ball will be drawn from the second urn.

Solution. We could advance the following four hypotheses:

H_1 = {3 white balls were transferred};

H_2 = {2 white balls and 1 black ball were transferred};

H_3 = { 1 white ball and 2 black balls were transferred};

H_4 = {3 black balls were transferred}.

But it is easier to solve the problem having only two hypotheses, viz.

H_1 = {the ball drawn from the second urn belonged to the first urn};

H_2 = {the ball drawn from the second urn belonged to the second urn}.

Since three balls from the second urn belonged to the first urn and $c + d$ balls belonged to the second urn, it follows that $P(H_1) = 3/(c + d + 3)$, $P(H_2) = (c + d)/(c + d + 3)$.

Let us find the conditional probabilities of the event A = {a white ball is drawn from the second urn} on the hypotheses H_1 and H_2. The probability of an appearance of a white ball belonging to the first urn does not depend on whether the ball was drawn directly from the first urn or after it had been transferred to the second urn. Therefore,

$$P(A|H_1) = \frac{a}{a+b}, \quad P(A|H_2) = \frac{c}{c+d},$$

whence

$$P(A) = \frac{3}{c+d+3}\frac{a}{a+b} + \frac{c+d}{c+d+3}\frac{c}{c+d}.$$

3.6. We have n urns, each of which contains a white and b black balls. A ball is drawn from the first urn and put into the second, then

a ball is drawn from the second urn and put into the third one and so on. Then a ball is drawn from the last urn. Find the probability that the ball is white.

Solution. The probability of an event $A_2 = $ {a white ball is drawn from the second urn after the transfer} can be found in the same way as it was done in Problem 3.5 (for $c = a$, $d = b$):

$$\mathrm{P}(A_2) = \frac{a}{a+b}\,\frac{a+1}{a+b+1} + \frac{b}{a+b}\,\frac{a}{a+b+1} = \frac{a}{a+b}.$$

Thus the probability that a white ball is drawn from the second urn after it has been transferred is the same as before the transfer. Consequently, the probability of drawing a white ball from the third, the fourth, ..., the nth urn is the same:

$$\mathrm{P}(A_n) = a/(a+b).$$

3.7. Instruments of the same kind are manufactured by two factories, the first factory produces 2/3 of all the instruments and the second factory produces 1/3. The reliability of an instrument manufactured by the first factory is p_1 and that of an instrument manufactured by the second factory is p_2. Find the total (average) reliability p of an instrument produced by the two factories.

Answer.

$$2/3 p_1 + 1/3\, p_2.$$

3.8. There are two batches of articles of the same kind. The first batch consists of N articles of which n are faulty. The second batch consists of M articles of which m are faulty. K articles are selected at random from the first batch and L articles from the second ($K < N$; $L < M$). These $K + L$ articles are mixed and a new batch is formed. An article is selected at random from the mixed batch. Find the probability that the article is faulty.

Solution. An event $A = $ {the article is faulty}. The hypotheses: $H_1 = $ {the article belongs to the first batch}; $H_2 = $ {the article belongs to the second batch}.

$$\mathrm{P}(H_1) = \frac{K}{K+L}, \quad \mathrm{P}(H_2) = \frac{L}{K+L}, \quad \mathrm{P}(A) = \frac{K}{K+L}\,\frac{n}{N} + \frac{L}{K+L}\,\frac{m}{M}.$$

3.9. On the hypothesis of the preceding problem three articles are selected from the new, mixed, batch. Find the probability that at least one of the three articles is faulty.

Solution. The hypotheses are

$H_0 = $ {the three articles belong to the first batch};

$H_1 = $ {two articles belong to the first batch and one to the second batch};

$H_2 = $ {one article belongs to the first batch and two to the second batch};

H_3 = {all the three articles belong to the second batch}.

$$\mathrm{P}(H_0)=\frac{K(K-1)(K-2)}{(K+L)(K+L-1)(K+L-2)},$$

$$\mathrm{P}(H_1)=\frac{3K(K-1)L}{(K+L)(K+L-1)(K+L-2)},$$

$$\mathrm{P}(H_2)=\frac{3KL(L-1)}{(K+L)(K+L-1)(K+L-2)},$$

$$\mathrm{P}(H_3)=\frac{L(L-1)(L-2)}{(K+L)(K+L-1)(K+L-2)},$$

$$\mathrm{P}(A|H_0)=1-\frac{(N-n)(N-n-1)(N-n-2)}{N(N-1)(N-2)},$$

$$\mathrm{P}(A|H_1)=1-\frac{(N-n)(N-n-1)(M-m)}{N(N-1)M},$$

$$\mathrm{P}(A|H_2)=1-\frac{(N-n)(M-m)(M-m-1)}{NM(M-1)},$$

$$\mathrm{P}(A|H_3)=1-\frac{(M-m)(M-m-1)(M-m-2)}{M(M-1)(M-2)},$$

$$\mathrm{P}(A)=\mathrm{P}(H_0)\,\mathrm{P}(A|H_0)+\mathrm{P}(H_1)\,\mathrm{P}(A|H_1)+\mathrm{P}(H_2)\,\mathrm{P}(A|H_2)+\mathrm{P}(H_3)\,\mathrm{P}(A|H_3).$$

3.10. There are two boxes of identical articles, some of which are sound and the others are faulty. The first box contains a sound articles and b faulty ones, and the second box contains c sound articles and d faulty ones. An article is drawn from the first box at random and put into the second box. After the articles in the second box are stirred, an article is drawn at random from the second box and put into the first box. Then an article is drawn from the first box at random. Find the probability that the article is sound.

Solution. The hypotheses are

H_1 = {the content of the articles in the first box has not changed};

H_2 = {one faulty article in the first box has been replaced by a sound one};

H_3 = {one sound article in the first box has been replaced by a faulty one}.

An event A = {a sound article is drawn from the first box after the transfer}.

$$\mathrm{P}(H_1)=\frac{a}{a+b}\,\frac{c+1}{c+d+1}+\frac{b}{a+b}\,\frac{d+1}{c+d+1},$$

$$\mathrm{P}(H_2)=\frac{b}{a+b}\,\frac{c}{c+d+1},$$

$$\mathrm{P}(H_3)=\frac{a}{a+b}\,\frac{d}{c+d+1}\,.$$

$$\mathrm{P}(A)=\left(\frac{a}{a+b}\,\frac{c+1}{c+d+1}+\frac{b}{a+b}\,\frac{d+1}{c+d+1}\right)\frac{a}{a+b}$$

$$+\frac{b}{a+b}\,\frac{c}{c+d+1}\,\frac{a+1}{a+b}$$

$$+\frac{a}{a+b}\,\frac{d}{c+d+1}\,\frac{a-1}{a+b}$$

$$=\frac{a\,(a+b)\,(c+d+1)+bc-ad}{(a+b)^2\,(c+d+1)}=\frac{a}{a+b}$$

$$+\frac{bc-ad}{(a+b)^2\,(c+d+1)}\,.$$

This solution shows that the probability that the selected article is sound does not change if the quantities of sound and faulty articles in the urns are equal: $c/a = d/b$, i.e. $bc - ad = 0$.

3.11. Two numbers are selected at random from the series $1, 2, \ldots, n$. What is the probability that the difference between the first and the second chosen number is no less than m $(m > 0)$?

Solution. An event A consists in the difference between the first selected number k and the second number l being no less than m (i.e. $k-l \geqslant m$). The hypotheses are

$$H_k=\{\text{the first selected number is } k\}\quad (k=m+1,\ \ldots,\ n).$$

$$\mathrm{P}(H_k)=1/n,\quad \mathrm{P}(A\,|\,H_k)=(k-m)/(n-1),$$

$$\mathrm{P}(A)=\sum_{k=m+1}^{n}\frac{k-m}{n\,(n-1)}=\frac{1}{n\,(n-1)}\,[1+2+\ldots+(n-m)]$$

$$=\frac{(n-m)\,(n-m+1)}{2n\,(n-1)}\,.$$

3.12. Four groups can be formed from N marksmen: a_1 excellent marksmen, a_2 good ones, a_3 fair ones and a_4 poor marksmen. A marksman from the ith group may hit a target in one shot with probability p_i ($i = 1, 2, 3, 4$). Two marksmen are selected at random to fire at the same target. Find the probability that the target will be hit at least once.

Solution. An event $A = \{$the target is hit at least once$\}$. The hypotheses are $H_i = \{$a marksman from the ith group is the first to be selected$\}$ $i = 1, 2, 3, 4$).

$$\mathrm{P}(H_i)=\frac{a_i}{N}\,,\quad \mathrm{P}(A)=\sum_{i=1}^{4}\frac{a_i}{N}\,\mathrm{P}(A\,|\,H_i),$$

where $\mathrm{P}(A \mid H_i)$ may again be found from the total probability formula on four hypotheses concerning the marksman who was the

second to be selected:

$$\mathrm{P}(A|H_i) = \frac{a_i - 1}{N-1}[1-(1-p_i)^2] + \sum_{j \neq i} \frac{a_j}{N-1}[1-(1-p_i)(1-p_j)].$$

3.13. A radar unit tracks a target which may use interference. If the target does not employ interference, then during one surveillance cycle the unit may detect it with probability p_0; if it employs interference, then the unit may detect it with probability $p_1 < p_0$. The probability that the interference will be used in one cycle is p and does not depend on the way and time the interference was used in the other cycles. Find the probability that the target will be detected at least once in n surveillance cycles.

Solution. The total probability that the target will be detected during one cycle is $(1-p)p_0 + pp_1$; the probability of at least one target detection in n cycles is $1 - [1-(1-p)p_0 - pp_1]^n$.

3.14. A diagnostic apparatus is functioning, into which the results of n analyses taken from a patient are fed. Each analysis (independently of the others) may prove to be erroneous with probability p. The probability $\mathcal{P}$ of a correct diagnosis is a nondecreasing function of the number m of correct analyses: $\mathcal{P} = \varphi(m)$. During the time τ of functioning of the apparatus diagnoses for k patients were made. Find the probability of the event A = {an erroneous diagnosis was made for a certain patient}.

Solution. We consider the event B = {an erroneous diagnosis was made for a certain patient}. We advance hypotheses concerning the number of correct analyses:

$$\begin{aligned}
H_0 &= \{\text{not a single correct analysis}\};\\
H_1 &= \{\text{exactly one correct analysis}\};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
H_m &= \{\text{exactly } m \text{ correct analyses}\};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
H_n &= \{n \text{ correct analyses}\}.
\end{aligned}$$

The probability of the event H_m for any m can be calculated by formula (2.0.15) (the theorem on repetition of trials) when p is replaced by $1-p$:

$$\mathrm{P}(H_0) = p^n, \quad \mathrm{P}(H_1) = n(1-p)p^{n-1}, \ldots,$$
$$\mathrm{P}(H_m) = C_n^m (1-p)^m p^{n-m}, \ldots, \qquad \mathrm{P}(H_n) = (1-p)^n.$$

By the total probability formula we have

$$\mathrm{P}(B) = \sum_{m=0}^{n} C_n^m (1-p)^m p^{n-m} \varphi(m), \quad \mathrm{P}(A) = 1 - [1 - \mathrm{P}(B)]^k.$$

3.15. Each article a factory produces may have a defect with probability p. Each article is checked by a sorter who may detect a defect, if any, with probability p_1 or may not detect it with probability

$1-p_1$. In addition, the sorter may make a mistake and reject a sound article; this occurs with probability p_2. The sorter can check N articles during one shift. Find the probability R that at least one article will be assessed incorrectly, i.e. a sound one rejected as defective or a defective one passed as sound (the results of the checks of the articles are independent).

Solution. The hypotheses are

$$H_1 = \{\text{an article has a defect}\};$$
$$H_2 = \{\text{an article does not have a defect}\}.$$

By the total probability formula, the probability of one article being assessed incorrectly is

$$\tilde{p} = p\,(1-p_1) + (1-p)\,p_2.$$

The probability that at least one article is assessed incorrectly is

$$R = 1 - (1-\tilde{p})^N.$$

3.16. There are a students obtaining "excellent" marks, b students obtaining "good" marks and c students obtaining "fair" marks in a group. In a forth-coming examination, students that usually get excellent marks receive only "excellent" marks, those who usually get "good" marks may receive either "excellent" or "good" marks with equal probabilities, and those who usually get "fair" marks may receive "good", "fair" or "poor" marks with equal probabilities. At the examination a student is selected at random. Find the probability of the event $A = \{$the student receives a "good" or an "excellent" mark$\}$.

Solution. The hypotheses are

$H_1 = \{$a student who usually gets "excellent" marks is selected$\}$;

$H_2 = \{$a student who usually gets "good" marks is selected$\}$;

$H_3 = \{$a student who usually gets "fair" marks is selected$\}$.

$$\mathrm{P}(H_1) = \frac{a}{a+b+c}, \quad \mathrm{P}(H_2) = \frac{b}{a+b+c}, \quad \mathrm{P}(H_3) = \frac{c}{a+b+c}.$$

$$\mathrm{P}(A) = \mathrm{P}(H_1)\cdot 1 + \mathrm{P}(H_2)\cdot 1 + \mathrm{P}(H_3)\,\frac{1}{3} = \frac{a+b+c/3}{a+b+c}.$$

3.17. The conditions are the same as in the preceding problem, but three students are selected at random. Find the probability that they will receive an "excellent", a "good", and a "fair" mark (in any sequence).

Solution. An event $A = \{$an "excellent", a "good" and a "fair" mark are obtained$\}$ is possible only for the following hypotheses:

$H_1 = \{$one student who usually obtains "fair" marks, one student who usually obtains "good" marks and one student who usually obtains "excellent" marks are selected$\}$;

$H_2 =$ {one student who usually obtains "fair" marks and two students who usually obtain "good" marks are selected};

$H_3 =$ {two students who usually obtain "fair" marks and one student who usually obtains "good" marks are selected};

$H_4 =$ {two students who usually obtain "fair" marks and one student who usually obtains "excellent" marks are selected}.

$$P(H_1) = \frac{6abc}{N(N-1)(N-2)}, \quad P(H_2) = \frac{3b(b-1)c}{N(N-1)(N-2)},$$

$$P(H_3) = \frac{3bc(c-1)}{N(N-1)(N-2)}, \quad P(H_4) = \frac{3ac(c-1)}{N(N-1)(N-2)}.$$

$$N = a + b + c.$$

$$P(A) = P(H_1)\cdot 1\cdot\frac{1}{2}\cdot\frac{1}{3} + P(H_2)\frac{2}{4}\cdot\frac{1}{3} + P(H_3)\frac{1}{2}\cdot\frac{2}{9} + P(H_4)\cdot 1\cdot\frac{2}{9}.$$

3.18. Each of the n passengers sitting in a bus may alight from it at the next stop with probability p. Moreover, at the next stop either no passengers alight with probability p_0 or one passenger boards the bus with probability $1 - p_0$. Find the probability that when the bus continues on his way after the stop, there will again be n passengers in it.

Solution. An event $A =$ {there are again n passengers in the bus after the stop}.

The hypotheses are

$$H_0 = \{\text{ no passengers board the bus}\};$$

$$H_1 = \{\text{ one passenger boards the bus}\}.$$

$$P(H_0) = p_0, \quad P(H_1) = 1 - p_0, \quad P(A \mid H_0) = (1-p)^n,$$

$$P(A \mid H_1) = np(1-p)^{n-1}, \quad P(A) = p_0(1-p)^n$$

$$+ (1 - p_0)\, np\, (1-p)^{n-1}.$$

3.19. An instrument consists of two units I and II which repeat each other (Fig. 3.19) and can operate at random under one of the two sets of conditions: favourable or unfavourable. Under favourable conditions the reliability of each unit is p_1 and under the unfavourable conditions it is p_2. The probability that the instrument will operate under favourable conditions is P_1 and under the unfavourable conditions $1 - P_1$. Find the total (average) reliability P of the instrument.

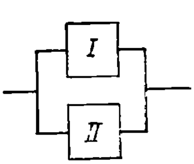

Fig. 3.19

Answer.

$$P = P_1[1 - (1-p_1)^2] + (1 - P_1)[1 - (1-p_2)^2].$$

3.20. There are instruments of the same kind on a shelf, among which there are a new instruments and b used instruments ($a \geqslant 2$, $b \geqslant 2$). Two instruments are selected at random and used for some time. Then they are replaced onto the shelf.

Then again two instruments are selected at random. Find the probability of the event A = {both instruments selected the second time are new}.

Solution. The hypotheses are

H_1 = {both instruments selected the first time were new};

H_2 = {both instruments selected the first time had been used};

H_3 = {one of the instruments selected the first time was new and the other had been used}.

$$P(H_1) = \frac{a(a-1)}{(a+b)(a+b-1)}, \quad P(H_2) = \frac{b(b-1)}{(a+b)(a+b-1)},$$

$$P(H_3) = \frac{2ab}{(a+b)(a+b-1)},$$

$$P(A) = \frac{a(a-1)(a-2)(a-3)+b(b-1)a(a-1)+2ab(a-1)(a-2)}{(a+b)^2(a+b-1)^2}.$$

3.21. A message can be sent over each of n communication channels, each channel possessing different properties (or being in different conditions), n_1 of the channels are in excellent condition, n_2 channels are in good condition, n_3 channels are in fair condition and n_4 are in poor condition ($n_1 + n_2 + n_3 + n_4 = n$). The probability of a message being transmitted correctly is equal to p_1, p_2, p_3 and p_4 for different channels respectively. To increase the reliability of the traffic, each message is repeated twice over two different channels which are selected at random. Find the probability that it will be transmitted correctly at least over one channel.

Solution. A = {at least one message is transmitted correctly}. We advance four hypotheses concerning a group to which the channel over which the first of the two messages is sent belongs: H_1, H_2, H_3, H_4, where H_i = {the first message is sent over a channel in the ith group}, $P(H_i) = n_i/n$ ($i = 1, 2, 3, 4$). By the total probability formula

$$P(A) = \sum_{i=1}^{4} P(H_i)\,P(A|H_i).$$

On the hypothesis H_i, the conditional probability of the event A is

$$P(A|H_i) = \frac{n_i-1}{n-1}[1-(1-p_i)^2] + \sum_{j\neq i} \frac{n_j}{n-1}[1-(1-p_i)(1-p_j)].$$

3.22. An instrument consists of two units I and II (Fig. 3.22), both of which must be faultless for the instrument to operate, and a voltage stabilizer S which can be either sound or faulty. When the stabilizer is sound, the reliabilities of units I and II are p_1 and p_2 respectively; when the stabilizer is faulty, they are p_1' and p_2'. The stabilizer is sound with probability p_S. Find the total reliability P of the instrument.

Answer.

$$P = p_S p_1 p_2 + (1 - p_S) p_1' p_2'.$$

3.23. The conditions are the same as in Problem 3.22 but units I and II repeat each other (Fig. 3.23).

Answer.

$$P = p_S [1 - (1 - p_1)(1 - p_2)] + (1 - p_S)[1 - (1 - p_1') \times (1 - p_2')].$$

3.24. There are n test papers, each of which contains two questions. A student does not know the answers to all the $2n$ questions but only knows those to $k < 2n$ questions. Find the probability p that he will

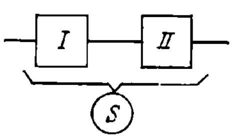

Fig. 3.22

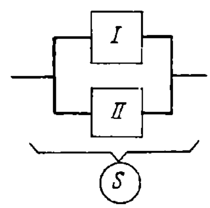

Fig. 3.23

pass the test if he either answers both questions in one test paper chosen at random or one question in his paper and one question (which the examiner chooses) in another paper.

Solution. The hypotheses are

H_1 = {the student knows the answers to both questions on his paper};

H_2 = {the student knows the answer to one question out of the two questions on his paper}.

$$p = \frac{k(k-1)}{2n(2n-1)} \cdot 1 + \frac{2k(2n-k)}{2n(2n-1)} \frac{k-1}{2n-2}.$$

3.25*. An artillery target may be either at point I with probability p_1 or at point II with probability $p_2 = 1 - p_1$ $(p_1 > 1/2)$. We have n shells, each of which can be fired at either point I or point II. Each shell may hit the target, independently of the other shells, with probability p. How many shells n_1 must be fired at point I to hit the target with a maximum probability?

Solution. The event A = {the target is hit when n_1 shells are fired at point I}. The hypotheses are

H_1 = {the target is at point I}; $\mathrm{P}(H_1) = p_1$;

H_2 = {the target is at point II}; $\mathrm{P}(H_2) = 1 - p_1$.

$$\mathrm{P}(A) = p_1 [1 - (1 - p)^{n_1}] + (1 - p_1)[1 - (1 - p)^{n - n_1}].$$

Considering P (A) to be a function of the continuous argument n_1, we find that

$$\frac{dP(A)}{dn_1}=[-p_1(1-p)^{n_1}+(1-p_1)(1-p)^{n-n_1}]\ln(1-p),$$
$$\frac{d^2P(A)}{dn_1^2}=-[p_1(1-p)^{n_1}+(1-p_1)(1-p)^{n-n_1}]\ln^2(1-p)<0,$$

and see that the function possesses a single maximum at the point

$$\tilde{n}_1=\frac{n}{2}+\ln\frac{1-p_1}{p_1}\Big/[2\ln(1-p)], \text{ where } dP(A)/dn_1=0.$$

Note that $\tilde{n}_1>n/2$ for $p_1>1/2$.

If the number obtained $\tilde{n}_1\leqslant n$ and is an integer, then it is the required number; if it is not an integer (but $\leqslant n$), then we have to calculate P (A) for the two nearest integers and choose the integer for which P (A) is greater. If the number $\tilde{n}_1$ is greater than n, then all n shells must be fired at point I (this happens for $\ln[(1-p_1)/p_1]\leqslant n\ln(1-p)$, i.e. $p_1\geqslant[1+(1-p)^n]^{-1}$.

3.26. A plane is landing. If the weather is favourable, the pilot landing the plane can see the runway. In this case the probability of a safe landing is p_1. If there is a low cloud ceiling, the pilot has to make a blind landing by instruments. The reliability (the probability of failure-free functioning) of the instruments needed for a blind landing is P. If the blind landing instruments function normally, the plane makes a safe landing with the same probability p_1 as in the case of a visual landing. If the blind landing instruments fail, then the pilot may make a safe landing with probability $p_1^*<p_1$.

Find the total probability of a safe landing if it is known that in k per cent of cases there is a low cloud ceiling.

Solution. A = {a safe landing}. The hypotheses are

$$H_1=\{\text{the cloud ceiling is high}\};$$
$$H_2=\{\text{the cloud ceiling is low}\}.$$

$$\mathrm{P}(H_1)=1-k/100,\quad \mathrm{P}(H_2)=k/100,\quad \mathrm{P}(A\mid H_1)=p_1.$$

P $(A\mid H_2)$ can also be found by the total probability formula:

$$\mathrm{P}(A\mid H_2)=Pp_1+(1-P)p_1^*,\quad \mathrm{P}(A)=\left(1-\frac{k}{100}\right)p_1$$
$$+\frac{k}{100}[Pp_1+(1-P)p_1^*].$$

3.27. The first of three urns contains a white and b black balls, the second contains c white and d black balls, and the third contains k white balls (and no black balls). A certain Petrov chooses an urn at random and draws a ball from it. The ball is white. Find the probabilities that the ball came from (a) the first, (b) the second, or (c) the third urn.

Solution. We use Bayes's formula (3.0.2) to solve the problem. The hypotheses are

$$H_1 = \{\text{the first urn is chosen}\};$$
$$H_2 = \{\text{the second urn is chosen}\};$$
$$H_3 = \{\text{the third urn is chosen}\}.$$

A priori (before the experiment) each hypothesis is equally possible: i.e. $\mathrm{P}(H_1) = \mathrm{P}(H_2) = \mathrm{P}(H_3) = 1/3$.

The event $A = \{\text{a white ball is drawn}\}$ occurred. We find the conditional probabilities:

$$\mathrm{P}(A \mid H_1) = a/(a+b), \quad \mathrm{P}(A \mid H_2) = c/(c+d), \ \mathrm{P}(A \mid H_3) = 1.$$

By Bayes's formula, the a posteriori probability that the ball was drawn from the first urn is

$$\mathrm{P}(H_1 \mid A) = \frac{1}{3}\mathrm{P}(A \mid H_1) \Big/ \Big[\frac{1}{3}\sum_{i=1}^{3}\mathrm{P}(A \mid H_i)\Big]$$
$$= \left(\frac{a}{a+b}\right) \Big/ \left(\frac{a}{a+b} + \frac{c}{c+d} + 1\right).$$

Similarly

$$\mathrm{P}(H_2 \mid A) = \left(\frac{c}{c+d}\right) \Big/ \left(\frac{a}{a+b} + \frac{c}{c+d} + 1\right),$$
$$\mathrm{P}(H_3 \mid A) = \left(\frac{a}{a+b} + \frac{c}{c+d} + 1\right)^{-1}.$$

3.28. An instrument consists of two units. Each unit must function for the instrument to operate. The reliability (the probability of failure-free performance for time t) of the first unit is p_1 and that of the second unit is p_2. The instrument is tested for a time t and fails. Find the probability that only the first unit failed and the second unit is sound.

Solution. Four hypotheses are possible before the experiment:

$$H_0 = \{\text{both units are sound}\};$$
$$H_1 = \{\text{the first unit failed and the second one is sound}\};$$
$$H_2 = \{\text{the first unit is sound and the second one failed}\};$$
$$H_3 = \{\text{both units failed}\}.$$

The probabilities of the hypotheses are

$$\mathrm{P}(H_0) = p_1 p_2, \quad \mathrm{P}(H_1) = (1-p_1)p_2, \quad \mathrm{P}(H_2) = p_1(1-p_2),$$
$$\mathrm{P}(H_3) = (1-p_1)(1-p_2).$$

The event $A = \{\text{the instrument failed}\}$ occurred:

$$\mathrm{P}(A \mid H_0) = 0, \quad \mathrm{P}(A \mid H_1) = \mathrm{P}(A \mid H_2) = \mathrm{P}(A \mid H_3) = 1.$$

By Bayes's formula we have

$$P(H_1 \mid A) = \frac{(1-p_1)\,p_2}{(1-p_1)\,p_2+p_1\,(1-p_2)+(1-p_1)\,(1-p_2)} = \frac{(1-p_1)\,p_2}{1-p_1p_2}.$$

3.29. As for Problem 3.26 except that it is known that the plane landed safely. Find the probability that the pilot used the blind-landing instruments.

Solution. If the pilot landed blind, then the cloud ceiling was low (hypothesis H_2). From Problem 3.26 we find that

$$P(H_2 \mid A) = \frac{\frac{k}{100}\,[Pp_1+(1-P)\,p_1^*]}{\left(1-\frac{k}{100}\right)p_1+\frac{k}{100}\,[Pp_1+(1-P)\,p_1^*]}.$$

3.30. A fisherman has three favourite places for fishing and he may visit each with equal probability. If he casts a line at the first place, the fish may bite with probability p_1, at the second place with probability p_2, and at the third place with probability p_3. It is known that the fisherman cast a line for three times and that he got only one bite. Find the probability that he fished at the first place.

Answer.

$$P(H_1 \mid A) = [p_1\,(1-p_1)^2]/\sum_{i=1}^{3} p_i\,(1-p_i)^2.$$

3.31. Each article a factory manufactures may have a defect with probability p. Each article is checked on the assembly line by one of two sorters with equal probability. The first sorter detects a defect with probability p_1 and the second with probability p_2. If an article is not rejected on the assembly line, it passes to the quality control department where defects, provided that they exist, are detected with probability p_3. An article is known to be rejected. Find the probability that it was rejected (1) by the first sorter, (2) by the second sorter or (3) by the quality control department.

Solution. Before the experiment four hypotheses are possible:

H_0 = {the article is not rejected};

H_1 = {the article is rejected by the first sorter};

H_2 = {the article is rejected by the second sorter};

H_3 = {the article is rejected by the quality control department}.

The event A = {the article is rejected} occurred. We do not need the hypothesis H_0 since $P(A \mid H_0) = 0$;

$$P(H_1) = pp_1/2, \quad P(H_2) = pp_2/2, \quad P(H_3) = p\,[1-(p_1+p_2)/2]p_3.$$

After the experiment the probabilities of the hypotheses are

$$P(H_1 \mid A) = \frac{p_1}{2p_3+(p_1+p_2)\,(1-p_3)}, \quad P(H_2 \mid A) = \frac{p_2}{2p_3+(p_1+p_2)\,(1-p_3)},$$

$$P(H_3 \mid A) = p_3\,[2-(p_1+p_2)]/[2p_3+(p_1+p_2)\,(1-p_3)].$$

3.32. Three out of ten students taking a test are excellently prepared, four are well prepared, two are adequately prepared and one is poorly prepared. There are 20 questions on the test paper. A student who was excellently prepared can answer all 20 questions, a student who was well prepared can answer 16 questions, a student who was adequately prepared can answer 10 questions and the student who was poorly prepared can answer only five questions. A student was called at random and answered three arbitrarily chosen questions. Find the probability that the student was (1) excellently prepared, and (2) poorly prepared.

Solution. The hypotheses are

$$H_1 = \{\text{the student was excellently prepared}\};$$
$$H_2 = \{\text{the student was well prepared}\};$$
$$H_3 = \{\text{the student was adequately prepared}\};$$
$$H_4 = \{\text{the student was poorly prepared}\}.$$

A priori: $\mathrm{P}(H_1) = 0.3$, $\mathrm{P}(H_2) = 0.4$, $\mathrm{P}(H_3) = 0.2$, $\mathrm{P}(H_4) = 0.1$. The event $A = \{\text{the student answered three questions}\}$. $\mathrm{P}(A \mid H_1) = 1$.

$$\mathrm{P}(A \mid H_2) = \frac{16}{20}\cdot\frac{15}{19}\cdot\frac{14}{18} \approx 0.491,$$

$$\mathrm{P}(A \mid H_3) = \frac{10}{20}\cdot\frac{9}{19}\cdot\frac{8}{18} \approx 0.105,$$

$$\mathrm{P}(A \mid H_4) = \frac{5}{20}\cdot\frac{4}{19}\cdot\frac{3}{18} \approx 0.009.$$

A posteriori:

(1) $$\mathrm{P}(H_1 \mid A) = \frac{0.3\cdot 1}{0.3\cdot 1 + 0.4\cdot 0.491 + 0.2\cdot 0.105 + 0.1\cdot 0.009} \approx 0.58,$$

(2) $\mathrm{P}(H_4 \mid A) = 0.1\cdot 0.009/0.518 \approx 0.002.$

3.33. A radar unit picks up a mixture of a legitimate signal and noise with probability p, and noise alone with probability $1 - p$. If a legitimate signal arrives with noise, then the unit registers the presence of a signal with probability p_1. If noise alone arrives, then the unit registers the presence of a signal with probability p_2. The unit is known to have registered the presence of a signal. Find the probability that there was a legitimate signal.

Answer.

$$pp_1/[pp_1 + (1 - p)p_2].$$

3.34. A passenger can buy a train ticket at one of three booking-offices. The probability that he buys a ticket at a booking-office depends on its location; the probabilities for the three are p_1, p_2 and p_3 respectively. The probability that all the tickets are sold by the time the passenger arrives is equal to P_1 for the first booking-office, to P_2 for the second, and to P_3 for the third. The passenger buys a ticket at a booking-office. Find the probability that it is the first one.

Solution.

$$P(H_1) = p_1, \quad P(H_2) = p_2, \quad P(H_3) = p_3, \quad A = \{\text{he buys a ticket}\}.$$

$$P(A \mid H_1) = 1 - P_1, \quad P(A \mid H_2) = 1 - P_2, \quad P(A \mid H_3) = 1 - P_3.$$

$$P(H_1 \mid A) = \frac{p_1(1-P_1)}{p_1(1-P_1) + p_2(1-P_2) + p_3(1-P_3)}.$$

3.35. A missile is fired at a plane on which there are two targets, I and II (Fig. 3.35). The probability of hitting target I is p_1, and that of hitting target II is p_2. It is known that target I was not hit. Find the probability that target II was hit.

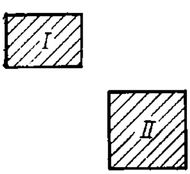

Fig. 3.35

Solution. The hypotheses are

$$H_1 = \{\text{target I is hit}\};$$
$$H_2 = \{\text{target II is hit}\};$$
$$H_3 = \{\text{neither target is hit}\}.$$

The event $A = \{\text{target I is not hit}\}$

$$P(H_1) = p_1, \quad P(H_2) = p_2, \quad P(H_3) = 1 - (p_1 + p_2),$$
$$P(A \mid H_1) = 0, \quad P(A \mid H_2) = 1, \quad P(A \mid H_3) = 1.$$

By Bayes's formula we have

$$P(H_2 \mid A) = p_2/[p_2 + 1 - (p_1 + p_2)] = p_2/(1 - p_1).$$

The problem can also be solved without resort to Bayes's formula:

$$P(H_2 \mid A) = P(H_2 A)/P(A) = P(H_2)/P(A) = p_2/(1 - p_1).$$

3.36. Signals are sent with probabilities Q_1, Q_2, Q_3 under one of three sets of conditions R_1, R_2, R_3; under each set of conditions a signal may reach the destination undistorted by interference with probabilities p_1, p_2, and p_3 respectively. Three signals were sent under one set of conditions (it is not known which) and one of the signals was distorted. Find the a posteriori probabilities that the signals were sent under the first, the second and the third set of conditions.

Solution. The hypotheses are

$$H_1 = \{\text{conditions } R_1\};$$
$$H_2 = \{\text{conditions } R_2\};$$
$$H_3 = \{\text{conditions } R_3\}.$$

The a priori probabilities are $P(H_1) = Q_1$, $P(H_2) = Q_2$, $P(H_3) = Q_3$.

The event $A = \{\text{one signal is distorted and the other two are not}\}$.

$$P(A \mid H_i) = 3p_i^2(1 - p_i) \quad (i = 1, 2, 3),$$

$$P(H_i \mid A) = Q_i p_i^2 (1 - p_i) \Big/ \sum_{j=1}^{3} Q_j p_j^2 (1 - p_j) \quad (j = 1, 2, 3).$$

3.37. The cause of a fatal air crash is being investigated, and four hypotheses are possible: H_1, H_2, H_3, H_4. Statistically $\mathrm{P}(H_1) = 0.2$, $\mathrm{P}(H_2) = 0.4$, $\mathrm{P}(H_3) = 0.3$ and $\mathrm{P}(H_4) = 0.1$. The event $A = \{\text{fuel ignition}\}$ was found to have occurred. By the same statistics, the conditional probabilities for the event A, given the hypotheses H_1, H_2, H_3, H_4, are $\mathrm{P}(A \mid H_1) = 0.9$, $\mathrm{P}(A \mid H_2) = 0$, $\mathrm{P}(A \mid H_3) = 0.2$ and $\mathrm{P}(A \mid H_4) = 0.3$. Find the a posteriori probabilities for the hypotheses.

Answer.

$\mathrm{P}(H_1 \mid A) = 2/3$, $\mathrm{P}(H_2 \mid A) = 0$, $\mathrm{P}(H_3 \mid A) = 2/9$, $\mathrm{P}(H_4 \mid A) = 1/9$.

3.38. A target being tracked may be in one of two states: H_1 and H_2. The a priori probabilities of these events are $\mathrm{P}(H_1) = 0.3$ and $\mathrm{P}(H_2) = 0.7$. Two sources of information yield contradictory information concerning the state of the target. The first source reports that the target is in state H_1 and the second reports that it is in state H_2. In general, the first source yields correct information about the state of the target in 90 per cent of cases and is erroneous in only 10 per cent of cases. The second source is less reliable, it yields correct information in 70 per cent of cases and is erroneous in 30 per cent of cases. Analysing the information received, find the new (a posteriori) probabilities of states H_1 and H_2.

Solution. The event $A = \{$the first source reports H_1 and the second reports $H_2\}$. On the hypotheses H_1 and H_2 the conditional probabilities of this event are

$\mathrm{P}(A \mid H_1) = \mathrm{P}\{$the first source sent correct information and the second was erroneous$\} = 0.9 \cdot 0.3 = 0.27$,

$\mathrm{P}(A \mid H_2) = \mathrm{P}\{$the first source was erroneous and the second sent correct information$\} = 0.1 \cdot 0.7 = 0.07$.

By Bayes's formula we have

$$\mathrm{P}(H_1 \mid A) = \frac{0.3 \cdot 0.27}{0.3 \cdot 0.27 + 0.7 \cdot 0.07} \approx 0.623,$$

$$\mathrm{P}(H_2 \mid A) = 1 - \mathrm{P}(H_1 \mid A) \approx 0.377.$$

3.39. Under the conditions of the preceding problem there are three equally reliable sources of information which yield correct information in 70 per cent of cases and are erroneous in 30 per cent of cases. Two sources report that the target is in state H_1 and one source reports that it is in state H_2. Find the a posteriori probabilities of the states H_1 and H_2.

Solution. $A = \{$the first and the second sources report H_1 and the third reports $H_2\}$.

$$\mathrm{P}(A \mid H_1) = 0.7 \cdot 0.7 \cdot 0.3 = 0.147$$

$$\mathrm{P}(A \mid H_2) = 0.3 \cdot 0.3 \cdot 0.7 = 0.063$$

$$\mathrm{P}(H_1 \mid A) = \frac{0.3 \cdot 0.147}{0.3 \cdot 0.147 + 0.7 \cdot 0.063} = 0.5, \quad \mathrm{P}(H_2 \mid A) = 0.5,$$

i.e. after the experiment the hypotheses H_1 and H_2 are equally possible.

3.40. Before the experiment, it is possible to make n mutually exclusive hypotheses which form a complete group: $H_1, H_2, \ldots, H_n$, each hypothesis having an a priori probability $P(H_1), P(H_2), \ldots, P(H_n)$. It became clear as a result of the experiment that one of the hypotheses from the group $H_1, \ldots, H_k$ was the true one and the other hypotheses were impossible, i.e. $H_1 + H_2 + \ldots + H_k = \Omega$ and $H_{k+1} + \ldots + H_n = \varnothing$. Find the a posteriori probabilities of the hypotheses.

Solution. The event observed in the experiment $A = \sum_{i=1}^{k} H_i$. By Bayes's formula we have

$$P(H_i \mid A) = P(H_i)/\sum_{i=1}^{k} P(H_i) \quad (i = 1, \ldots, k),$$

i.e. *if it becomes apparent as a result of the experiment that only some of the hypotheses $H_1, \ldots, H_k$ are possible and the others are not, then to obtain the a posteriori probabilities, we must divide each of the a priori probabilities $P(H_1), P(H_2), \ldots, P(H_n)$ by their sum.*

3.41. An instrument consisting of two units, I and II, is being tested. The reliabilities (the probabilities of failure-free performance during time τ) of units I and II are known to be $p_1 = 0.8$ and $p_2 = 0.9$. The units may fail independently of each other. After the elapse of time τ it was found that the instrument was not sound. Taking this fact into account, find the probabilities of the hypotheses

$$H_1 = \{\text{only the first unit is faulty}\};$$
$$H_2 = \{\text{only the second unit is faulty}\};$$
$$H_3 = \{\text{both units are faulty}\}.$$

Solution. Four hypotheses rather than three were possible before the experiment, including $H_0 = \{\text{both units are sound}\}$. The experiment shows that one of the hypotheses H_1, H_2, H_3 holds true; $A = H_1 + H_2 + H_3$. The a priori probabilities of these hypotheses are

$$P(H_1) = 0.2 \cdot 0.9 = 0.18, \quad P(H_2) = 0.8 \cdot 0.1 = 0.08,$$

$$P(H_3) = 0.2 \cdot 0.1 = 0.02, \quad \sum_{i=1}^{3} P(H_i) = 0.28.$$

The a posteriori probabilities are

$$P(H_1 \mid A) = 0.18/0.18 \approx 0.643, \quad P(H_2 \mid A) \approx 0.286 \text{ and}$$
$$P(H_3 \mid A) \approx 0.071.$$

3.42. The instrument whose characteristics are given in Problem 3.41 is tested during time τ and is found to be faulty. To locate the fault, the instrument is put through three tests T_1, T_2 and T_3. The first two tests give positive results and the third test gives a negative result, i.e. the event $B = \{+ + -\}$ occurs, where "+" signifies a positive

result and "—" signifies a negative result. The conditional probabilities of the positive result for each test T_1, T_2, T_3, given the hypotheses H_1, H_2, H_3, are known; we designate them as p_{ij}, where i is the number of the test and j is the number of the hypothesis: $p_{11} = 0.4$, $p_{12} = 0.6$, $p_{13} = 0.9$, $p_{21} = 0.5$, $p_{22} = 0.4$, $p_{23} = 0.6$, $p_{31} = 0.7$, $p_{32} = 0.6$ and $p_{33} = 0.3$. The results of the tests are independent.

Indicate the most probable of the possible states of the instrument, for which purpose find the a posteriori probabilities of the hypotheses, given that the experiment yielded events A and B (A = {the instrument is faulty} $= H_1 + H_2 + H_3$).

Solution. We take the data of the preceding problem as the a priori probabilities to estimate the results of the tests:

$$\mathrm{P}(H_1 \mid A) = 0.643,\ \mathrm{P}(H_2 \mid A) = 0.286 \text{ and } \mathrm{P}(H_3 \mid A) = 0.071.$$

We calculate the conditional probabilities of the event B given these hypotheses:

$$\mathrm{P}(B \mid H_1) = 0.4\cdot 0.6\cdot 0.1 = 0.024,\ \mathrm{P}(B \mid H_2) = 0.5\cdot 0.6\cdot 0.4 = 0.120,$$
$$\mathrm{P}(B \mid H_3) = 0.7\cdot 0.6\cdot 0.7 = 0.294.$$

By Bayes's formula we have

$$\mathrm{P}(H_1 \mid A\cdot B) = \frac{0.643\cdot 0.024}{0.643\cdot 0.024 + 0.286\cdot 0.120 + 0.071\cdot 0.294} \approx 0.298,$$
$$\mathrm{P}(H_2 \mid A\cdot B) \approx 0.662,\quad \mathrm{P}(H_3 \mid A\cdot B) \approx 0.040.$$

The most probable state of the instrument is H_2 = {only the second unit failed}.

3.43. A collection of n parts, any number of which may be defective, is chosen to manufacture an instrument. The parts are supplied by two factories I and II. The statistics show that the articles produced by factory I may have a defect with probability p_1 and those by factory II with p_2. The instrument is assembled from parts produced by the same factory. In the laboratory where the instrument is being assembed there are three boxes of parts, two of which contain parts from factory I and one contains parts from factory II. The box from which the articles are taken is selected at random. The assembled instrument is tested. If no less than m of the n parts are found to be defective, the instrument is rejected and it is reported as unsatisfactory to the manufacturer. The instrument was rejected by the quality control department. Find the probability that its condition is reported as unsatisfactory to factory I.

Solution. The hypotheses are

H_1 = {the instrument was assembled from parts manufactured by factory I};

H_2 = {the instrument was assembled from parts manufactured by factory II}.

The a priori probabilities are $\mathrm{P}(H_1) = 2/3$ and $\mathrm{P}(H_2) = 1/3$.

The event A = {the instrument was rejected} = {no less than m articles out of the n were defective} occurred.

$$P(A \mid H_1) = \sum_{i=m}^{n} p_1^i (1-p_1)^{n-i}, \quad P(A \mid H_2) = \sum_{i=m}^{n} p_2^i (1-p_2)^{n-i}.$$

By Bayes's formula the a posteriori probabilities are

$$P(H_1 \mid A) = \frac{\frac{2}{3}\sum_{i=m}^{n} p_1^i (1-p_1)^{n-i}}{\frac{2}{3}\sum_{i=m}^{n} p_1^i (1-p_1)^{n-i} + \frac{1}{3}\sum_{i=m}^{n} p_2^i (1-p_2)^{n-i}},$$

$$P(H_2 \mid A) = 1 - P(H_1 \mid A).$$

3.44. There are two boxes with parts of the same type. The first contains a sound parts and b defective ones and the second contains c sound parts and d defective ones. A box is selected at random and one part is drawn from it and is found to be sound. Find the probability that the next part to be drawn from the box will also be sound.

Solution. The hypotheses are

H_1 = {the first box was selected};

H_2 = {the second box was selected}.

The event A = {a sound part was drawn first}.

$$P(H_1) = P(H_2) = 1/2, \quad P(H_1 \mid A) = \left(\frac{a}{a+b}\right) \Big/ \left(\frac{a}{a+b} + \frac{c}{c+d}\right),$$

$$P(H_2 \mid A) = \left(\frac{c}{c+d}\right) \Big/ \left(\frac{a}{a+b} + \frac{c}{c+d}\right).$$

The event B = {a sound part was drawn second}.

$$P(B \mid A) = P(H_1 \mid A)\, P(B \mid H_1A) + P(H_2 \mid A)\, P(B \mid H_2A).$$

Given that the first box was selected and a sound part was drawn from it, the conditional probability of drawing a second sound part is

$$P(B \mid H_1A) = (a-1)/(a+b-1),$$

similarly, $P(B \mid H_2A) = (c-1)/(c+d-1)$.

Hence the required probability is

$$P(B \mid A) = \left(\frac{a}{a+b} + \frac{c}{c+d}\right)^{-1} \left[\frac{a(a-1)}{(a+b)(a+b-1)} + \frac{c(c-1)}{(c+d)(c+d-1)}\right].$$

3.45. There are three communication channels over which messages are sent at random (with equal probability over each channel). The probability of a message being distorted when it is sent over the first channel is p_1, over the second channel, p_2, over the third channel, p_3. A channel is selected and k messages are sent over it, none of which are

distorted. Find the probability that the $(k+1)$th message, sent over the same channel, will not be distorted.

Solution. The hypotheses are

H_1 = {the messages were sent over the first channel};

H_2 = {the messages were sent over the second channel};

H_3 = {the messages were sent over the third channel}.

$$\mathrm{P}(A \mid H_1) = (1-p_1)^k, \quad \mathrm{P}(A \mid H_2) = (1-p_2)^k, \mathrm{P}(A \mid H_3) = (1-p_3)^k$$

$$\mathrm{P}(H_i \mid A) = \frac{1/3\,(1-p_i)^k}{1/3\,[(1-p_1)^k + (1-p_2)^k + (1-p_3)^k]}$$

$$= \frac{(1-p_i)^k}{(1-p_1)^k + (1-p_2)^k + (1-p_3)^k} \quad (i = 1, 2, 3).$$

The event A = {k messages are not distorted} and B = {the $(k+1)$th message is not distorted}.

$$\mathrm{P}(B \mid A) = \frac{(1-p_1)^{k+1} + (1-p_2)^{k+1} + (1-p_3)^{k+1}}{(1-p_1)^k + (1-p_2)^k + (1-p_3)^k}.$$

3.46. There are m batches of articles with $N_1, N_2, \ldots, N_m$ articles in each batch. The ith batch contains n_i defective articles and $N_i - n_i$ sound ones ($i = 1, 2, \ldots, m$). A batch is selected at random and k articles from it are checked. They all prove to be sound. Find the probability that the next l articles taken from the same batch will also be sound.

Sotution. The hypotheses $H_1, H_2, \ldots, H_m$, where H_i = {the ith batch is selected} have equal a priori probabilities $\mathrm{P}(H_i) = 1/m$ ($i = 1, 2, \ldots, m$).

On the hypothesis H_i the conditional probability of the observed event A = {all the k articles checked were sound} is

$$\mathrm{P}(A \mid H_i) = C^k_{N_i - n_i}/C^k_{N_i} \quad (i = 1, 2, \ldots, m). \tag{3.46.1}$$

By Bayes's formula the a posteriori probabilities of the hypotheses are

$$\mathrm{P}(H_i \mid A) = \mathrm{P}(A \mid H_i)/\sum_{i=1}^{m} \mathrm{P}(A \mid H_i) \quad (i = 1, 2, \ldots, m). \tag{3.46.2}$$

The probability of an event B = {the next l articles taken from the same batch are sound} can be calculated from the total probability formula using the a posteriori probabilities given in (3.46.2):

$$\mathrm{P}(B) = \sum_{i=1}^{m} \mathrm{P}(H_i \mid A)\,\mathrm{P}(B \mid H_i A), \text{ where}$$

$$\mathrm{P}(B \mid H_i A) = \frac{N_i - n_i - k}{N_i - k}\,\frac{N_i - n_i - k - 1}{N_i - k - 1}$$

$$\ldots \frac{N_i - n_i - k - l + 1}{N_i - k - l + 1} \quad (i = 1, 2, \ldots, m). \tag{3.46.3}$$

Remark. Formulas (3.46.1) and (3.46.3) are valid only for $k < N_i - n_i$ and $l < N_i - n_i - k$. If these conditions are not satisfied, the corresponding probabilities will be zero.

CHAPTER 4

Discrete Random Variables

4.0. The concept of a *random variable* is one of the most significant in probability theory. A random variable is a variable which, as a result of an experiment with a random outcome, assumes a certain value. Examples: (a) an experiment is to fire four shots at a target, the random variable is a number of hits; (b) an experiment is the operation of a computer, the random variable is the time the computer operates until it fails.

In the set-theoretical interpretation of the fundamental concepts of probability theory, which we have adopted, a random variable X is a function of an elementary event ω: $X = \varphi(\omega)$, where $\omega \in \Omega$. The value of that function depends on which elementary event ω occurs as a result of an experiment.

We shall denote random variables by capital letters and nonrandom variables by small letters.

The *law of probability distribution* of a random variable is the rule used to find the probability of the event related to a random variable, for instance, the probability that the variable assumes a certain value or falls in a certain interval. If a random variable X has a given distribution law, then it is said to "have such and such a distribution".

The most general form of the distribution law is a *distribution function*, which is the probability that a random variable X assumes a value smaller than a given value x, i.e.

$$F(x) = \mathrm{P}\{X < x\}. \tag{4.0.1}$$

The distribution function $F(x)$ for any random variable possesses the following properties: $F(-\infty) = 0$, $F(+\infty) = 1$, the function $F(x)$ does not decrease with an increase in x. The distributions of discrete random variables have a very simple form. A random variable is said to be *discrete* if it has a finite or countable set of possible values.

In the examples considered above, the random variable X, i.e. the number of hits out of four shots being fired, is discrete and may take the values 0, 1, 2, 3, 4. The second random variable T, the time a computer operates until its first failure, is nondiscrete, its possible values continuously fill a certain part of the abscissa axis and their set is uncountable.

The simplest distribution for a discrete random variable X is an *ordered sample*, or *ordered series*, which is a table whose top row contains all the values of the random variable $x_1, x_2, \ldots, x_i, \ldots$ in the ascending order and the bottom row contains the corresponding probabilities $p_1, p_2, \ldots, p_i, \ldots$:

$$X: \left|\begin{array}{c|c|c|c|c} x_1 & x_2 & \ldots & x_i & \ldots \\ \hline p_1 & p_2 & \ldots & p_i & \ldots \end{array}\right., \tag{4.0.2}$$

where $p_i = \mathrm{P}\{X = x_i\}$; $\sum\limits_i p_i = 1$.

A graphical representation of an ordered sample (Fig. 4.0.1) is known as a *frequency polygon*.

The distribution function $F(x)$ of a discrete random variable X is a discontinuous step, or jump, function (Fig. 4.0.2), whose jumps correspond to the possible values $x_1, x_2, \ldots$ of the random variable X and are equal to the probabilities p_1,

$p_2, \ldots$ of these values respectively; between the jumps the function $F(x)$ remains constant. At a point of discontinuity the function $F(x)$ is equal to the value with which it approaches the point of discontinuity from the left (in Fig. 4.0.2 those values are shown by dots). The function $F(x)$ is said to be "continuous on the left", i.e. when it approaches any point from the left, it does not suffer discontinuity, but when it approaches the point from the right, it may become discontinuous.

The probability that a random variable X falls on an interval from α to β (including α) is expressed in terms of a distribution function by the formula

$$P\{\alpha \leqslant X < \beta\} = F(\beta) - F(\alpha), \tag{4.0.3}$$

or, in another notation,

$$P\{X \in [\alpha, \beta)\} = F(\beta) - F(\alpha), \tag{4.0.4}$$

where the '[' sign signifies that the point α is included in the interval from α to β and the ')' sign signifies that the point β is not included into it.

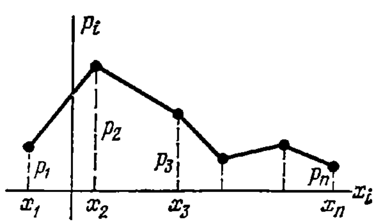

Fig. 4.0.1

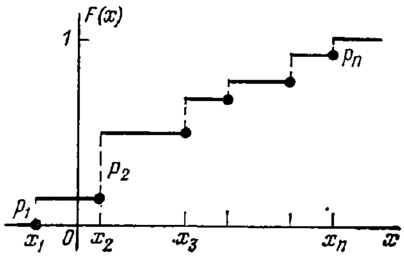

Fig. 4.0.2

The *mathematical expectation*, or *mean value*, of a discrete random variable X is the sum of the products of all its possible values x_i and their probabilities p_i, i.e.

$$M[X] = \sum_i x_i p_i. \tag{4.0.5}$$

A random variable may not have a mean value if this sum diverges. When the mean value of a random variable X has to be denoted by one letter, we write

$$M[X] = m_x.$$

A *centred random variable* is the difference between the random variable X and its mean:

$$\overset{\circ}{X} = X - m_x. \tag{4.0.6}$$

The *variance* of a random variable X is the mean value of the square of the corresponding centred random variable:

$$\mathrm{Var}[X] = M[\overset{\circ}{X}{}^2]. \tag{4.0.7}$$

For a discrete random variable X the variance can be calculated by the formula

$$\mathrm{Var}[X] = \sum_i (x_i - m_x)^2 p_i. \tag{4.0.8}$$

For the variance of a random variable X we shall also use a designation Var_x.

The *mean square deviation*, or *standard deviation*, of a random variable X is the square root of its variance:

$$\sigma[X] = \sigma_x = \sqrt{\mathrm{Var}_x} \tag{4.0.9}$$

(the arithmetic, or positive, value of the root is always implied).

The kth *moment about the origin* of a random variable X is the kth-power mean value of that variable:

$$\alpha_k [X] = \mathrm{M} [X^k]. \tag{4.0.10}$$

For a discrete random variable X the first moment about the origin can be calculated by the formula

$$\alpha_k [X] = \sum_i x_i^k p_i. \tag{4.0.11}$$

The kth *central moment* of a random variable is the kth-power mean value of the corresponding centred value:

$$\mu_k [X] = \mathrm{M} [\overset{\circ}{X}{}^k]. \tag{4.0.12}$$

For a discrete random variable X the central moment can be calculated by the for) mula

$$\mu_k [X] = \sum_i (x_i - m_x)^k \cdot p_i. \tag{4.0.13}$$

The mean value of a random variable X is its first moment about the origin and the variance is the second control moment:

$$\mathrm{M} [X] = \alpha_1 [X], \quad \mathrm{Var} [X] = \mu_2 [X]. \tag{4.0.14}$$

Central moments are expressed in terms of moments about the origin:

$$\left.\begin{array}{l} \mu_2 [X] = \alpha_2 [X] - m_x^2, \\ \mu_3 [X] = \alpha_3 [X] - 3m_x\alpha_2 [X] + 2m_x^3, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \tag{4.0.15}$$

Of especial importance is the first formula which expresses the variance in terms of the second moment about the origin:

$$\mathrm{Var} [X] = \alpha_2 [X] - m_x^2, \tag{4.0.16}$$

or, in another notation,

$$\mathrm{Var} [X] = \mathrm{M} [X^2] - (\mathrm{M} [X])^2, \tag{4.0.17}$$

i.e. *the variance is equal to the mean value of the square of the random variable minus the square of the mean.*

The *indicator of an event* A is a discrete random variable U which possesses two possible values: 0 and 1.

The random variable is equal to 0 if the event A does not occur and to unity if it occurs:

$$U = \begin{cases} 0 & \text{for} \quad \omega \notin A, \\ 1 & \text{for} \quad \omega \in A. \end{cases} \tag{4.0.18}$$

The ordered sample of the indicator U of an event A has the form

$$U\colon \left| \begin{array}{c|c} 0 & 1 \\ \hline q & p \end{array} \right|,$$

where p is the probability of the event in the experiment and $q = 1 - p$.

The mean and the variance of the quantity U are

$$\mathrm{M} [U] = p \quad \text{and} \quad \mathrm{Var} [U] = pq \tag{4.0.19}$$

respectively.

In the theory of probability, the use of the indicators of events substantially simplifies the solution of problems (see Chapter 7).

When calculating the characteristics of random variables, it is often convenient to use the formula for complete expectation: *if n mutually exclusive hypotheses H_1, H_2 ..., H_n can be made concerning the conditions of an experiment, then the complete expectation of the random variable X can be calculated by the formula*

$$\mathrm{M}[X]=\sum_{i=1}^{n}\mathrm{P}(H_i)\,\mathrm{M}[X\,|\,H_i], \tag{4.0.20}$$

where $\mathrm{M}[X \mid H_i]$ is the conditional expectation of the variable X on the hypothesis H_i.

The formula for complete expectation can be employed in calculations of moments about the origin of any order:

$$\alpha_k[X]=\sum_{i=1}^{n}\mathrm{P}(H_i)\,\mathrm{M}[X^k\,|\,H_i]. \tag{4.0.21}$$

In principle, this formula can also be used in calculating central moments of any order:

$$\mu_k[X]=\sum_{i=1}^{n}\mathrm{P}(H_i)\,\mathrm{M}[\overset{\circ}{X}{}^k\,|\,H_i], \tag{4.0.22}$$

but it must be borne in mind that the quantity $\overset{\circ}{X}$ in (4.0.22) must be calculated as $\overset{\circ}{X} = X - m_x$, where m_x is the *unconditional* expectation of the random variable X, as expressed by formula (4.0.20), rather than the conditional expectation on the hypothesis H_i.

There are several frequently encountered types of distributions of random variables.

1. **The binomial distribution.** A random variable X is said to be *binomially distributed* if it may take the values 0, 1, ..., m, ..., n, and the respective probabilities are

$$P_m = \mathrm{P}\{X = m\} = C_n^m p^m q^{n-m}, \tag{4.0.23}$$

where $0 < p < 1$, $q = 1 - p$ and $m = 0, 1, \ldots, n$. Distribution (4.0.23) depends on the two parameters, p and n.

From the theorem on repetition of trials [formula (2.0.15)] it follows that the *number X of occurrences of an event in n independent trials has a binomial distribution.* For the random variable X, which has a binomial distribution with parameters p and n, we have

$$\mathrm{M}[X] = np, \qquad \mathrm{Var}[X] = npq, \tag{4.0.24}$$

where $q = 1 - p$.

2. **The Poisson distribution.** A discrete random variable X *has a Poisson distribution* if its possible values are 0, 1, 2, ..., m, ..., and the probability of the event $\{X = m\}$ is expressed by the formula

$$P_m=\mathrm{P}\{X=m\}=\frac{a^m}{m!}e^{-a} \quad (m=0,\ 1,\ 2,\ \ldots), \tag{4.0.25}$$

where $a > 0$. The Poisson distribution depends on one parameter a. For a random variable X which has a Poisson distribution

$$\mathrm{M}[X] = \mathrm{Var}[X] = a. \tag{4.0.26}$$

The Poisson distribution is the limit for a binomial distribution as $p \to 0$ and $n \to \infty$, given that $np = a = \text{const}$. We can use this distribution to make approximations when we have a large number of independent trials in each of which an event A occurs with small probability.

The Poisson distributions can also be used in the case of a number of points falling in a given region of space (one-dimensional, two-dimensional or three-dimensional) if the location of the points in space is random and satisfies certain restrictions.

The one-dimensional variant is encountered when flows of events are considered. A *flow of events*, or traffic, is a sequence of homogeneous events occurring one after another at random moments in time (for more detail see Chapter 10).

The average number of events λ occurring per unit time is known as the *intensity* of the flow. The quantity λ can be either constant or variable: $\lambda = \lambda(t)$.

A flow of events is said to be *without aftereffects* if the probability of the number of events falling on an interval of time does not depend on the number of events falling on any other non-overlapping interval.

A flow of events is *ordinary* if the probability of two or more events occurring on an elementary interval Δt is negligibly small as compared to the probability of the occurrence of just one event.

An ordinary flow of events without an aftereffect is known as a Poisson flow or traffic. If certain events form a Poisson flow, then the number X of events falling on an arbitrary time interval $(t_0, t_0+\tau)$ has a Poisson distribution:

$$P_m = \frac{a^m}{m!} e^{-a} \quad (m = 0, 1, 2, \ldots), \tag{4.0.27}$$

where a is the mean value of the number of points falling on the interval: $a = \int_{t_0}^{t_0+\tau} \lambda(t)\, dt$ and $\lambda(t)$ is the intensity of the flow.

If $\lambda = \text{const}$, then the Poisson flow is said to be a stationary or *elementary* flow of the Poisson type. For an elementary flow, the number of events falling on any time interval of length τ has a Poisson distribution with paramter $a = \lambda\tau$.

A *random field of points* is a collection of points randomly scattered over a plane (or in space).

The *intensity* (or *density*) *of a field* λ is the mean number of points which fall in a unit area (unit volume).

A field of points is said to be of the Poisson type if it possesses the following properties: (1) the probability of a number of points falling in any region of a plane (space) is independent of the number of points falling in any other non-overlapping region; (2) the probability of two or more points falling in the elementary region $\Delta x \Delta y$ is negligibly small as compared to the probability of one point falling in that region (the property of ordinariness).

The number X of points in a Poisson field falling in any region S of a plane (space) has a Poisson distribution

$$P_m = \frac{a^m}{m!} e^{-a} \quad (m = 0, 1, \ldots), \tag{4.0.28}$$

where a is the mean value of the number of points falling in the region S. If the intensity of the field $\lambda(x, y) = \lambda = \text{const}$, the field is said to be *homogeneous* (a property similar to the stationariness of a flow). For a homogeneous field with intensity λ we have $a = s\lambda$, where s is the area (volume) of the region S. If a field is inhomogeneous, then $a = \iint_{(S)} \lambda(x, y)\, dx\, dy$ (for a plane) and $a = \iiint_{(S)} \lambda(x, y, z) \times dx\, dy\, dz$ (for space).

When doing problems related to a Poisson distribution, it is convenient to use tables of the function $P(m, a) = \frac{a^m}{m!} e^{-a}$ (see Appendix 1) and $R(m, a) = \sum_{k=0}^{m} \frac{a^k}{k!} e^{-a}$ (see Appendix 2). The last function is the probability that the random variable X, which has a Poisson distribution with parameter a, assumes a value not exceeding m: $R(m, a) = \mathrm{P}\{X \leqslant m\}$.

3. A **geometric distribution.** A random variable X is said to have a *geometric distribution* if its possible values are 0, 1, 2, ..., m, ... and the probabilities of these values are

$$P_m = q^{m-1}p \quad (m=0,\ 1,\ 2,\ \ldots), \tag{4.0.29}$$

where $0 < p < 1$, $q = 1 - p$.

For a sequence of values m, the probabilities P_m form an infinitely decreasing geometric progression with a common ratio q. In practical applications, geometric distributions are encountered when a number of independent trials are carried out to reach a result A; in each trial the result is achieved with probability p. The random variable X is the number of "useless" trials (*till the first trial* in which the event A occurs).

The ordered series of the random variable X has the form

$$X: \begin{array}{|c|c|c|c|c|c}0 & 1 & 2 & \ldots & m & \ldots \\ \hline p & qp & q^2p & \ldots & q^m p & \ldots\end{array}.$$

The mean value of the random variable X, which has a geometric distribution, is

$$\mathrm{M}\,[X] = q/p, \tag{4.0.30}$$

and its variance is

$$\mathrm{Var}\,[X] = q/p^2. \tag{4.0.31}$$

The random variable $Y = X + 1$ is often considered, which is the number of trials made till a result A is achieved, the *successful trial inclusive*. The ordered series of the random variable Y has the form

$$Y: \begin{array}{|c|c|c|c|c|c}1 & 2 & 3 & \ldots & m & \ldots \\ \hline p & qp & q^2p & \ldots & q^{m-1}p & \ldots\end{array}, \tag{4.0.32}$$

$$\mathrm{M}\,[Y] = 1/p;\ \mathrm{Var}\,[Y] = q/p^2. \tag{4.0.33}$$

We shall call the distribution of the random variable $Y = X + 1$ a geometric distribution beginning with unity.

4. A **hypergeometric distribution.** A random variable X with possible values $0, 1, \ldots, m, \ldots, a$ has a *hypergeometric distribution* with parameters n, a, b if

$$P_m = \mathrm{P}\,\{X = m\} = C_a^m C_b^{n-m}/C_{a+b}^n \ (m = 0, 1, \ldots, a)^{*)}. \tag{4.0.34}$$

A hypergeometric distribution occurs when there is an urn which contains a white and b black balls and n balls are drawn from it. The random variable X is the number of white balls drawn and its distribution is expressed by formula (4.0.34).

The mean of a random variable which has distribution (4.0.34) is

$$\mathrm{M}\,[X] = na/(a+b), \tag{4.0.35}$$

and its variance is

$$\mathrm{Var}\,[X] = \frac{nab}{(a+b)^2} + n(n-1)\left[\frac{a}{a+b}\,\frac{a-1}{a+b-1} - \left(\frac{a}{a+b}\right)^2\right]. \tag{4.0.36}$$

*) When formula (4.0.34) is used, we must assume that $C_k^r = 0$ if $r > k$.

Problems and Exercises

4.1. Construct the distribution function of the indicator U of an event A whose probability is p.

Solution.

$$F(x) = \mathrm{P}\{U < x\} \begin{cases} 0 & \text{for} \quad x \leqslant 0, \\ q & \text{for} \quad 0 < x \leqslant 1, \\ 1 & \text{for} \quad x > 1, \end{cases}$$

where $q = 1 - p$ (see Fig. 4.1).

4.2. A coin is tossed three times. A random variable X is the heads obtained. Construct for it (1) the ordered series; (2) the frequency polygon; (3) the distribution function. Find $\mathrm{M}[X]$, $\mathrm{Var}[X]$ and σ_x.

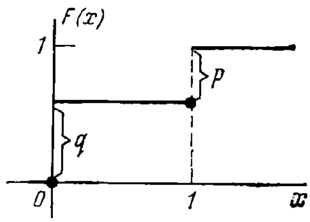

Fig. 4.1

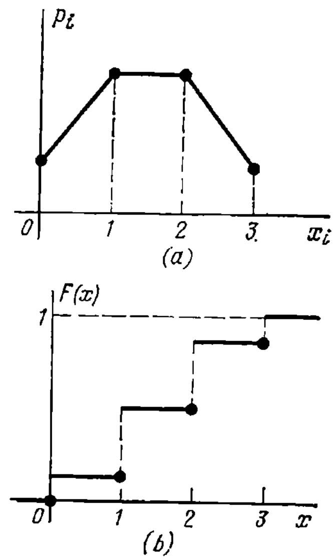

Fig. 4.2

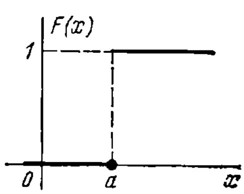

Fig. 4.3

Solution. The variable X has a binomial distribution with parameters $n = 3$ and $p = 1/2$. The ordered series is

$$X: \left|\begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \hline 1/8 & 3/8 & 3/8 & 1/8 \end{array}\right|.$$

The frequency polygon is shown in Fig. 4.2*a* and the distribution function is shown in Fig. 4.2*b*.

4.3. Considering a nonrandom variable a to be a special case of a random variable, construct for it (1) the ordered series, (2) the distribution function, (3) find its mean and variance.

Answer. (1) a: $\left|\begin{array}{c} a \\ \hline 1 \end{array}\right|$, (2) the distribution function is shown in Fig. 4.3; (3) $\mathrm{M}[a] = a$; $\mathrm{Var}[a] = 0$.

4.4. A constant nonrandom variable a is added to a random variable X. How does this affect the characteristics of the random variable: (1) the mean value; (2) the variance; (3) the mean square deviation; (4) the second moment about the origin?

Answer. (1) A term a is added; (2) it does not change; (3) it does not change; (4) a term $a^2 + 2am_x$ is added (since $\alpha_2[X] = \text{Var}[X] + m_x^2$).

4.5. A random variable X is multiplied by a. How does this affect its characteristics: (1) the mean value; (2) the variance; (3) the mean square deviation; (4) the second moment about the origin?

Answer. (1) it is multiplied by a; (2) it is multiplied by a^2; (3) it is multiplied by $|a|$; (4) it is multiplied by a^2.

4.6. We toss a coin n times and consider a random variable X which is the number of heads obtained. Construct the ordered series of the random variable and find its characteristics, i.e. m_x, Var_x, σ_x, $\mu_3[X]$.

Answer.

$$X: \left| \begin{array}{c|c|c|c|c|c} 0 & 1 & \ldots & m & \ldots & n \\ \hline (1/2)^n & C_n^1(1/2)^n & \ldots & C_n^m(1/2)^n & \ldots & (1/2)^n \end{array} \right. ,$$

$$m_x = n/2; \quad \text{Var}_x = n/4, \quad \sigma_x = \sqrt{n}/2, \quad \mu_3[X] = 0$$

(since the distribution is symmetric about $m_x = n/2$).

4.7. We carry out n independent trials in each of which an event A may occur with probability p. Write the ordered series of the random variable X which is the number of occurrences of the complementary event $\bar{A}$ in n trials, and find its mean and variance.

Answer.

$$X: \left| \begin{array}{c|c|c|c|c|c} 0 & 1 & \ldots & m & \ldots & n \\ \hline p^n & C_n^1 q p^{n-1} & \ldots & C_n^m q^m p^{n-m} & \ldots & q^n \end{array} \right| ,$$

where $q = 1 - p$; $m_x = nq$; $\text{Var}_x = npq$.

4.8. We carry out n independent trials, in each of which an event A may occur with probability p. We consider a random variable R which is the frequency of occurrence of the event A in n trials, i.e. the ratio of the number of occurrences of the event A in n trials to the total number of trials n. Write the ordered series of the random variable and find its mean and variance.

Answer.

$$R: \left| \begin{array}{c|c|c|c|c|c} 0 & 1/n & \ldots & m/n & \ldots & 1 \\ \hline q^n & C_n^1 p q^{n-1} & \ldots & C_n^m p^m q^{n-m} & \ldots & p^n \end{array} \right| ,$$

where $q = 1 - p$, $m_x = p$, $\text{Var}_x = pq/n$.

4.9*. We carry out n independent trials. The probability of occurrence of an event A is the same in all the trials and equal to p. Find the most probable number m^* of the occurrences of the event A.

Solution. We find the condition for which $m^* = 0$. If $m^* = 0$, then $q^n > C_n^1 pq^{n-1}$ or $q > np$, whence $p < 1/(n+1)$. If $m^* = n$, then $p^n > C_n^1 qp^{n-1}$, $p > nq$, whence $p > n/(n+1)$.

Let us consider the case when $0 < m^* < n$; in this case two inequalities must be simultaneously satisfied:

$$C_n^{m^*} p^{m^*} q^{n-m^*} \geqslant C_n^{m^*+1} p^{m^*+1} q^{n-m^*-1},$$
$$C_n^{m^*} p^{m^*} q^{n-m^*} \geqslant C_n^{m^*-1} p^{m^*-1} q^{n-m^*+1}.$$

These two inequalities are equivalent to the following inequalities:

$$(m^*+1)\, q \geqslant (n-m^*)\, p; \quad (n-m^*+1)\, p \geqslant m^* q,$$

whence m^* must be an integer satisfying the inequalities

$$(n+1)\, p - 1 \leqslant m^* \leqslant (n+1)\, p.$$

It can be verified that this condition is also satisfied if $p < 1/(n+1)$ $(m^* = 0)$ or in another extreme case, if $p > n/(n+1)$ $(m^* = n)$. Since the right-hand side exceeds the left-hand side by unity, there is only one integer m^* between them. The only exception is the case when $(n+1)\, p$ and $(n+1)\, p - 1$ are integers. Then there are two very improbable values $(n+1)\, p$ and $(n+1)\, p - 1$. If np is an integer, then $m^* = np$.

4.10. Two marksmen fire at two targets. Each of them fires one shot at his target independently of the other marksman. The probability of hitting the target is p_1 for the first marksman and p_2 for the second. Two random variables are considered, viz. X_1, the number of times the first marksman hits his target, and X_2, the number of times the second marksman hits the target, and their difference $Z = X_1 - X_2$. Construct the ordered series of the random variable Z and find its characteristics m_Z and Var_Z.

Solution. The random variable Z has three possible values: -1, 0 and $+1$.

$$\mathrm{P}\{Z=-1\} = \mathrm{P}\{X_1=0\}\,\mathrm{P}\{X_2=+1\} = q_1 p_2,$$
$$\mathrm{P}\{Z=0\} = \mathrm{P}\{X_1=0\}\,\mathrm{P}\{X_2=0\}$$
$$+\,\mathrm{P}\{X_1=1\}\,\mathrm{P}\{X_2=1\} = q_1 q_2 + p_1 p_2,$$
$$\mathrm{P}\{Z=1\} = \mathrm{P}\{X_1=1\}\,\mathrm{P}\{X_2=0\} = p_1 q_2,$$

where $q_1 = 1 - p_1$; $q_2 = 1 - p_2$.

The ordered series of the variable Z has the form

$$Z: \left|\begin{array}{c|c|c} -1 & 0 & 1 \\ \hline q_1 p_2 & q_1 q_2 + p_1 p_2 & p_1 q_2 \end{array}\right|.$$

$$m_z = (-1)\, q_1 p_2 + 0\,(q_1 q_2 + p_1 p_2) + 1 p_1 p_2 = -q_1 p_2 + 1 p_1 q_2 = p_1 - p_2.$$

We can find the variance in terms of the second moment about the origin [see (4.0.16)]:

$$\alpha_2[Z] = (-1)^2 \cdot q_1p_2 + 0^2 \cdot (q_1q_2 + p_1p_2) + 1^2 \cdot p_1q_2$$
$$= q_1p_2 + p_1q_2 = p_1 + p_2 - 2p_1p_2$$
$$\mathrm{Var}_Z = \alpha_2[Z] - m_Z^2 = p_1 + p_2 - 2p_1p_2 - (p_1 - p_2)^2$$
$$= p_1q_1 + p_2q_2.$$

4.11. Two shots are fired independently at a target. The probability of hitting the target on each shot is p. The following random variables are considered: X, which is the difference between the number of hits and the number of misses; Y, which is the sum of the number of hits and the number of misses. Construct an ordered series for each of the random variables X and Y. Find their characteristics: m_x, Var_x, m_y, Var_y.

Solution. The ordered series of the variable X has the form

$$X: \left|\begin{array}{c|c|c} -2 & 0 & 2 \\ \hline q^2 & 2pq & p^2 \end{array}\right|, \quad \text{where} \quad q = 1 - p.$$

$$m_x = -2q^2 + 2p^2 = 2(p - q); \quad \alpha_2[X] = 4(q^2 + p^2),$$
$$\mathrm{Var}_x = \alpha_2[X] - m_x^2 = 8pq.$$

The variable Y is actually not random and has one value 2; its ordered series is

$$Y: \left|\begin{array}{c} 2 \\ \hline 1 \end{array}\right|; \quad m_y = 2; \quad \mathrm{Var}_y = 0.$$

4.12. We have n lamps at our disposal, each of which may have a defect with probability p. A lamp is screwed into a holder and the current is switched on. A defective lamp immediately burns out and is replaced by another one. A random variable X, the number of lamps that must be tried, is considered. Construct its ordered series and find its mean value m_x.

Solution. The ordered series of the variable X has the form

$$X: \left|\begin{array}{c|c|c|c|c|c|c} 1 & 2 & 3 & \dots & i & \dots & n \\ \hline q & pq & p^2q & \dots & p^{i-1}q & \dots & p^{n-1} \end{array}\right|, \quad \text{where } q = 1 - p. \tag{4.12.1}$$

$$m_x = \sum_{i=1}^{n-1} ip^{i-1}q + np^{n-1} = q\frac{d}{dp}\left(\frac{p-p^n}{1-p}\right) + np^{n-1} = \frac{1-p^n}{1-p}.$$

4.13. A random variable X has a Poisson distribution with mean value $a = 3$. Construct the frequency polygon and the distribution function of the random variable X. Find the probability that (1) the random variable X will assume a value smaller than its mean; (2) the random variable X will assume a positive value.

Answer. (1) 0.423; (2) 0.960.

4.14. When a computer is operating it fails (goes down) from time to time. The faults can be considered to occur in an elementary flow. The average number of failures per day is 1.5.

Find the probabilities of the following events:

A = {the computer does not go down for two days};

B = {it goes down at least once during a day};

C = {it goes down no less than three times during a week}.

Answer. $\mathrm{P}(A) = 0.050$; $\mathrm{P}(B) = 0.777$; $\mathrm{P}(C) = 0.998$.

4.15. A shell landing and exploding at a certain position covers the target by a homogeneous Poisson field of splinters with intensity $\lambda = 2.5$ splinters per sq metre. The area of the projection of the target onto the plane on which the field of splinters falls is $S = 0.8$ m^2. If a splinter hits the target, it destroys it completely and for certain. Find the probability that the target will be destroyed.

Answer. 0.865.

4.16. The same problem, but each splinter hitting the target may destroy it with probability 0.6.

Solution. We consider a "field of affecting splinters" with density $\lambda^* = 0.6\,\lambda = 1.5$ splinters per sq metre rather than the assigned field of splinters. The mean value of the number of affecting splinters hitting the target is $a^* = \lambda^* S = 1.2$ splinters, hence the probability of destruction $R_1 = 1 - e^{-a^*} = 1 - 0.301 = 0.699$.

4.17. A vacuum valve operates without a failure for a random time T. When it fails, it is immediately replaced by a new one. The flow of failures is elementary, and has intensity μ. Find the probability of A = {the valve will not be replaced during the time τ}, B = {the valve will be replaced three times} and C {= the valve will be replaced no less than three times}.

Solution. The mean value of the number of failures X during the time τ is $a = \mu\tau$.

$$\mathrm{P}(A) = P_0 = e^{-\mu\tau}, \quad \mathrm{P}(B) = P_3 = \frac{(\mu\tau)^3}{3!} e^{-\mu\tau},$$

$$\mathrm{P}(C) = 1 - (P_0 + P_1 + P_2) = 1 - e^{-\mu\tau}\,[1 - \mu\tau - 0.5\,(\mu\tau)^2].$$

4.18. A device consists of three units, the first of which contains n_1 elements, the second, n_2 elements, an the third, n_3 elements. The first unit is obviously necessary for the device to function; the second and the third units repeat each other. The flows of failures of the elements are elementary. For the elements of the first unit the intensity of the flow of failures is λ_1 while for the elements of the second and third units it is λ_2. The first unit fails if no less than two elements fail. The second unit (as well as the repeating third unit) fails if at least one element fails. For the whole device to fail, it is sufficient for the first unit to fail, or for both the second and third units to fail. Find the probability that the device will fail during the time τ.

Solution. The probabilities that one element from the first, the second or the third unit will fail during the time τ are

$$p_1 = 1 - e^{-\lambda_1 \tau} \quad \text{and} \quad p_2 = p_3 = 1 - e^{-\lambda_2 \tau}$$

respectively. The probability that the first unit will fail during the time τ is

$$\mathcal{P}_1 = 1 - (1 - p_1)^{n_1} - n_1 p_1 (1 - p_1)^{n_1 - 1}.$$

The probabilities that the second and the third unit will fail are

$$\mathcal{P}_2 = 1 - (1 - p_2)^{n_2} \quad \text{and} \quad \mathcal{P}_3 = 1 - (1 - p_3)^{n_3}.$$

The probability that the whole device will fail is

$$\mathcal{P} = \mathcal{P}_1 + (1 - \mathcal{P}_1) \mathcal{P}_2 \mathcal{P}_3.$$

4.19. An earth satellite, orbitting for n days, may collide with a meteorite at random. The meteorites which cut the orbit and collide with the satellite form a stationary Poisson flow with intensity $\varkappa$ (meteorites a day). A meteorite which hits the satellite may hole its shell with probability p_0. A meteorite which holes the shell disables some of the instruments in the satellite with probability p_1. Find the probabilities of

A = {during the time of orbitting the shell of the satellite will be holed};

B = {the instruments will fail during the orbitting of the satellite};

C = {during the orbitting of the satellite only its shell will be holed but the instruments will not be disabled}.

Solution. The mean value of the number of meteorites which hole the shell $a_0 = \varkappa n p_0$. The mean value of the number of meteorites which hole the shell and damage the instruments $a_1 = \varkappa n p_1 p_0$.

$$\mathrm{P}(A) = 1 - e^{-a_0} = 1 - e^{-\varkappa n p_0}, \quad \mathrm{P}(B) = 1 - e^{-a_1} = 1 - e^{-\varkappa n p_0 p_1},$$

$$\mathrm{P}(C) = \mathrm{P}(A) - \mathrm{P}(B) = e^{-\varkappa n p_0 p_1} - e^{-\varkappa n p_0}.$$

4.20. A group of hunters have gathered to hunt a wolf and set off forming a random chain that can be described as an elementary flow of points with intensity λ on the x-axis (λ hunters per unit length, see Fig. 4.20). The wolf runs at right angles to the chain. A hunter fires a shot at the wolf only if the wolf runs at a distance no larger than R_0 from him and, having fired a shot, may kill it with probability p. Find the probability that the wolf will be killed if the animal does not know where the hunters are and the chain is long enough for the wolf not to be able to run beyond it.

Solution. If the wolf runs in the direction indicated by the arrow, it is fired at if at least one hunter gets into a strip $2R_0$ wide connected with

the path of the wolf's displacement. Every hunter who fires at the wolf may be successful, i.e. kill the wolf with probability p. Let us pass from the chain of hunters which has intensity λ to the "chain of successful hunters" which has intensity $\lambda^* = \lambda p$. The wolf will be killed if at least one successful hunter gets into a strip $2R_0$ long thrown at random onto the abscissa axis. The probability of this is

$$P(A) = 1 - e^{-2R_0\lambda p}.$$

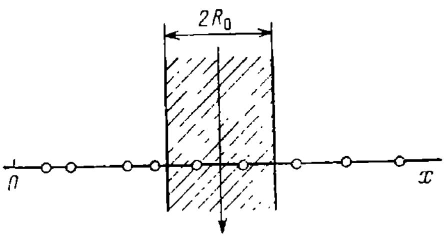

Fig. 4.20

4.21. A car is subjected to an inspection and maintenance. The number of faults detected during the inspection has a Poisson distribution with parameter a. If no faults are detected, the maintenance lasts for an average of two hours. If one or two faults are detected, another half-hour is spent eliminating each defect. If more than two faults are detected, the car takes four hours on the average to be repaired. Find the distribution of the average time T of maintenance and repair of the car and its mean value $M[T]$.

Solution.

$$T: \left|\begin{array}{c|c|c|c} 2 & 2.5 & 3 & 6 \\ \hline e^{-a} & ae^{-a} & \frac{a^2}{2}e^{-a} & 1-e^{-a}\left(1+a+\frac{a^2}{2}\right) \end{array}\right|.$$

$$M[T] = e^{-a}\left(2+2.5a+\frac{3}{2}a^2\right)+6\left[1-e^{-a}\left(1+a+\frac{a^2}{2}\right)\right]$$
$$= 6 - e^{-a}(4+3.5a+1.5a^2).$$

4.22*. A group of animals is examined by a vet and each may be found to be sick with probability p. The examination procedure involves a blood test on a sample blood taken from n animals and mixed. The mixture will yield abnormal test results if at least one of the n animals is sick. A large number N of animals must be examined. Two methods of examination are suggested:

(1) all N animals are examined, in which case N blood tests must be made;

(2) groups of animals are examined, first mixing the blood of n animals. If the test yields normal results, then all the animals in the group are considered healthy and the next n animals are examined. If the results are abnormal, each of the n animals is examined and then the next group of animals is examined ($n > 1$).

Find which of the methods is more advantageous in the sense of the minimum average number of tests. Find $n = n^*$ at which the minimum average number of tests is needed to examine the animals.

Solution. A random variable X_n, which is the number of tests needed

for a group of n animals examined by the second method, has the following ordered series:

$$X_n: \left| \begin{array}{c|c} 1 & n+1 \\ \hline q^n & 1-q^n \end{array} \right|; \quad q=1-p.$$

The average number of tests for a group of n animals examined by the second method is

$$\mathrm{M}\,[X_n] = q^n + (n+1)(1-q^n) = n + (1 - nq^n).$$

When the first method is used, n tests are needed for n animals. For $nq^n < 1$ the first method is, evidently, more advantageous than the second, while for $nq^n > 1$ the second method is preferable.

Let us find the q for which the second method becomes more advantageous. What then is the optimal value $n = n^*$? It follows from the inequality $nq^n > 1$ that $q > 1/\sqrt[n]{n}$ and hence that $q > 0.694$ since the minimum of $1/\sqrt[n]{n}$ for an integer n is attained at $n = 3$. Let us assume that $q > 0.694$ and find the value $n = n^*$ for which the average number of tests per animal is a minimum:

$$R_n = \mathrm{M}\,[X_n]/n = 1 - q^n + 1/n.$$

We must find the least positive root of the equation

$$dR_n/dn = -q^n \ln q - 1/n^2 = 0,$$

and having done so take the two closest integers, substitute them into the formula for R_n and then choose the optimal n^*. By using the substitution $-\ln q = a$, $an = x$, the equation $-q^n \ln q = 1/n^2$ can be reduced to an equation $x^2 e^{-x} = a$ $(a = -\ln q < \ln 3/3 = 0.366)$. For small values of a (and, hence, for small $p = 1 - q$) the last equation has a solution $x \approx \sqrt{a}$, whence $n^* \approx 1/\sqrt{a}$. If the values of a are not small, then a direct comparison of the quantities R_2, R_3 and R_4 shows that R_3 is always smaller than R_2 and that $R_3 < R_4$ for $0.694 < q < 0.876$; consequently, for $0.694 < q < 0.876$ the optimal $n^* = 3$. We can show that for $q > 0.889$ $(p < 0.111)$ the formula $n^* \approx 1/\sqrt{p} + 0.5$ is a close approximation.

4.23. A number of trials are made to switch on an engine. Each trial may be successful (the engine is switched on), independently of the other trials, with probability $p = 0.6$. Each trial lasts for a time τ. Find the distribution of the total time T which will be needed to switch on the engine, its mean value and variance.

Solution. The number of trials X is a variable having a geometric distribution beginning with unity (4.0.32) and $T = X\tau$ has the distribution

$$T: \left| \begin{array}{c|c|c|c|c|c} \tau & 2\tau & 3\tau & \dots & m\tau & \dots \\ \hline p & qp & q^2p & \dots & q^{m-1}p & \dots \end{array} \right.,$$

$$\mathrm{M}\,[T] = \tau \mathrm{M}\,[X] = \tau/p, \quad \mathrm{Var}\,[T] = \tau^2\,\mathrm{Var}\,[X] = \tau^2 q/p^2 \quad (q = 1-p).$$

4.24. Under the conditions of the preceding problem the trials are dependent and p_i is the probability that the engine can be switched on after $i - 1$ unsuccessful trials, p_i being a function of i, i.e. $p_i = \varphi(i)$. Find the distribution of the random variable $T = \tau X$.

Answer.

T:

τ	2τ	3τ	$\ldots$	$m\tau$	$\ldots$
p_1	q_1p_2	$q_1q_2p_3$	$\ldots$	$\prod_{i=1}^{m-1} q_i p_m$	$\ldots$

where $q_i = 1 - p_i$.

4.25. When a reliable instrument, consisting of homogeneous parts, is assembled, each part is subjected to a thorough inspection as a result of which it is either found to be sound (with probability p) or is rejected (with probability $q = 1 - p$). The classification of each part is independent. The store of parts is practically unlimited. The selection of parts and their inspection go on until k high-quality parts are selected. A random variable X is the number of rejected parts. Find the distribution of the random variable X: $P_m = \mathrm{P}\{X = m\}$.

Solution. The possible values of the random variable X are 0, 1, . . ., m, . . . We find their probabilities:

$P_0 = \mathrm{P}$ {the first k parts are sound} $= p^k$;

$P_1 = \mathrm{P}$ {one part out of the first k is rejected, the $(k+1)$th part is sound} $= C_k^1 q^1 p^{k-1} p = C_k^1 q^1 p^k$;

. . . .

$P_m = \mathrm{P}$ {m parts out of the first $k + m - 1$ parts are rejected, the $(m + k)$th part is sound} $= C_{k+m-1}^m q^m p^k$ $(m = 1, 2, \ldots)$.

4.26. Two random variables X and Y may be either 0 or 1 independently of each other. Their ordered series are

X:

0	1
q_x	p_x

Y:

0	1
q_y	p_y

Construct the ordered series: (1) of their sum $Z = X + Y$; (2) of their difference $U = X - Y$; (3) or their product $V = XY$.

Solution. (1) The random variable Z has three possible values: 0, 1 and 2.

$$\mathrm{P}\{Z=0\} = \mathrm{P}\{X=0,\ Y=0\} = q_xq_y,$$

$$\mathrm{P}\{Z=1\} = \mathrm{P}\{X=1,\ Y=0\} + \mathrm{P}\{X=0,\ Y=1\} = p_xq_y + q_xp_y,$$

$$\mathrm{P}\{Z=2\} = \mathrm{P}\{X=1,\ Y=1\} = p_xp_y,$$

Z:

0	1	2
q_xq_y	$p_xq_y+q_xp_y$	p_xp_y

By analogy we find that

$$U: \left|\begin{array}{c|c|c} -1 & 0 & 1 \\ \hline q_x p_y & p_x p_y + q_x q_y & p_x q_y \end{array}\right|,$$

$$V: \left|\begin{array}{c|c} 0 & 1 \\ \hline q_x q_y + p_x q_y + q_x p_y & p_x p_y \end{array}\right|.$$

4.27. A random variable X has an ordered series

$$X: \left|\begin{array}{c|c|c} 1 & 2 & 4 \\ \hline 0.5 & 0.2 & 0.3 \end{array}\right|.$$

Construct the ordered series of the random variable $Y = 1/(3 - X)$.

Answer.

$$Y: \left|\begin{array}{c|c|c} -1 & 0.5 & 1 \\ \hline 0.3 & 0.5 & 0.2 \end{array}\right|.$$

4.28. A random variable X has an ordered series

$$X: \left|\begin{array}{c|c|c|c|c} -2 & -1 & 0 & 1 & 2 \\ \hline 0.2 & 0.3 & 0.2 & 0.2 & 0.1 \end{array}\right|.$$

Construct the ordered series of its square: $Y = X^2$.

Answer.

$$Y: \left|\begin{array}{c|c|c} 0 & 1 & 4 \\ \hline 0.2 & 0.5 & 0.3 \end{array}\right|.$$

4.29. When a message is sent over a communication channel, noise hinders the decoding of the message. It may be impossible to decode the message with probability p. The message is repeated until it is decoded. The duration of the transmission of a massage is 2 min. Find: (1) the mean value of the time T needed for the transmission of the message; (2) the probability that the transmission of the message will take a time exceeding the time t_0 at our disposal.

Solution. (1) The random variable X, the number of trials to send a message, has a geometric distribution beginning with unity; $T = 2X$ min. The ordered series T of the random variable is

$$T: \left|\begin{array}{c|c|c|c|c|c} 2 & 4 & 6 & \dots & 2m & \dots \\ \hline q & pq & p^2 q & \dots & p^{m-1}q & \dots \end{array}\right. .$$

$$(2)\ \mathrm{P}\{T > t_0\} = \sum_{m=[t_0/2]+1}^{\infty} p^{m-1} q = q \sum_{m=[t_0/2]+1}^{\infty} p^{m-1}, \qquad (4.29)$$

where $[t_0/2]$ is the largest integer in $t_0/2$. Summing up the geometric progression (4.29), we obtain

$$\mathrm{P}\{T > t_0\} = \frac{q}{1-q} \frac{p^{[t_0/2]+1}}{1-p} = p^{[t_0/2]}.$$

4.30. A discrete random variable X has an ordered series

$$X: \left|\begin{array}{c|c|c|c} x_1 & x_2 & \ldots & x_n \\ \hline p_1 & p_2 & \ldots & p_n \end{array}\right|.$$

A random variable Z is the minimum value of two numbers—the value of the random variable X and the number a, i.e. $Z = \min\{X, a\}$, where $x_1 \leqslant a \leqslant x_n$. Find the ordered series of the random variable Z.

Solution. By definition

$$Z = \begin{cases} X & \text{for} \quad X \leqslant a; \\ a & \text{for} \quad X > a. \end{cases}$$

The ordered series of the random variable Z coincides with that of the random variable X for the values $x_1, x_2, \ldots$, which are smaller than or equal to a; P $\{Z = a\}$ can be calculated as unity minus the sum of all the other probabilities:

$$\mathrm{P}\{Z = a\} = 1 - \sum_{x_i \leqslant a} p_i.$$

4.31. The ordered series of a discrete random variable X is

$$X: \left|\begin{array}{c|c|c|c|c} 1 & 3 & 5 & 7 & 9 \\ \hline 0.1 & 0,2 & 0.3 & 0.3 & 0.1 \end{array}\right|.$$

Find the ordered series of the random variable $Z = \min\{X, 4\}$.

Answer. Proceeding from the solution of the preceding problem, we have

$$Z: \left|\begin{array}{c|c|c} 1 & 3 & 4 \\ \hline 0.1 & 0.2 & 0.7 \end{array}\right|.$$

4.32. Two random variables X and Y, independently of each other, assume a value according to the following ordered series:

$$X: \left|\begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \hline 0.2 & 0.3 & 0.4 & 0.1 \end{array}\right|, \quad Y: \left|\begin{array}{c|c|c} 1 & 3 & 4 \\ \hline 0.7 & 0.2 & 0.1 \end{array}\right|.$$

Construct the ordered series of the random variable $Z = \min\{X, Y\}$.

Solution.

$$\mathrm{P}\{Z=0\} = \mathrm{P}\{X=0\} = 0.2,$$
$$\mathrm{P}\{Z=1\} = \mathrm{P}\{X=1\} + \mathrm{P}\{Y=1, X>1\} = 0.3 + 0.7 \cdot 0.5 = 0.65,$$
$$\mathrm{P}\{Z=2\} = \mathrm{P}\{X=2, Y>2\} = 0.4 \cdot 0.3 = 0.12,$$
$$\mathrm{P}\{Z=3\} = \mathrm{P}\{X=3, Y \geqslant 3\} = 0.1 \cdot 0.3 = 0.03.$$

$$Z: \left|\begin{array}{c|c|c|c} 0 & 1 & 2 & 3 \\ \hline 0.20 & 0.65 & 0.12 & 0.03 \end{array}\right|.$$

4.33. Under the conditions of the preceding problem, find the ordered series of the random variable $U = \max\{X, Y\}$.

Answer.

$$U: \left|\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ \hline 0.35 & 0.28 & 0.27 & 0.10 \end{array}\right|.$$

4.34. An n-digit binary number is written in a computer cell and each sign of the number assumes the value 0 or 1 with equal probability independently of the other signs. The random variable X is the number of signs "1" in the notation of the binary number. Find the probabilities of the events $\{X = m\}$, $\{X \geqslant m\}$, $\{X < m\}$.

Solution. The random variable X has a binomial distribution with parameters n, $p = 1/2$, $\mathrm{P}\{X = m\} = C_n^m (1/2)^n$,

$$\mathrm{P}\{X \geqslant m\} = (1/2)^n \sum_{k=m}^{n} C_n^m, \ \mathrm{P}\{X < m\} = 1 - \mathrm{P}\{X \geqslant m\}.$$

4.35. A proper three-place decimal fraction X is considered, each sign of which may assume each of the values 0, 1, . . ., 9 with equal probability independently of the other signs. Construct the ordered series of the random variable X and find its mean value.

Solution. The possible values of the random variable X are 0.000, 0.001, 0.002, . . ., 0.999. The probability of each of them is $p = (0.1)^3 = 0.001$. The ordered series of the random variable X has the form

$$X: \left|\begin{array}{c|c|c|c|c} 0.000 & 0.001 & 0.002 & \ldots & 0.999 \\ \hline 0.001 & 0.001 & 0.001 & \ldots & 0.001 \end{array}\right|.$$

$$\mathrm{M}[X] = \sum_{i=0}^{999} x_i p_i = 0.001 \sum_{i=0}^{999} x_i, \text{ where } x_i = i \cdot 0.001.$$

The numbers x_i form an arithmetic progression consisting of 1000 terms with a common difference 0.001. Summing up the progression, we obtain

$$\mathrm{M}[X] = \frac{0+0.999}{2} \cdot 1000 \cdot 0.001 = 0.4995 \approx 0.5.$$

4.36. A random variable Y is a random proper binary fraction with n decimal places. Each sign, independently of the other signs, may be either 0 or 1 with probability 1/2. Find the ordered series of the random variable Y and its mean value $\mathrm{M}[Y]$.

Solution. As in the preceding problem, all the values of the binary number from 0.00 . . . 0 to 0.11 . . . 1 are equally probable and each of them has a probability $1/2^n$. The mean value of the random variable Y (in the decimal notation) $\mathrm{M}[Y] = 0.5 - 1/2^{n+1}$.

4.37. A message is sent over a communication channel in the binary code and consists of a sequence of symbols 0 and 1, alternating with an equal probability and independently of each other, say, 0, 0, 1, 1, 1. 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, ...

A run of the same symbols is considered, say, 0, 0, 0 or 1, 1, 1, 1 (or simply 0 or 1 if the symbols are not repeated). Any one of these runs is taken at random. A random variable X is the number of symbols in a run. Find its ordered series, mean value and variance. Find $\mathrm{P}\{X \geqslant k\}$.

Solution. The random variable X has a geometric distribution beginning with unity:

$$X: \left|\begin{array}{c|c|c|c|c} 1 & 2 & \ldots & m & \ldots \\ \hline 0.5 & 0.5^2 & \ldots & 0.5^m & \ldots \end{array}\right|, \quad \mathrm{M}[X] = 1/0.5 = 2,$$

$$\mathrm{Var}[X] = 0.5/0.5^2 = 2, \quad \mathrm{P}\{X \geqslant k\} = \sum_{m=k}^{\infty} 0.5^m = 0.5^{k-1}.$$

4.38. Under the conditions of the preceding problem P {"0"} = 0.7, P {"1"} = 0.3. Find the average number of symbols $\mathrm{M}[X]$ in a run of zeros, the average number of symbols $\mathrm{M}[Y]$ in a run of unities and the overall average number of symbols $\mathrm{M}[Z]$ in a run of symbols chosen at random. Find the variances $\mathrm{Var}[X]$, $\mathrm{Var}[Y]$, $\mathrm{Var}[Z]$.

Solution. $\mathrm{M}[X] = 1/0.7 = 10/7$, $\mathrm{M}[Y] = 1/0.3 = 10/3$. By the formula for the complete expectation we find that $\mathrm{M}[Z] = 0.7 \cdot \frac{10}{7} + 0.3 \cdot \frac{10}{3} = 2$, i.e. the same as in the preceding problem. $\mathrm{Var}[X] = 0.3/0.7^2 = 0.612$, $\mathrm{Var}[Y] = 0.7/0.3^2 = 7.778$.

To find $\mathrm{Var}[Z]$, we first find $\alpha_2[Z] = \mathrm{M}[Z^2]$ by the formula for the complete expectation [see (4.0.20)] *):
$\alpha_2[Z] = 0.7\,\alpha_2[X] + 0.3\,\alpha_2[Y]$, where $\alpha_2[X] = \mathrm{Var}[X] + (\mathrm{M}[X])^2 \approx 2.66$, $\alpha_2[Y] = \mathrm{Var}[Y] + (\mathrm{M}[Y])^2 \approx 18.9$, then $\alpha_2[Z] \approx 7.53$, $\mathrm{Var}[Z] = \alpha_2[Z] - (\mathrm{M}[Z])^2 \approx 3.53$, i.e. the last variance is larger than that in the preceding problem.

The probability $\mathrm{P}\{X \geqslant k\}$ can be found by the total probability formula with hypotheses H_0 = {the first symbol is "0"}, H_1 = {the first symbol is "1"}:

$$\mathrm{P}(H_0) = 0.7,\ \mathrm{P}(H_1) = 0.3,\ \mathrm{P}\{X \geqslant k \mid H_0\} = 0.7^{k-1},$$
$$\mathrm{P}\{X \geqslant k \mid H_1\} = 0.3^{k-1}, \quad \text{then} \quad \mathrm{P}\{X \geqslant k\} = 0.7 \cdot 0.7^{k-1}$$
$$+ 0.3 \cdot 0.3^{k-1} = 0.7^k + 0.3^k.$$

4.39. A device consists of m units of type I and n units of type II. The reliability (a failure-free performance for an assigned time τ) of each type I unit is p_1 and that of a type II unit is p_2. The units fail independently of one another. For the device to operate, any two type I units and any two type II units must operate simultaneously for time τ. Find the probability $\mathscr{P}$ of the failure-free operation of the device.

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*) By the formula for the complete expectation we seek the second moment about the origin rather than the variance itself (since conditional expectations are different for different hypotheses).

Solution. The event A = {failure-free operation of the device} is a product of two events:

A_1 = {no less than two out of m type I units operate without failure};

A_2 = {no less than two out of n type II units operate without failure}.

The number X_1 of type I units operating without failure is a random variable distributed according to the binomial law with parameters m and p_1; the event A_1 corresponds to the random variable X_1 assuming a value no smaller than 2. Therefore

$$\begin{aligned}P(A_1) &= P\{X_1 \geqslant 2\} = 1 - P\{X_1 < 2\}\\ &= 1 - P\{X_1 = 0\} - P\{X_1 = 1\}\\ &= 1 - (q_1^m + mq_1^{m-1}p_1)\ (q_1 = 1 - p_1).\end{aligned}$$

Similarly, $P(A_2) = 1 - (q_2^n + nq_2^{n-1}p_2)\ (q_2 = 1 - p_2)$.

The probability of a failure-free performance of the device

$$\mathscr{P} = P(A) = P(A_1)P(A_2) = [1 - (q_1^m + mq_1^{m-1}p_1)]\,[1 - (q_2^n + nq_2^{n-1}p_2)].$$

4.40. An instrument is used (activated) several times before it fails (an instrument that has failed is not repaired). The probability that

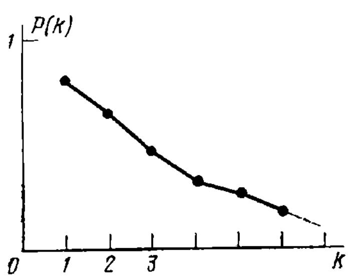


Fig. 4.40

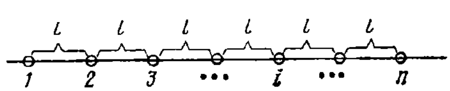


Fig. 4.41

having been used k times an instrument does not fail is $P(k)$. The function $P(k)$ is given (see Fig. 4.40). The instrument is known to have been activated n times. Find the probability Q_m that it can be activated another m times and the mean value of the number X of future activations.

Solution. Q_m is the conditional probability that the instrument will not fail when activated the next m times provided that it has not failed the first n times. By multiplication rule for probabilities $P(n + m) = P(n)\,Q_m$, whence

$$Q_m = \frac{P(n+m)}{P(n)}, \quad M[X] = \sum_{m=1}^{\infty} mQ_m.$$

4.41. A worker operates n machine-tools of the same type which are located along a straight line with intervals l (Fig. 4.41). From time to time the tools stop (with equal probability and independently of one another) and must be adjusted. Having adjusted one tool, the worker stays put until another tool stops; then he goes to that tool (if the same tool fails, he stays put). A random variable X is the distance the worker covers between two adjustments. Find the mean value of the random variable X.

Solution. We apply the complete expectation formula with hypothesis H_i = {the worker stays at the ith tool} ($i = 1, \ldots, n$). According to the conditions of the problem all the hypotheses are equally probable: $\mathrm{P}(H_1) = \mathrm{P}(H_2) = \ldots = \mathrm{P}(H_n) = 1/n$. Let us find the conditional expectation $\mathrm{M}[X|H_i]$ for the ith hypothesis. Each tool (ith inclusive) stops with probability $1/n$; hence

$$\mathrm{M}[X \mid H_i] = \frac{l}{n}\left[\sum_{k=1}^{i-1}(i-k) + \sum_{k=i}^{n}(k-i)\right] = \frac{l}{2n}[i(i-1) + (n-i)(n-i+1)].$$

By the complete expectation formula

$$\mathrm{M}[X] = \sum_{i=1}^{n}\mathrm{P}(H_i)\,\mathrm{M}[X \mid H_i] = \sum_{i=1}^{n}\frac{1}{n}\,\frac{l}{2n}[i(i-1) + (n-i)(n-i+1)]$$

$$= \frac{l}{2n^2}\left[\sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n}(n-i+1)(n-i)\right]. \qquad (4.41)$$

Using the familiar formula $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, we calculate

$$\sum_{i=1}^{n} i(i-1) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n(n+1)(n-1)}{3}.$$

Using this formula to calculate both sums in (4.41), we make sure that they are equal and

$$\mathrm{M}[X] = \frac{l}{n^2}\sum_{i=1}^{n} i(i-1) = \frac{l}{n^2}\,\frac{n(n+1)(n-1)}{3} = \frac{l(n^2-1)}{3n}.$$

4.42. A number of independent trials are made, in each of which an event A may occur with probability p; the trials are continued until the event A occurs after which they are terminated. A random variable X is the number of trials. Construct its ordered series and frequency polygon; derive formulas for its mean value, variance and mean square deviation.

Solution. The random variable X has a geometric distribution beginning with unity [see (4.0.32)]. The ordered series of the variable X

has the form

$$X: \left|\begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & \ldots & m & \ldots \\ \hline p & qp & q^2p & \ldots & q^{m-1}p & \ldots \end{array}\right. .$$

where $q = 1 - p$. The frequency polygon for $p = 0.6$ is shown in Fig. 4.42.

$$\mathrm{M}[X] = \sum_{m=1}^{\infty} mq^{m-1}p = p\sum_{m=1}^{\infty} mq^{m-1}.$$

We note that mq^{m-1} is the derivative of q^m with respect to q, hence

$$\sum_{m=1}^{\infty} mq^{m-1} = \sum_{m=1}^{\infty} \frac{d}{dq} q^m = \frac{d}{dq} \sum_{m=1}^{\infty} q^m.$$

The last sum is the sum of the terms of an infinitely decreasing geometric progression with a common ratio q; summing it up, we find that

$$\sum_{m=1}^{\infty} mq^{m-1} = \frac{d}{dq}\left(\frac{q}{1-q}\right) = \frac{1}{(1-q)^2}, \tag{4.42}$$

whence $\mathrm{M}[X] = p/(1-q)^2 = 1/p$.

The variance of the random variable X can be found in terms of its second moment about the origin

$$\alpha_2[X] = \mathrm{M}[X^2] = \sum_{m=1}^{\infty} m^2q^{m-1}p.$$

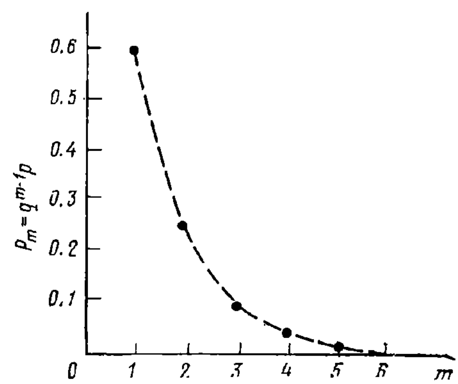


Fig. 4.42

To calculate it, we multiply series (4.42) by q, whence we get

$$\sum_{m=1}^{\infty} mq^m = \frac{q}{(1-q)^2}.$$

Differentiating the series, we obtain

$$\sum_{m=1}^{\infty} m^2q^{m-1} = \frac{1+q}{(1-q)^3}.$$

Multiplying this expression by $p = 1 - q$, we get $\mathrm{M}[X^2] = (1 + q)/(1 - q)^2$. The variance of the random variable X is [see (4.0.17)]

$$\mathrm{Var}[X] = \mathrm{M}[X^2] - (\mathrm{M}[X])^2 = (1+q)/(1-q)^2 - 1/(1-q)^2 = q/p^2,$$

$$\sigma_x = \sqrt{\mathrm{Var}[X]} = \sqrt{q}/p.$$

Thus, for a geometric distribution beginning with unity

$$M[X] = 1/p, \quad \mathrm{Var}[X] = q/p^2, \quad \sigma = \sqrt{q}/p.$$

The random variable $Y = X - 1$, which has a geometric distribution beginning with a zero, possesses the characteristics

$$M[Y] = M[X] - 1 = q/p, \quad \mathrm{Var}[Y] = \mathrm{Var}[X] = q/p^2, \quad \sigma_y = \sqrt{q}/p.$$

4.43. Several trials are needed to tune up an intricate electronic circuit. The probability that the circuit will be tuned in the first trial is P_1, in the second P_2, ..., in the kth trial P_k, ... The probabilities $P_1, P_2, \ldots, P_k, \ldots, P_n$ are given. After the nth unsuccessful trial to tune up the circuit, the trials are terminated. Find the mean value and the variance of the random variable X, which is the total number of trials.

Solution. The ordered series of the random variable X has the form

$$X: \left\| \begin{array}{c|c|c|c|c|c} 1 & 2 & \ldots & k & \ldots & n \\ \hline P_1 & P_2 & \ldots & P_k & \ldots & 1-\sum_{k=1}^{n-1} P_k \end{array} \right\|,$$

$$M[X] = \sum_{k=1}^{n-1} iP_i + n\left[1 - \sum_{k=1}^{n-1} P_k\right],$$

$$\mathrm{Var}[X] = M[X^2] - (M[X])^2 = \sum_{i=1}^{n-1} i^2 P_i + n^2\left[1 - \sum_{i=1}^{n-1} P_i\right] - (M[X])^2.$$

4.44. An instrument is assembled from k types of parts including m_i parts of type i $(i = 1, \ldots, k)$ $\left(\sum_{i=1}^{k} m_i = m,\ m\right.$ being the total number of parts). The probability that a part of type i taken at random has a defect is q_i. The instrument operates only when none of the parts are defective. (1) Find the probability P that the instrument will operate; (2) find the probability R_2 that there will be no less than two defective parts in the instrument.

Solution. (1) P is the probability that in m independent trials the event A = {a part is defective} will not occur: $P = \prod_{i=1}^{k} (1-q_i)^{m_i} = P_0$; (2) $R_2 = 1 - (P_0 + P_1)$, where P_0 is the probability that none of the m parts are defective; P_1 is the probability that there is exactly one defective part. P_1 can be found as the sum of probabilities that there is exactly one defective part among the m_i parts of type i, the other parts being sound, i.e.

$$\begin{aligned} P_1 &= m_1 q_1 (1-q_1)^{m_1-1} \prod_{i \neq 1} (1-q_j)^{m_j} \\ &\quad + m_2 q_2 (1-q_2)^{m_2-1} \prod_{j \neq 2} (1-q_j)^{m_j} + \ldots \\ &= \sum_{i=1}^{k} m_i q_i (1-q_i)^{m_i-1} \prod_{j \neq i} (1-q_j)^{m_j}. \end{aligned}$$

4.45. A total of k messages are sent over a communication channel which contain $n_1, n_2, \ldots, n_k$ binary symbols (0 or 1) respectively. The symbols assume the values 0 and 1 independently of one another with probability 0.5. Each symbol may be distorted (replaced by the opposite value) with probability p. When the messages are coded, a code is used which can correct errors in one or two symbols (practically with a complete trustworthiness). An error in at least one symbol (after the correction) makes the whole message erroneous. Find the probability R that at least one of the k messages will be erroneous.

Solution. The random variable X_i defined as the number of incorrect symbols in the ith message, has a binomial distribution (4.0.23) with parameters n_i and p. The probability that the ith message is erroneous is equal to the probability that no less than three symbols of the n_i symbols in the message are incorrect:

$$P\{X_i \geqslant 3\} = 1 - P\{X_i < 3\} = 1 - \sum_{j=0}^{2} C_{n_i}^j p^j (1-p)^{n_i - j}.$$

The probability that at least one of the k messages is erroneous is

$$R = 1 - \prod_{i=1}^{k} \sum_{j=0}^{2} C_{n_i}^j p^j (1-p)^{n_i - j}.$$

4.46. Four identical parts are needed to assemble an instrument; we have ten parts at our disposal, six of which are sound and four are faulty. All the parts are identical in appearance. Five parts are selected at random (one extra part is taken "just in case"). Find the probability **that no less than four of the five parts are sound.**

Solution. The random variable X is the number of sound parts among the five chosen and has a hypergeometric distribution with parameters 5, 6, 4 [see (4.0.34)]. The probability that m of the five parts are sound is $P_m = C_6^m C_4^{5-m} / C_{10}^5$; hence

$$P_4 = 5/21, \quad P_5 = 1/42, \quad P\{X \geqslant 4\} = P_4 + P_5 = 11/42.$$

4.47. There are seven radio valves, three of which are faulty and indistinguishable from the others. Four valves are taken at random and are screwed into four holders. Find and construct (as a frequency polygon) an ordered series of X radio valves which will function. Find its mean value, variance and mean square deviation.

Solution. The variable X has a hypergeometric distribution with parameters $n = 4$, $a = 4$, $b = 3$.

$$P_1 = C_4^1 C_3^3 / C_7^4 = 4/35, \qquad P_2 = C_4^2 C_3^2 / C_7^4 = 18/35,$$
$$P_3 = C_4^3 C_3^1 / C_7^4 = 12/35, \quad P_4 = C_4^4 / C_7^4 = 1/35.$$

The ordered series of the random variable X has the form

X:

1	2	3	4
4/35	18/35	12/35	1/35

The frequency polygon of the random variable X is given in Fig. 4.47. Using formulas (4.0.35) and (4.0.36), we calculate: $M[X] \approx 2.28$ and $\mathrm{Var}[X] \approx 0.38$.

4.48*. A number of independent trials are made in each of which an event A occurs with probability p. Our aim is to obtain the event A k times. The maximum possible number of trials is n (with $n \geqslant 2k$). The trials are terminated either when the event A occurs k times or when it is clear that the event cannot occur k times, i.e. when the opposite

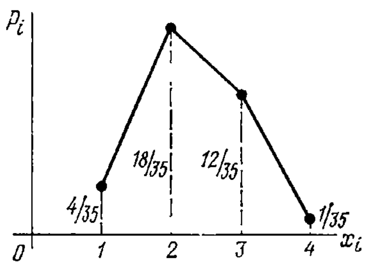


Fig. 4.47

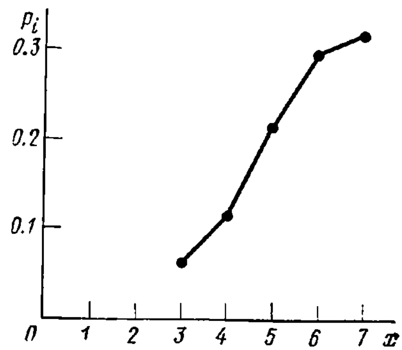


Fig. 4.49

event $\bar{A}$ occurs $n - k + 1$ times. A random variable X is the number of trials that will be made. Find the distribution of the random variable X.

Solution. The possible values of the random variable X are k, $k + 1, \ldots, n - 1, n$. Let us find the corresponding probabilities $P_m = \mathrm{P}\{X = m\}$.

For $k \leqslant m \leqslant n - k$ the trials will be terminated after the mth trial because the event A has occurred k times. Therefore,

$P_m = \mathrm{P}$ {as a result of $m - 1$ trials the event A occurs $k - 1$ times and the mth trial yields the event A} $=$

$$C_{m-1}^{k-1} p^{k-1} q^{m-k} p = C_{m-1}^{k-1} p^k q^{m-k}. \tag{4.48.1}$$

In particular, for $m = k$ we have $\mathrm{P}_k = p^k$.

For $n - k < m \leqslant n$ the trials can be terminated after the mth trial either because the event A occurs for the kth time or because the opposite event $\bar{A}$ occurs $n - k + 1$ times. The probability of the first variant has been calculated; the probability of the second variant is

$\mathrm{P} =$ {the first $m - 1$ trials yield $n - k$ occurrences of the event $\bar{A}$ and the mth trial yields the event $\bar{A}$} $=$

$$C_{m-1}^{n-k} p^{m-1-n+k} q^{n-k} q = C_{m-1}^{n-k} p^{m-1-n+k} q^{n-k+1}.$$

Adding the probabilities of the two variants together, we obtain

$$P_m = C_{m-1}^{k-1} p^k q^{m-k} + C_{m-1}^{n-k} p^{m-1-n+k} q^{n-k+1}. \tag{4.48.2}$$

Thus, for $k \leqslant m \leqslant n-k$, the distribution of the random variable X is given by formula (4.48.1) and for $n-k<m \leqslant n$, by formula (4.48.2)

4.49. Under the conditions of the preceding problem $n=7$, $k=3$ and $p=0.4$. Construct the ordered series of the random variable X.

Solution. The possible values of the random variable X are 3, 4, 5, 6, 7.

$$\mathrm{P}_3=p^3=0.4^3=0.064, \quad P_4=C_{4-1}^{3-1}p^3q^{4-3}=3\cdot 0.4^3\cdot 0.6=0.115,$$

$$\mathrm{P}_5=C_{5-1}^{3-1}p^3q^{5-3}+C_{5-1}^{7-3}p^{5-1-7+3}q^{7-3+1}=6\cdot 0.4^3\cdot 0.6^2+0.6^5=0.216,$$

$$\mathrm{P}_6=C_{6-1}^{3-1}p^3q^{6-3}+C_{6-1}^{7-3}p^{6-1-7+3}q^{7-3+1}=10\cdot 0.4^3\cdot 0.6^3+5\cdot 0.4\cdot 0.6^5=0.293,$$

$$\mathrm{P}_7=C_{7-1}^{3-1}p^3q^{7-3}+C_{7-1}^{7-3}p^{7-1-7+3}q^{7-3+1}=15\cdot 0.4^3\cdot 0.6^4+15\cdot 0.4^2\cdot 0.6^5=0.312.$$

X:

3	4	5	6	7
0.064	0.115	0.216	0.293	0.312

The frequency polygon is given in Fig. 4.49.

CHAPTER 5

Continuous and Mixed Random Variables

5.0. The set of the possible values of a discrete random variable is finite and countable but a nondiscrete random variable is characterized by an uncountable set of possible values. Examples of nondiscrete random variables are the detection range of a radar, the time a train is late, and the error in measuring an angle with a protractor. The sets of the possible values of all these random variables is uncountable since they continuously fill a certain section along the abscissa axis.

Remember that the distribution function of a random variable X is defined thus

$$F(x) = \mathrm{P}\{X < x\}. \tag{5.0.1}$$

Hence a distribution function exists for any random variable, whether discrete or nondiscrete.

If the distribution function $F(x)$ of a random variable X is continuous for any x and, in addition, has a derivative $F'(x)$ everywhere, except, maybe, at individual particular points (Fig. 5.0.1), then the random variable X is said to be continuous.

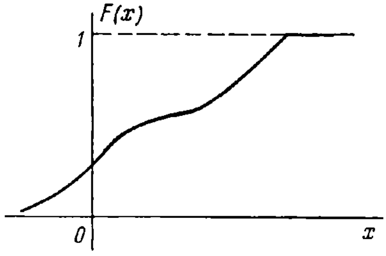


Fig. 5.0.1

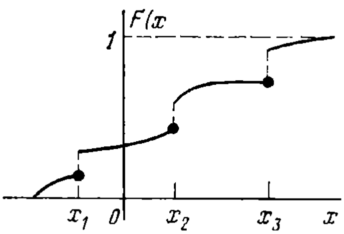


Fig. 5.0.2

If the distribution function $F(x)$ continuously increases on certain intervals but has discontinuities at particular points (Fig. 5.0.2), then the random variable is said to be mixed. The function $F(x)$ is continuous on the left for either a mixed or a discrete random variable.

The probability of a particular individual value of a continuous random variable is equal to zero. The probability of a particular individual value of a mixed random variable, which lies on the interval of continuity of $F(x)$, is also zero, while the probability of each of the values $x_1, x_2, \ldots,$ at which the function $F(x)$ jumps, is numerically equal to the value of the corresponding jump.

For any random variable (discrete, continuous or mixed) the probability that it will fall on the interval of the abscissa axis from α to β (including α but excluding β) is expressed by the formula

$$\mathrm{P}\{\alpha \leqslant X < \beta\} = F(\beta) - F(\alpha). \tag{5.0.2}$$

Since $\mathrm{P}\{X = \alpha\} = 0$ for a continuous random variable, the equality sign in (5.0.2) can be neglected:

$$\mathrm{P}\{\alpha < X < \beta\} = F(\beta) - F(\alpha), \tag{5.0.3}$$

or, using another notation,

$$\mathrm{P}\{X \in (\alpha, \beta)\} = F(\beta) - F(\alpha). \tag{5.0.4}$$

The *probability density* (or distribution density, or density function) of a continuous random variable X is a derivative of the distribution function:

$$f(x) = F'(x). \tag{5.0.5}$$

A *probability element* for a continuous random variable X is the quantity $f(x)\,dx$, which is approximately the probability that the random variable X will fall on the elementary interval dx adjoining the point x:

$$f(x)\,dx \approx \mathrm{P}\{x < X < x + dx\}. \tag{5.0.6}$$

The density $f(x)$ of any random variable is nonnegative ($f(x) \geqslant 0$) and possesses the property

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \tag{5.0.7}$$

The graph of the density $f(x)$ is known as a *distribution curve.*

The probability that a continuous random variable X falls on the interval from α to β is expressed as

$$\mathrm{P}\{\alpha < X < \beta\} = \int_{\alpha}^{\beta} f(x)\,dx. \tag{5.0.8}$$

The distribution function of a continuous random variable X can be expressed in terms of its density:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx. \tag{5.0.9}$$

The *mathematical expectation* of a continuous random variable X with density $f(x)$ is its mean value, which can be calculated from the formula

$$\mathrm{M}[X] = \int_{-\infty}^{\infty} x f(x)\,dx. \tag{5.0.10}$$

The mathematical expectation of a mixed random variable with a distribution unction $F(x)$ can be calculated from the formula

$$\mathrm{M}[X] = \sum_{i} x_i p_i + \int_{(c)} x F'(x)\,dx, \tag{5.0.11}$$

where the sum extends over all the points of discontinuity of the distribution function and the integral over all the intervals of its continuity. When $\mathrm{M}[X]$ has to be denoted by one letter, we shall write $\mathrm{M}[X] = m_x$.

The variance of a continuous random variable X is

$$\mathrm{Var}[X] = \mathrm{M}[\overset{\circ}{X}{}^2] = \mathrm{M}[(X - m_x)^2]$$

and can be calculated by the formula

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - m_x)^2 f(x)\,dx. \tag{5.0.12}$$

The variance of a mixed random variable is expressed by the formula

$$\operatorname{Var}[X]=\sum_i (x_i-m_x)^2 p_i+\int_{(c)} (x-m_x)^2 F'(x)\,dx, \tag{5.0.13}$$

where the sum extends over all the points of discontinuity of the function $F(x)$ and the integral over all the intervals of its continuity.

The square root of variance is known as the *mean square deviation* of a random variable:

$$\sigma_x=\sqrt{\operatorname{Var}[X]}. \tag{5.0.14}$$

The *coefficient of variation*

$$v_x=\sigma_x/m_x \tag{5.0.15}$$

is sometimes used as a characteristic of the degree of randomness of a nonnegative random variable.

Note that the coefficient of variation depends on the origin of the random variable.

The mean square deviation can be used to approximate the range of the possible values of a random variable. This approximation is known as the *three sigma rule* which states that the *range of the practically possible values of a random variable X lies within the limits*

$$m_x \pm 3\sigma_x. \tag{5.0.16}$$

This rule is also valid for a discrete random variable.

The kth moments about the origin for a continuous and for a mixed random variable

$$\alpha_k[X]=\mathrm{M}[X^k]$$

are expressed by the formulas

$$\alpha_k[X]=\int_{-\infty}^{\infty} x^k f(x)\,dx, \tag{5.0.17}$$

$$\alpha_k[X]=\sum_i x_i^k p_i+\int_{(c)} x^k F'(x)\,dx, \tag{5.0.18}$$

respectively.

The central moments can be calculated from similar formulas

$$\mu_k[X]=\int_{-\infty}^{\infty} (x-m_x)^k f(x)\,dx, \tag{5.0.19}$$

$$\mu_k[X]=\sum_i (x_i-m_x)^k p_i+\int_{(c)} (x-m_x)^k F'(x)\,dx. \tag{5.0.20}$$

The central moments can be expressed in terms of the moments about the origin as was the case for a discrete random variable (see Chapter 4). The expression for variance in terms of the second moment about the origin is of especial practical interest:

$$\operatorname{Var}[X]=\alpha_2[X]-m_x^2, \tag{5.0.21}$$

or, in another notation,

$$\operatorname{Var}[X]=\mathrm{M}[X^2]-(\mathrm{M}[X])^2. \tag{5.0.22}$$

If the probability of an event A depends on the value assumed by a continuous random variable X with density $f(x)$, then the total probability of the event A

can be calculated from the *integral formula for total probability*

$$P(A) = \int_{-\infty}^{\infty} P(A|x) f(x)\, dx, \tag{5.0.23}$$

where $P(A \mid x) = P\{A \mid X = x\}$ is the conditional probability of the event A on the hypothesis $\{X = x\}$.

Bayes's formula also has an analogue for continuous random variables. If an event A occurs as a result of an experiment and the probability of the event depends on the value the continuous random variable X assumes, then the conditional probability density of the random variable, the occurrence of the event A being taken into account, is

$$f_A(x) = f(x)\, P(A \mid x)/P(A), \tag{5.0.24}$$

or, with due regard for formula (5.0.23),

$$f_A(x) = f(x)\, P(A|x) \Big/ \int_{-\infty}^{\infty} P(A|x) f(x)\, dx. \tag{5.0.25}$$

This formula is known as *Bayes's formula*.

There is also an *integral formula for total expectation* for continuous random variables, i.e.

$$M[X] = \int_{-\infty}^{\infty} M[X|y] f(y)\, dy, \tag{5.0.26}$$

where X is any random variable; Y is a continuous random variable with density $f(y)$ and $M[X \mid y]$ is the conditional expectation of the random variable X, provided that the random variable Y has assumed a value y.

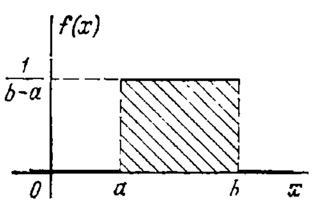


Fig. 5.0.3

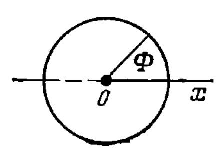


Fig. 5.0.4

Let us consider the distribution and properties of continuous random variables that are often encountered in practical applications.

1. **Uniform distribution.** A random variable X has a *uniform distribution* on the interval from a to b if its density on that interval is constant:

$$f(x) = \begin{cases} 1/(b-a) & \text{for } x \in (a, b), \\ 0 & \text{for } x \notin (a, b). \end{cases} \tag{5.0.27}$$

The values of $f(x)$ at points a and b are not defined, but this is inessential since the probability of falling in any of them is zero and, hence, the probability of any event related to the random variable X does not depend on the value of the density $f(x)$ at the points a and b*). The graph of the probability density of a uniform distribution is shown in Fig. 5.0.3.

*) From now on, when we define the density $f(x)$ by different formulas on different intervals of the x-axis, we shall not indicate the values of $f(x)$ on the boundaries of the intervals.

The mean value, variance and mean square deviation of the random variable X, which has density (5.0.27), are

$$\mathrm{M}[X]=\frac{a+b}{2}, \quad \mathrm{Var}[X]=\frac{(b-a)^2}{12}, \quad \sigma_x=\frac{b-a}{2\sqrt{3}}, \tag{5.0.28}$$

respectively.

A uniform distribution is inherent in the measurement errors when using an instrument with large divisions, when the value obtained is rounded off to the nearest integer (or to the nearest smaller number, or nearest larger number). For instance, the error (in centimetres) when measuring the length of a pencil with a ruler with cm divisions has a uniform distribution on the interval $(-1/2, 1/2)$ if the value is rounded off to the nearest integer, and on the interval $(0, 1)$ if it is rounded down

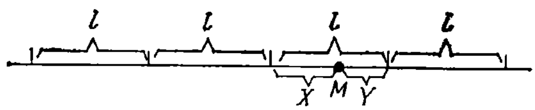


Fig. 5.0.5

to the nearest smaller value. The error (in min) made using a watch with a jumping minute hand also has a uniform distribution [the interval $(0, 1)$]. The rotation angle Φ of a well-balanced wheel (Fig. 5.0.4) has a uniform distribution on the interval $(0, 2\pi)$ if it is set in motion and stops by friction. In Buffon's needle problem (see problem 1.45) the angle which defines the direction of the needle also has a uniform distribution since the needle is thrown at random so that none of the values of φ is more likely.

Typical conditions for the occurrence of a uniform distribution are the following: a point M is thrown at random onto the x-axis divided into equal intervals of length l (Fig. 5.0.5). Each of the random intervals X and Y into which the point M divides the interval in which it has fallen has a uniform distribution on the interval $(0, l)$.

From now on, we shall often use a briefer notation

$$f(x)=1/(b-a) \quad \text{for} \quad x \in (a, b) \tag{5.0.29}$$

for probability density rather than the more detailed notation as in (5.0.27).

2. **Exponential distribution.** A random variable X has an *exponential distribution* if its density is defined by the formula

$$f(x)=\begin{cases} \lambda e^{-\lambda x} & \text{for} \quad x>0, \\ 0 & \text{for} \quad x<0, \end{cases} \tag{5.0.30}$$

where λ is the parameter of the exponential distribution (Fig. 5.0.6). An exponential distribution is especially significant in the theory of Markov processes and the queueing theory (see Chapters 10 and 11).

If there is a simple flow with intensity λ on the time axis Ot (see Chapter 4), then the time interval T between two neighbouring events has an exponential distribution with parameter λ.

The mean value, variance and mean square deviation of the variable X which has an exponential distribution are

$$\mathrm{M}[X]=1/\lambda, \quad \mathrm{Var}[X]=1/\lambda^2, \quad \sigma_x=1/\lambda \tag{5.0.31}$$

respectively. The coefficient of variation of an exponential distribution is unity:

$$v_x=\sigma_x/m_x=1.$$

We shall often replace the more detailed notation for the exponential distribution (5.0.30) by a briefer one, i.e.

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad x > 0$$

or the even briefer notation

$$f(x) = \lambda e^{-\lambda x} \quad (x > 0).$$

A table of the function e^{-x} is given in Appendix 3.

3. **Normal distribution.** A random variable X has a *normal distribution* (or has a Gaussian probability distribution) if its density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \tag{5.0.33}$$

(Fig. 5.0.7). The mean value of a random variable which has this distribution is m, the variance is σ^2, and the mean square deviation is σ. The probability that a

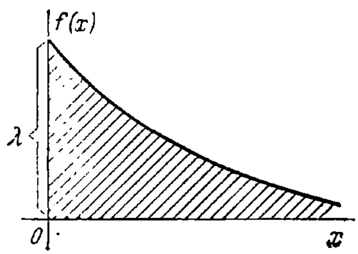


Fig. 5.0.6

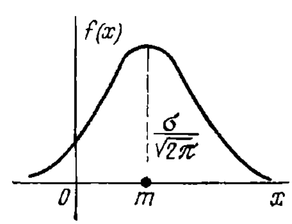


Fig. 5.0.7

random variable X, which has a normal distribution with parameters m and σ, will fall on the interval from α to β is expressed by the formula

$$P\{X \in (\alpha, \beta)\} = \Phi\left(\frac{\beta - m}{\sigma}\right) - \Phi\left(\frac{\alpha - m}{\sigma}\right), \tag{5.0.34}$$

where $\Phi(x)$ is the error function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt \tag{5.0.35}$$

The error function has the following properties: (1) $\Phi(0) = 0$; (2) $\Phi(-x) = -\Phi(x)$ (an odd function); (3) $\Phi(\infty) = 0.5$. The tables of the error function are given in Appendix 5.

If the interval (α, β) is symmetric about a point m, then the probability of falling in it

$$P\{|X - m| < l\} = 2\Phi(l/\sigma), \tag{5.0.36}$$

where $l = (\beta - \alpha)/2$, i.e. half the length of the interval.

Normal distribution arises when the variable X results from the summation of a large number of independent (or weakly dependent) random variables which are comparable as concerns their effect on the scattering of the sum (for more detail see Chap. 8). Tables of the normal density for $m = 0$ and $\sigma = 1$ are given in Appendix 4.

Problems and Exercises

5.1. We are given an arbitrary value of an argument. (1) Can the distribution function be greater than unity? (2) Can the probability density function be greater than unity? (3) Can the distribution function be negative? (4) Can the probability density function be negative?

Answer. (1) No, (2) yes, (3) no, (4) no.

5.2. What is the dimensionality of (1) the distribution function; (2) the probability density function; (3) the mean value; (4) the variance; (5) the mean square deviation; (6) the third moment about the origin?

Answer. (1) Dimensionless, (2) the inverse of the dimension of the random variable, (3) the dimension of the random variable, (4) the dimension of the square of the random variable, (5) the dimension of the random variable, (6) the dimension of the cube of the random variable.

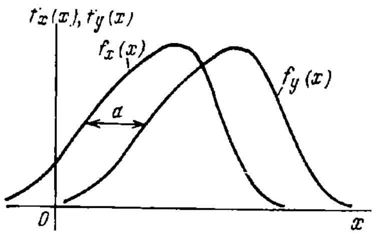


Fig. 5.3

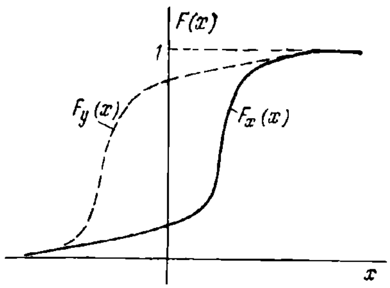


Fig. 5.4

5.3. Given the graph of the probability density function $f_x(x)$ of a random variable X (Fig. 5.3), construct the probability density function $f_y(x)$ of a random variable $Y = X + a$, where a is a nonrandom variable. Write the expression for $f_y(x)$.

Solution. The distribution curve of the random variable Y is the same as the curve of distribution of $f_x(x)$ but shifted to the right by a, i.e., $f_y(x) = f_x(x - a)$.

5.4. Given the graph of the distribution function $F_x(x)$ of a random variable X (Fig. 5.4), construct the distribution function $F_y(x)$ of the random variable $Y = -X$.

Solution. $F_y(x) = \mathrm{P}\{Y < x\} = \mathrm{P}\{-X < x\} = \mathrm{P}\{X > -x\} = 1 - \mathrm{P}\{X < -x\} = 1 - F_x(-x)$.

To construct the graph of the function $F_y(-x)$, we must consider the mirror reflection of the curve of $F_x(x)$ about the axis of ordinates and subtract each ordinate from unity (see the dash line in Fig. 5.4).

5.5. Construct the distribution function $F(x)$ for a random variable which has a uniform distribution on the interval (a, b).

Solution.

$$F(x)=\int_{-\infty}^{x} f(x)\,dx,$$

$$f(x)=\begin{cases} 0 & \text{for } x<a, \\ 1/(b-a) & \text{for } a<x<b, \\ 0 & \text{for } x>b, \end{cases}$$

$$F(x)=\begin{cases} 0 & \text{for } x\leqslant a, \\ (x-a)/(b-a) & \text{for } a<x\leqslant b, \\ 1 & \text{for } x>b \end{cases}$$

(Fig. 5.5).

5.6. The random variable X is distributed according to the "right triangle law" in the interval $(0, a)$ (see Fig. 5.6). (1) Write the expression

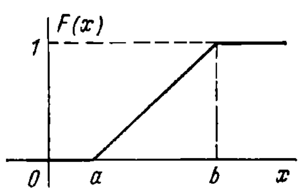


Fig. 55.

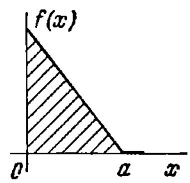


Fig. 5.6

for the probability density function $f(x)$; (2) find the distribution function $F(x)$; (3) find the probability that the random variable X will fall on the interval from $a/2$ to a; (4) find the characteristics of the variable X: m_x, Var_x, σ_x, $\mu_3[X]$.

Answer.

$$(1)\quad f(x)=\begin{cases} 2(1-x/a)/a & \text{for } x\in(0, a), \\ 0 & \text{for } x\notin(0, a), \end{cases}$$

or more concisely $f(x)=2(1-x/a)/a$ for $x\in(0, a)$,

$$(2)\quad F(x)=\begin{cases} 0 & \text{for } x\leqslant 0, \\ x(2-x/a)/a & \text{for } 0<x\leqslant a, \\ 1 & \text{for } x>a, \end{cases}$$

$$(3)\quad \mathrm{P}\{X\in(a/2, a)\}=F(a)-F(a/2)=1/4,$$

(4) $m_x=a/3$, $\mathrm{Var}_x=a^2/18$, $\sigma_x=a/(3\sqrt{2})$, by formula (4.0.15)

$$\mu_3[X]=\alpha_3[X]-3m_x\alpha_2[X]+2m_x^3=a^3/135.$$

5.7. A random variable X has a **Simpson distribution** (obeys the "law of an isosceles triangle") on the interval from $-a$ to a (see Fig. 5.7a). (1) Find the expression for the probability density function;

(2) construct a graph of the distribution function; (3) find m_x, Var_x, σ_x, $\mu_3[X]$; (4) find the probability that the random variable X will fall in the interval $(-a/2, a)$.

Answer. (1)

$$f(x)=\begin{cases}\frac{1}{a}\left(1-\frac{x}{a}\right) & \text{for} \quad 0<x<a,\\ \frac{1}{a}\left(1+\frac{x}{a}\right) & \text{for} \quad -a<x<0,\\ 0 & \text{for} \quad x<-a \text{ or } x>a,\end{cases}$$

or, more concisely,

$$f(x)=\frac{1}{a}\left(1-\frac{|x|}{a}\right) \quad \text{for} \quad x\in(-a, a),$$

(2) for $x \in (-a, a)$ the graph of the distribution function is formed by two parts of the parabola (Fig. 5.7*b*);

(3) $m_x = 0$, $\mathrm{Var}_x = a^2/6$, $\sigma_x = a/\sqrt{6}$, $\mu_3[X] = 0.$,

(4) $\mathrm{P}\{X \in (-a/2, a) = 7/8$.

5.8. The random variable X has a **Cauchy distribution**, i.e. $f(x) = a/\pi(1+x^2)$. (1) Find the factor a; (2) find the distribution function

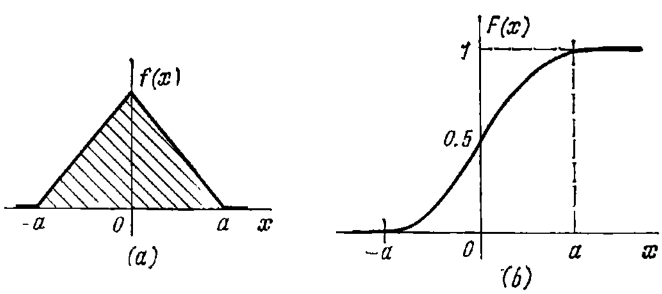


Fig. 5.7

$F(x)$; (3) find the probability that the random variable X will fall on the interval $(-1, +1)$; (4) find whether the numerical characteristics, i.e. mean value and variance, exist for the random variable X.

Answer. (1) $a = \frac{1}{\pi}$; (2) $F(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}$; (3) $\mathrm{P}\{-1 < X < 1\} = 1/2$; (4) the characteristics m_x and Var_x do not exist since the integrals for them diverge.

5.9. A random variable X has an exponential distribution with parameter μ, i.e. $f(x) = \mu e^{-\mu x}$ for $x > 0$. (1) Construct the distribution curve; (2) find the distribution function $F(x)$ and construct its graph; (3) find the probability that the random variable X will assume a value smaller than the mean.

Answer. (1) See Fig. 5.9*a*; (2) $F(x)=\begin{cases} 0 & \text{for} \quad x\leqslant 0, \\ 1-e^{-\mu x} & \text{for} \quad x>0 \end{cases}$ (see Fig. 5.9*b*);
(3) $m_x = 1/\mu$; $P\{X < 1/\mu\} = F(1/\mu) = 1 - e^{-1} \approx 0.632$.

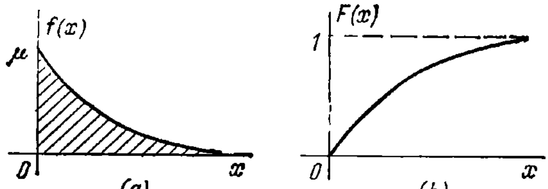


Fig. 5.9

5.10. A random variable X has a **Laplace distribution**, i.e. $f(x) = ae^{-\lambda|x|}$, where λ is a positive parameter. (1) Find the factor a; (2) construct graphs for the density and the distribution function; (3) find m_x and Var_x.

Answer.

$$(1)\ a=\lambda/2, \quad (2)\ F(x)=\begin{cases} \frac{1}{2}e^{\lambda x} & \text{for} \quad x\leqslant 0, \\ 1-\frac{1}{2}e^{-\lambda x} & \text{for} \quad x>0. \end{cases}$$

The graphs of the density and the distribution function are given in Fig. 5.10*a*, *b*;
(3) $m_x = 0$; $\mathrm{Var}_x = 2/\lambda^2$.

5.11. A random variable R, which is the distance from a bullet hole to the centre of the target, has a **Rayleigh distribution**, i.e. $f(r) = Are^{-h^2r^2}$ for $r > 0$ (Fig. 5.11).

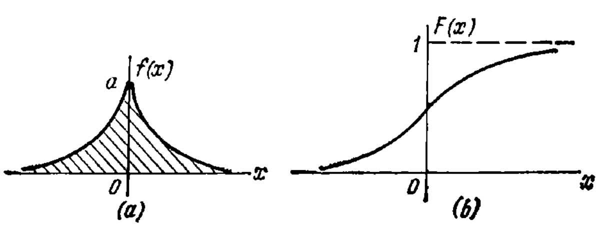


Fig. 5.10

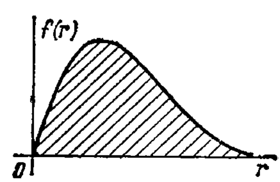


Fig. 5.11

Find (1) the factor A; (2) the mode $\mathcal{M}$ of the random variable R, i.e. the abscissa of the maximum of its probability density function; (3) m_r and Var_r; (4) the probability that the distance from the bullet hole to the centre of the target will be less than the mode.

Answer. (1) $A=2h^2$; (2) $\mathcal{M}=1/(h\sqrt{2})$; (3) $m_r=\mathcal{M}\sqrt{\pi/2}=\sqrt{\pi}/(2h)$; $\mathrm{Var}_r=(4-\pi)/(4h^2)=\mathcal{M}^2(4-\pi)/2$; (4) $P\{R<\mathcal{M}\}\approx 0.393$.

5.12. A random variable X has a density $f_1(x)$ with probability p_1 or a density $f_2(x)$ with probability p_2 $(p_1 + p_2 = 1)$. Write expressions for the density and distribution function of the variable X. Find its mean value and variance.

Solution. By the total probability formula with hypotheses $H_1 =$ {the variable X has a density $f_1(x)$} and $H_2 =$ {the variance X has a density $f_2(x)$}, we have

$$F(x) = \mathrm{P}\{X < x\} = p_1 F_1(x) + p_2 F_2(x),$$

where $F_1(x) = \int_{-\infty}^{x} f_1(x)\,dx$; $F_2(x) = \int_{-\infty}^{x} f_2(x)\,dx$;

$$f(x) = F'(x) = p_1 f_1(x) + p_2 f_2(x).$$

By the formula for the complete expectation

$$m_x = p_1 \int_{-\infty}^{\infty} x f_1(x)\,dx + p_2 \int_{-\infty}^{\infty} x f_2(x)\,dx = p_1 m_{x_1} + p_2 m_{x_2},$$

where m_{x_1} and m_{x_2} are the expectations for the distributions $f_1(x)$ and $f_2(x)$.

We find the variance from the second moment about the origin:

$$\mathrm{Var}_x = p_1 \alpha_{21} + p_2 \alpha_{22} - m_x^2 = p_1 \int_{-\infty}^{\infty} x^2 f_1(x)\,dx + p_2 \int_{-\infty}^{\infty} x^2 f_2(x)\,dx - m_x^2,$$

where α_{21} and α_{22} are the second moments about the origin for the distributions $f_1(x)$ and $f_2(x)$.

5.13. Balls for the ball-bearings are sorted by passing them over two holes. If a ball does not pass through a hole of diameter d_1 but passes through a hole of diameter $d_2 > d_1$, then it is accepted, otherwise it is rejected. The diameter D of a ball is a normally distributed random variable with characteristics $m_d = (d_1 + d_2)/2$ and $\sigma_d = (d_2 - d_1)/4$. Find the probability p that the ball will be rejected.

Solution. The interval (d_1, d_2) is symmetric about m_d. Setting $l = (d_2 - d_1)/2$ and using formula (5.0.36), we find the probability that the ball will not be rejected:

$$\mathrm{P}\{|D - m_d| < (d_2 - d_1)/2\} = 2\Phi\left(\frac{d_2 - d_1}{2\sigma_d}\right),$$

whence

$$p = 1 - 2\Phi\left(\frac{d_2 - d_1}{2\sigma_d}\right) = 1 - 2\Phi\left(\frac{2(d_2 - d_1)}{d_2 - d_1}\right) = 1 - 2\Phi(2) = 0.0455.$$

5.14. Under the conditions of the preceding problem, find the mean square deviation σ_d of the diameter of the ball given that 10 per cent of all the balls are rejected.

Solution. The probability of a reject

$$p=1-2\Phi\left(\frac{d_2-d_1}{2\sigma_d}\right)=0.1, \quad \Phi\left(\frac{d_2-d_1}{2\sigma_d}\right)=0.45.$$

Using the tables for the error function (Appendix 5), we find the argument for which the error function equals 0.45:

$$(d_2-d_1)/(2\sigma_d) \approx 1.65, \quad \sigma_d \approx (d_2-d_1)/3.3.$$

5.15. When a computer is operating, faults may occur at random moments. The time T a computer operates until the first fault occurs has an exponential distribution with parameter ν : $\varphi(t) = \nu e^{-\nu t} (t > 0)$. When a fault occurs, it is detected at once and the necessary corrections are made within a time t_0 after which the computer begins to operate again. Find the density $f(t)$ and the distribution function $F(t)$ of the time interval Z between successive troubles. Find the mean value and variance. Find the probability that Z is larger than $2t_0$.

Solution. $Z=T+t_0$;

$$f(t)=\begin{cases} \nu e^{-\nu(t-t_0)} & \text{for} \quad t>t_0, \\ 0 & \text{for} \quad t<t_0, \end{cases} \qquad F(t)=\begin{cases} 1-e^{-\nu(t-t_0)} & \text{for} \quad t>t_0, \\ 0 & \text{for} \quad t<t_0, \end{cases}$$

$$\mathrm{M}[Z]=1/\nu+t_0, \quad \mathrm{Var}[Z]=1/\nu^2, \quad \mathrm{P}\{Z>2t_0\}=1-F(2t_0)=e^{-\nu t_0}.$$

5.16. The time T between two computer malfunctions has an exponential distribution with parameter λ, i.e. $f(t) = \lambda e^{-\lambda t}$ for $t > 0$. A certain problem requires a failure-free computer run for a time τ. If a malfunction occurs during the time τ, the solution of the problem must be restarted. A malfunction is detected only in a time τ after the solution has begun. We consider a random variable Θ which is the time during which the problem will be solved. Find its distribution and expectation (the average time needed to solve the problem).

Answer. The random variable Θ is discrete and has an ordered series

$$\Theta: \left|\begin{array}{c|c|c|c|c} \tau & 2\tau & \ldots & i\tau & \ldots \\ \hline p & pq & \ldots & pq^{i-1} & \ldots \end{array}\right.,$$

where $p=e^{-\lambda\tau}$, $q=1-p=1-e^{-\lambda\tau}$; $\mathrm{M}[\Theta]=\tau/p=\tau e^{-\lambda\tau}$.

5.17. Under the conditions of the preceding problem, find the probability that no less than m problems $(m < k)$ will be solved during time $t = k\tau$.

Solution. We designate as $P_{m,k}$ the probability that exactly m problems will be solved during the time $t = k\tau$. $P_{m,k}$ is the probability that exactly m of the k time intervals τ will be without malfunctions. The probability that there will be no malfunctions during the time interval τ is $p = \mathrm{P}(T > \tau) = e^{-\lambda\tau}$. According to the theorem on the repetition of trials

$$P_{m,k}=C_k^m p^m q^{k-m}=C_k^m e^{-m\lambda\tau}(1-e^{-\lambda\tau})^{k-m}.$$

The probability that no less than m problems will be solved is

$$R_{m,k}=\sum_{i=m}^{k} P_{i,k}=\sum_{i=m}^{k} C_k^i e^{-i\lambda\tau}(1-e^{-\lambda\tau})^{k-i}$$

or more conveniently,

$$R_{m,k}=1-\sum_{i=0}^{m-1} C_k^i e^{-i\lambda\tau}(1-e^{-\lambda\tau})^{k-i}.$$

5.18. Prove that the distribution of the intervals between the successive events in a simple flow with intensity λ is exponential with parameter λ.

Solution. We first find the distribution function $F(t)$ of the random variable T, which is the distance between the successive events:

$F(t) = \mathrm{P}\{T < t\} = \mathrm{P}$ {at least one event in the elementary flow occurs during the time t} $= 1 - e^{-\lambda t}$ for $t > 0$; hence $f(t) = F'(t) = \lambda e^{-\lambda t}$ for $t > 0$.

5.19. Given a Poisson field of points on a plane with a constant density λ, find the distribution and the numerical characteristics m_r, Var_r of the distance R from any point in the field to its nearest neighbour.

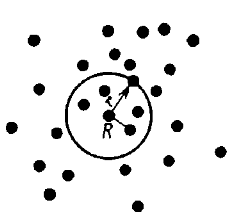


Fig. 5.19

Solution. We find the distribution function $F(r)$ of the variable R by drawing a circle of radius r about a point in the field (Fig. 5.19). For the distance R from the point to its nearest neighbour to be smaller than r, at least one other point must fall within the circle. The properties of the Poisson fields are such that the probability of this event is independent of whether there is a point in the centre of the field or not. Therefore,

$$F(r)=1-e^{-\pi r^2\lambda} \quad (r>0),$$

whence

$$f(r)=2\pi\lambda r e^{-\pi\lambda r^2} \quad (r>0).$$

This distribution is known as a **Rayleigh distribution** (see Problem 5.11

$$m_r=\int_0^\infty r 2\pi\lambda r e^{-\pi\lambda r^2}\,dr=\frac{1}{2\sqrt{\lambda}}.$$

$$\alpha_2[R]=\int_0^\infty r^2 2\pi\lambda r e^{-\pi\lambda r^2}\,dr=\frac{1}{\pi\lambda},$$

$$\mathrm{Var}_r=\alpha_2[R]-m_r^2=1/(\pi\lambda)-1/(4\lambda)=(4-\pi)/(4\pi\lambda).$$

5.20. Points are located at random in a three-dimensional space. The number of points in a certain volume v of the space is a random variable which has a Poisson distribution with expectation $a = \lambda v$,

where λ is an average number of points per unit volume. We have to find the distribution of the distance R from any point in the space to the nearest random point.

Solution. The distribution function $F(r)$ is defined as the probability that at least one point falls in a sphere of radius r:

$$F(r) = \mathrm{P}\{R < r\} = 1 - \mathrm{e}^{-\lambda v(r)},$$

where $v(r) = \frac{4}{3}\pi r^3$ is the volume of the sphere of radius r. Hence

$$f(r) = 4\pi r^2 \lambda \mathrm{e}^{-\lambda \frac{4}{3}\pi r^3} \quad (r > 0).$$

5.21. The stars in a certain cluster form a three-dimensional Poisson field of points with density λ (the average number of stars per unit volume). An arbitrary star is fixed and the nearest neighbour, the next (second) nearest neighbour, the third nearest neighbour, and so on, are considered. Find the distribution of the distance R_n from the fixed star to its nth nearest neighbour.

Answer. The distribution function $F_n(r)$ has the form

$$F_n(r) = 1 - \sum_{k=0}^{n-1} \frac{a^k}{k!} \mathrm{e}^{-a}, \quad \text{where} \quad a = \frac{4}{3}\pi r^3 \lambda \quad (r > 0),$$

the density function is

$$f_n(r) = \frac{dF_n(r)}{dr} = \frac{a^{n-1}}{(n-1)!} \mathrm{e}^{-a} 4\pi\lambda r^2 \quad (r > 0).$$

5.22. Trees in a forest grow at random places which form a Poisson field with density λ (the average number of trees per unit area). An arbitrary point O is chosen in the forest and the following random variables are considered:

R_1, the distance from O to the nearest tree;

R_2, the distance from O to the next (second nearest) tree;

. . . .

R_n, the distance from O to the nth nearest tree.

Find the distribution for each variable.

Solution. We found the distribution function of the random variable in Problem 5.19 to be

$$F_1(r) = 1 - \mathrm{e}^{-\pi r^2 \lambda} \quad (r > 0).$$

The distribution function $F_2(r) = \mathrm{P}\{R_2 < r\}$ is equal to the probability that no less than two trees fall in a circle of radius r:

$$F_2(r) = 1 - \mathrm{e}^{-\pi r^2 \lambda} - \pi r^2 \lambda \mathrm{e}^{-\pi r^2 \lambda} \quad (r > 0).$$

Reasoning by analogy, we obtain

$$F_n(r) = \mathrm{P}\{R_n < r\} = 1 - \sum_{k=0}^{n-1} \frac{a^k}{k!} \mathrm{e}^{-a} \quad (r > 0),$$

where $a = \pi r^2 \lambda$.

To get the density $f_n(r)$, we differentiate $F_n(r)$ with respect to r, i.e.

$$f_n(r) = \frac{dF_n(r)}{da}\frac{da}{dr} = \left(-\sum_{k=0}^{n-1} k\frac{a^{k-1}}{k!}e^{-a} + \sum_{k=0}^{n-1}\frac{a^k}{k!}e^{-a}\right)2\pi\lambda r$$

$$= \frac{a^{n-1}}{(n-1)!}e^{-a}2\pi\lambda r \quad (r > 0).$$

5.23. An automatic traffic signal at a crossing is set so that a green light turns on for a minute and then a red light turns on for 0.5 min, then a green light turns on again for a minute and a red light for half a minute, and so on. A certain Petrov arrives at the crossing in a car at a random moment unrelated to the operation of the traffic signal. (1) Find the probability that he will pass through the crossing without

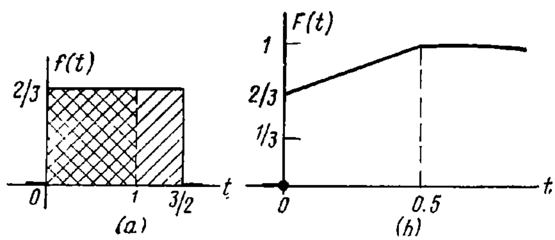


Fig. 5.23

stopping; (2) find the distribution and the numerical characteristics of the waiting time at the crossing T_w; (3) construct the distribution function $F(t)$ of the waiting time T_w.

Solution. The moment the car arrives at the crossing is uniformly distributed in the interval which is equal to the traffic signal cycle, i.e. to $1 + 0.5 = 1.5$ min (Fig. 5.23*a*).

For the car to pass through the crossing without stopping, it must arrive during the interval (0, 1). The probability that a random variable uniformly distributed in the interval (0, 1.5), will fall in the interval (0, 1) is 2/3. The waiting time T_w is a mixed random variable. It is equal to zero with probability 2/3 or may assume any value between 0 and 0.5 min with uniform density with probability 1/3. The graph of the distribution function $F(t)$ is shown in Fig. 5.23*b*.

The average waiting time at the crossing

$$\mathrm{M}[T_w] = 0\cdot 2/3 + 0.25\cdot 1/3 \approx 0.083 \text{ min}.$$

The variance of the waiting time

$$\mathrm{Var}[T_w] = \alpha_2[T_w] - (\mathrm{M}[T_w])^2 = 0^2\cdot\frac{2}{3} + \frac{1}{3}\int_0^{0.5} t^2\frac{1}{0.5}\,dt$$

$$-(0.083)^2 \approx 0.0208\,\text{min}^2;\quad \sigma_{t_w} \approx 0.144\text{ min}.$$

5.24. *The normal distribution function.* A random variable X has a normal distribution with parameters m and σ. Find its distribution function $F(x)$.

Solution.

$$F(x) = \mathrm{P}\{X < x\} = \mathrm{P}\{-\infty < X < x\}$$
$$= \Phi\left(\frac{x-m}{\sigma}\right) - \Phi(-\infty) = \Phi\left(\frac{x-m}{\sigma}\right) + 0.5.$$

5.25*. Show that a function of the form

$$f_s(x) = ax^s e^{-\alpha^2 x^2} \quad \text{for} \quad x > 0$$

[$\alpha > 0$ and $a > 0$ are constants and s is a natural number ($s = 1, 2, 3, \ldots$)] possesses the properties of a probability density function. Find the parameters a and α given a mean value m_x, and find Var_x.

Solution. The parameters a and α can be found from the conditions

$$\int_0^\infty ax^s e^{-(\alpha x)^2}\,dx = 1, \qquad \int_0^\infty ax^{s+1} e^{-(\alpha x)^2}\,dx = m_x.$$

Using the substitution $(\alpha x)^2 = t$, we reduce the integrals to the gamma function

$$\int_0^\infty x^s e^{-(\alpha x)^2}\,dx = \frac{1}{2\alpha^{s+1}} \int_0^\infty e^{-t} t^{\frac{s-1}{2}}\,dt = \Gamma\left(\frac{s+1}{2}\right)\Big/(2\alpha^{s+1}),$$

where $\Gamma(m) = \int_0^\infty e^{-t} t^{m-1}\,dt$ $(m > 0)$ with $\Gamma(m+1) = m\Gamma(m)$; for the integral $n = 1, 2, \ldots$ we obtain $\Gamma(n+1) = n!$, $\Gamma(n + 1/2) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$, $(2n-1)!! = 1, 3, 5, \ldots, (2n-1)$.

From the given conditions we find that

$$a\Gamma\left(\frac{s+1}{2}\right)\Big/(2\alpha^{s+1}) = 1, \quad a\Gamma\left(\frac{s+2}{2}\right)\Big/(2\alpha^{s+2}) = m_x,$$

whence

$$a = \frac{2\alpha^{s+1}}{\Gamma\left(\frac{s+1}{2}\right)}, \quad \alpha = \frac{1}{m_x}\,\frac{\Gamma\left(\frac{s+2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}.$$

The second moment about the origin

$$\alpha_2[X] = \int_0^\infty ax^{s+2} e^{-(\alpha x)^2}\,dx = a\,\frac{\Gamma\left(\frac{s+3}{2}\right)}{2\alpha^{s+3}} = a\,\frac{\frac{s+1}{2}\,\Gamma\left(\frac{s+1}{2}\right)}{2\alpha^{s+1}\alpha^2} = \frac{s+1}{2\alpha^2},$$

whence

$$\mathrm{Var}_x = \alpha_2[X] - m_x^2 = \frac{s+1}{2\alpha^2} - m_x^2 = m_x^2\left\{\frac{(s+1)\,\Gamma^2\left(\frac{s+1}{2}\right)}{2\Gamma^2\left(\frac{s+2}{2}\right)} - 1\right\}.$$

Some of the distributions of the form $f_s(x)$ have definite names, e.g. $f_1(x)$ is known as a **Rayleigh distribution**, and $f_2(x)$, as a **Maxwell distribution**. For a Rayleigh distribution ($s = 1$) we have $f_1(x) = ax\,e^{-\alpha^2x^2}$ ($x > 0$)

$$\alpha = \frac{1}{m_x}\frac{\sqrt{\pi}}{2}, \quad a = 2\alpha^2 = \frac{\pi}{2m_x^2}, \quad \mathrm{Var}_x = m_x^2\left[\frac{4}{\pi} - 1\right].$$

For a Maxwell distribution ($s = 2$) we have $f_2(x) = ax^2e^{-\alpha^2x^2}$

$$\alpha = \frac{2}{m_x\sqrt{\pi}}, \quad a = \frac{4\alpha^3}{\sqrt{\pi}} = \frac{32}{\pi^2m_x^3}, \quad \mathrm{Var}_x = m_x^2\left(\frac{3\pi}{8} - 1\right).$$

Remark. All the distributions of the form

$$f_s(x) = ax^se^{-\alpha^2x^2} \quad (x > 0)$$

for a given s have a single parameter, i.e. they depend only on one parameter, which may be either the mean value or the variance.

5.26*. *Moments of a normal distribution.* Given a random variable X which has a normal distribution with parameters m and σ, find the expression for the quantity $\alpha_s[X]$, which is the sth moment about the origin.

Solution. Let us express the moments $\alpha_s[X] = \mathrm{M}[X^s]$ about the origin in terms of the central moments $\mu_s[X] = \mathrm{M}[(X - m)^s]$:

$$\alpha_s = \mathrm{M}[(X - m + m)^s] = \sum_{k=0}^{s} C_s^k\mu_k[X]\,m^{s-k}, \quad \mu_0[X] = 1.$$

For central moments with odd $s = 2n + 1$, we have

$$\mu_s[X] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x - m)^s e^{-\frac{1}{2}\frac{(x-m)^2}{\sigma^2}}dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} y^s e^{-\frac{1}{2\sigma^2}y^2}dy = 0.$$

and with even $s = 2n$, by the formulas in the preceding problem, we have

$$\mu_s[X] = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} y^s e^{-\frac{1}{2\sigma^2}y^2}dy = \frac{2}{\sigma\sqrt{2\pi}}\int_{0}^{\infty} y^s e^{-\left(\frac{y}{\sigma\sqrt{2}}\right)^2}dy$$

$$= \frac{2}{\sigma\sqrt{2\pi}}\,\frac{\Gamma\left(\frac{2n+1}{2}\right)}{2}\,(\sigma\sqrt{2})^{2n+1} = (2n - 1)!!\,\sigma^{2n}.$$

For example,

$$\mu_2[X] = \sigma^2, \qquad \alpha_2[X] = m^2 + \sigma^2,$$
$$\alpha_3[X] = m^3 + 3\sigma^2m,$$

$$\mu_4[X] = 3\sigma^4, \qquad \alpha_4[X] = m^4 + 6\sigma^2 m^2 + 3\sigma^4,$$
$$\alpha_5[X] = m^5 + 10\sigma^2 m^3 + 5\cdot 3\sigma^4 m,$$
$$\mu_6[X] = 15\sigma^6, \qquad \alpha_6[X] = m^6 + 15\sigma^2 m^4 + 15\cdot 3\sigma^4 m^2 + 15\sigma^6.$$

5.27. A random variable X has a normal distribution with parameters m and σ. Write an expression for its distribution function $F(x) = \mathrm{P}\{X < x\}$. Write an expression for the distribution function $\Psi(y) = \mathrm{P}\{Y < y\}$ of the random variable $Y = -X$.

Solution. In accordance with the solution of Problem 5.24

$$F(x) = \Phi\left(\frac{x-m}{\sigma}\right) + 0.5,$$
$$\Psi(y) = \mathrm{P}\{Y < y) = \mathrm{P}\{-X < y\}$$
$$= \mathrm{P}(X > -y) = 1 - F(-y)$$
$$= \Phi\left(\frac{y+m}{\sigma}\right) + 0.5.$$

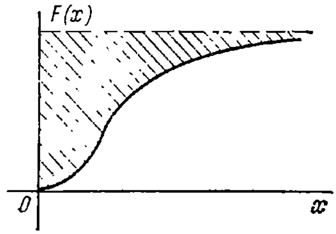


Fig. 5.28

5.28*. The distribution function $F(x)$ of a nonnegative random variable X is given by a graph (Fig. 5.28). The mean value of the random variable X is m_x. Show that m_x can be represented geometrically as the area of the hatched figure in Fig. 5.28 (bounded by the curve $y = F(x)$, the straight line $y = 1$ and the axis of ordinates).

Solution. We have

$$m_x = \int_0^\infty xf(x)\,dx = \int_0^\infty xF'(x)\,dx = -\int_0^\infty x[1-F(x)]'\,dx.$$

Integrating by parts, we get

$$m_x = -x[1-F(x)]\Big|_0^\infty + \int_0^\infty [1-F(x)]\,dx.$$

Let us prove that the first term is zero:

$$x[1-F(x)]\Big|_0^\infty = \lim_{x\to\infty} x[1-F(x)] = 0.$$

Indeed, for the nonnegative random variable X, which has a finite mean value, it follows from the convergence of the integral $\int_0^\infty xf(x)\,dx$ that $\int_K^\infty xf(x)\,dx \to 0$ as $K \to \infty$ and, since

$$K\int_K^\infty f(x)\,dx \leqslant \int_K^\infty xf(x)\,dx$$

it follows that $K[1-F(K)] \to 0$ as $K \to \infty$. Consequently, $\lim_{x\to\infty} x[1-F(x)] = 0$. Hence

$$m_x = \int_0^\infty [1-F(x)]\,dx,$$

and this is precisely the area hatched in Fig. 5.28.

5.29. A random variable X has a normal distribution with mean value $m = 0$ (Fig. 5.29). Given an interval (α, β) which does not include the origin, for what value of the mean square deviation σ does the probability that the random variable X will fall in the interval (α, β) attain a maximum?

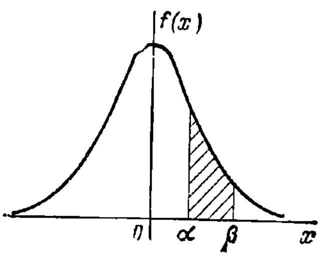


Fig. 5.29

Solution. We find the value of σ by differentiating the probability of falling in the interval (α, β) with respect to σ and equating the derivative to zero. We have

$$P\{\alpha < X < \beta\} = \Phi\left(\frac{\beta}{\sigma}\right) - \Phi\left(\frac{\alpha}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}\left\{\int_0^{\frac{\beta}{\sigma}} e^{-\frac{t^2}{2}}dt - \int_0^{\frac{\alpha}{\sigma}} e^{-\frac{t^2}{2}}dt\right\},$$

$$\frac{d}{d\sigma}[P\{\alpha < X < \beta\}] = \frac{1}{\sqrt{2\pi}}\left\{e^{-\frac{\beta^2}{2\sigma^2}}\left(-\frac{\beta}{\sigma^2}\right) - e^{-\frac{\alpha^2}{2\sigma^2}}\left(-\frac{\alpha}{\sigma^2}\right)\right\} = 0,$$

hence,

$$\beta e^{-\frac{\beta^2}{2\sigma^2}} = \alpha e^{-\frac{\alpha^2}{2\sigma^2}}$$

and, consequently,

$$\sigma = \sqrt{\frac{\beta^2-\alpha^2}{2(\ln\beta - \ln\alpha)}} = \sqrt{\frac{\beta+\alpha}{2}\,\frac{\beta-\alpha}{\ln\beta-\ln\alpha}}.$$

For a small interval $(a-\varepsilon, a+\varepsilon)$

$$\sigma \approx a[1-(\varepsilon/a)^2/6] \approx a.$$

For instance, for $\varepsilon/a < 0.24$ the formula $\sigma \approx a$ yields an error, smaller than one per cent.

5.30. A random variable X has a normal distribution with expectation m and mean square deviation σ. We must approximate the normal distribution by a uniform distribution in the interval (α, β) with the boundaries α and β chosen such that the mean value and variance of X are retained constant.

Solution. For a uniform distribution on the interval (α, β)

$$M[X] = (\alpha+\beta)/2, \quad \sigma[X] = (\beta-\alpha)/(2\sqrt{3}),$$

$$(\alpha+\beta)/2 = m, \quad (\beta-\alpha)/2\sqrt{3}) = \sigma.$$

Solving these equations for α and β, we have

$$\alpha = m - \sigma\sqrt{3}, \quad \beta = m + \sigma\sqrt{3}.$$

5.31. A continuous random variable X has a probability density function $f(x)$. As a result of an experiment it is found that an event

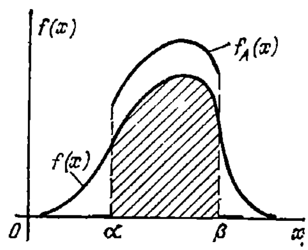


Fig. 5.31

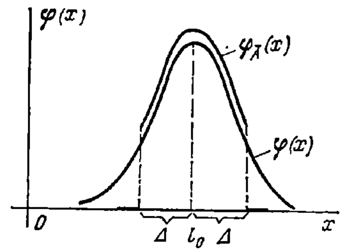


Fig. 5.32

$A = \{X \in (\alpha, \beta)\}$ [the random variable X falls on the interval (α, β)] occurred. Find the conditional density $f_A(x)$ of the random variable X provided that the event A occurs.

Solution. $P(A) = \int_\alpha^\beta f(x)\,dx$. By Bayes's formula (5.0.24) we have

$$f_A(x) = \begin{cases} f(x) \Big/ \left\{ \int_\alpha^\beta f(x)\,dx \right\} & \text{for} \quad x \in (\alpha, \beta), \\ 0 & \text{for} \quad x \notin (\alpha, \beta). \end{cases} \tag{5.31}$$

The distribution curves $f(x)$ and $f_A(x)$ are shown in Fig. 5.31. The ordinate of the curve $f_A(x)$ at each point is equal to that of the curve $f(x)$ divided by the area S hatched in Fig. 5.31. Since $S < 1$, it follows that $f_A(x) > f(x)$ for any $x \in (\alpha, \beta)$.

5.32. A factory manufactures homogeneous articles whose rated size is l_0 but for which random size deviations are actually observed that have a normal distribution with mean value $m = 0$ and mean square deviation σ. The sorters reject every article whose size differs from the rated size by more than a tolerance Δ. Find the probability of the event $A = \{$an article will be rejected$\}$. Find and construct the probability density function for the size of the article which passes the sorting.

Solution.

$$P(A) = P\{|X| > \Delta\} = 1 - 2\Phi(\Delta/\sigma), \quad P(\bar{A}) = 2\Phi(\Delta/\sigma).$$

By formula (5.31) we have

$$f_{\bar{A}}(x) = \frac{f(x)}{P(\bar{A})} = \frac{1}{2\sigma\sqrt{2\pi}\,\Phi(\Delta/\sigma)} e^{-\frac{x^2}{2\sigma^2}} \quad \text{for} \quad |x| < \Delta.$$

The size L of an article which passed the sorting is equal to the error X plus the rated size l_0 and has the conditional density

$$\varphi_{\bar{A}}(x) = \frac{1}{2\sigma\sqrt{2\pi}\,\Phi(\Delta/\sigma)}\, e^{-\frac{(x-l_0)^2}{2\sigma^2}} \quad \text{for} \quad x \in (l_0-\Delta,\ l_0+\Delta),$$

shown in Fig. 5.32.

5.33. The reliability p of a device depends both on the total elapsed time τ from the moment the device was turned on and on whether the voltage regulator failed at some moment $t < \tau$. If the voltage regulator does not fail until the moment τ, the reliability is defined by the function $p = p_0(\tau)$, if it fails at the moment $t < \tau$, then the reliability is a function of two arguments $p = p_1(\tau, t)$. The time of the failure-free performance of the voltage regulator is a random variable T with density $f(t)$. Find the distribution function $F_\Theta(x)$ of the time Θ of the failure-free performance of the device and its expectation m_Θ.

Solution. The complete reliability of the device (the probability of its failure-free operation for the time τ) can be found from the total probability integral formula:

$$p(\tau) = \int_0^\tau p(t, \tau)\, f(t)\, dt + p_0(\tau) \int_\tau^\infty f(t)\, dt$$

The function of time distribution Θ

$$F_\Theta(x) = \mathrm{P}\{\Theta < x\} = 1 - p(x).$$

Since the quantity Θ is nonnegative, it follows (see Problem 5.28) that

$$m_\Theta = \mathrm{M}[\Theta] = \int_0^\infty [1 - F_\Theta(x)]\, dx = \int_0^\infty p(\tau)\, d\tau.$$

5.34. A message S is expected over a communication channel. The moment T the message is received is accidental and has density function $f(t)$. At a moment τ it is found that the message has not yet arrived. Under this condition find the distribution density $\varphi(t)$ of the time Θ which remains until the arrival of the message S.

Solution. According to Bayes's formula $f_A(t)$, which is the conditional density of the quantity T, provided that an event $A = \{$the message has not arrived by the moment $\tau\}$ occurred, is

$$f_A(t) = \begin{cases} \dfrac{f(t)}{\mathrm{P}(A)} = \dfrac{f(t)}{1 - \int_0^\tau f(t)\, dt} & \text{for} \quad t > \tau, \\ 0 & \text{for} \quad t < \tau. \end{cases}$$

Since $\Theta = T - \tau$, it follows that

$$\varphi(t) = \begin{cases} \dfrac{f(t)}{1 - \int_0^\tau f(t)\,dt} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

Fig. 5.34. shows $f(t)$ and $\varphi(t)$; for the distribution $\varphi(t)$ the origin coincides with the point τ.

5.35. The moment T an event A occurs is a random variable with an exponential distribution, i.e. $f(t) = \lambda e^{-\lambda t}$ $(t > 0)$. It became known

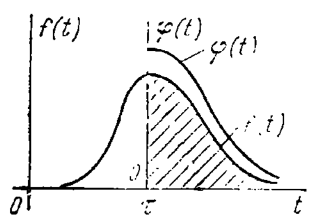


Fig. 5.34

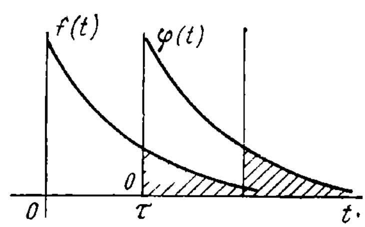


Fig. 5.35

at a moment τ that the event A had not occurred. Find the conditional distribution density $\varphi(t)$ of the time Θ remaining till the occurrence of the event.

Solution. The event B = {the event A has not occurred by the moment τ} occurred. Thus

$$P(B) = 1 - \int_0^\tau f(t)\,dt = e^{-\lambda\tau}.$$

$$f_B(t) = f(t)/P(B) = \lambda e^{-\lambda t}/e^{-\lambda\tau} = \lambda e^{-\lambda(t-\tau)} \quad (t > \tau).$$

The random variable $\Theta = T - \tau$ has the conditional density

$$\varphi(t) = \lambda e^{-\lambda\tau} \quad \text{for} \quad t > 0,$$

i.e. an exponential distribution which coincides with $f(t)$ (see Fig. 5.35). Thus, the conditional distribution of the time Θ which remains till the occurrence of the event does not depend, for an exponential distribution of T, on the elapsed time. An exponential distribution is the only distribution which possesses this property. Recall that the time interval between two successive events in an elementary flow has exactly an exponential distribution: the time remaining till the occurrence of the next event does not depend on how long we have been waiting for it (this follows from the absence of aftereffects in an elementary flow).

5.36. The probability of a failure of a radio valve at the moment it is turned on depends on the voltage V in the circuit and is equal to $q(V)$. The voltage V is random and has a normal distribution with

parameters v_0 and σ_v. Find the total probability q of a failure of the valve at the moment it is turned on.

Solution. By the integral total probability formula (5.0.23) we have

$$q=\int_{-\infty}^{\infty} q(v)\,f(v)\,dv=\frac{1}{\sqrt{2\pi}\,\sigma_v}\int_{-\infty}^{\infty} q(v)e^{-\frac{(v-v_0)^2}{2\sigma_v^2}}\,dv.$$

5.37. A random voltage V, which has a probability density $f(v)$, is passed through a voltage limiter which cuts off all voltages smaller than v_1 and larger than v_2, in the first case raising the voltage to v_1 and in the second case lowering it to v_2. Find the distribution of the random variable $\tilde{V}$, the voltage which has passed through the limiter, and determine its mean value and variance.

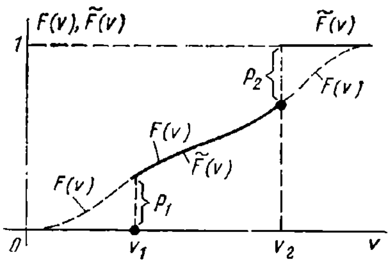


Fig. 5.37

Solution.

$$\tilde{V}=\begin{cases} v_1 & \text{for} \quad V<v_1, \\ V & \text{for} \quad v_1<V<v_2, \\ v_2 & \text{for} \quad V>v_2. \end{cases}$$

The random variable $\tilde{V}$ is a mixed quantity and its two values v_1 and v_2 have nonzero probabilities p_1 and p_2. For all the values between v_1 and v_2 the distribution function $\tilde{F}(v)$ of the variable $\tilde{V}$ is continuous and

$$F(v)=\int_{-\infty}^{v} f(v)\,dv, \quad p_1=\int_{-\infty}^{v_1} f(v)\,dv, \quad p_2=\int_{v_2}^{\infty} f(v)\,dv.$$

The graph of the function $F(v)$ is shown in Fig. 5.37.

$$\mathrm{M}[V]=v_1p_1+v_2p_2+\int_{v_1}^{v_2} vf(v)\,dv,$$

$$\mathrm{Var}[V]=\alpha_2[V]-(\mathrm{M}[V])^2,$$

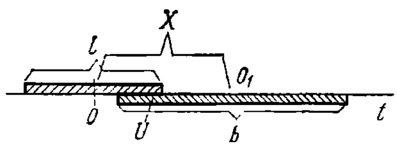
$$\alpha_2[V]=v_1^2p_1+v_2^2p_2+\int_{v_1}^{v_2} v^2f(v)\,dv.$$

5.38. A message of length l is being broadcast over a radio channel (Fig. 5.38*a*). In order to eraze the message, a noise pulse train of length $b>l$ is injected into the channel so that the centre O_1 of the train coincides with the centre O of the message.

Because of accidental errors, the centre of the pulse train proves to be displaced by X relative to the centre of the message. The random variable X has a normal distribution with parameters $m=0$ and $\sigma=l/2$. Find the distribution of the random variable U, the length of the

message which is erazed, and the mean value m_u, the variance Var_u, and the mean square deviation σ_u.

Solution. The random variable U is mixed; it assumes the values 0 and l with nonzero probabilities. On the interval from 0 to l the distribution function $F(u)$ is continuous. The noise train b does not touch



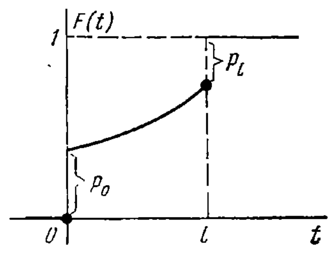


Fig. 5.38

the message l ($U = 0$) if its centre is further than $(b + l)/2$ from the origin.

$$p_0 = \mathrm{P}\{U=0\} = \mathrm{P}\left\{|X| > \frac{b+l}{2}\right\} = 1-\mathrm{P}\left\{|X| < \frac{b+l}{2}\right\}$$
$$= 1-2\Phi\left(\frac{b+l}{2}\right) = 1-2\Phi\left(\frac{b+l}{l}\right).$$

For the noise b to eraze the whole message l ($U = l$), its centre O_1 must be less than $(b - l)/2$ distant from O:

$$p_l = \mathrm{P}\{U=l\} = \mathrm{P}\left\{|X| < \frac{b-l}{2}\right\} = 2\Phi\left(\frac{b-l}{l}\right).$$

For $(b - l)/2 < X < (b + l)/2$ a part U of the message l is erazed ($0 < U < l$). We find the distribution function of the random variable U: $\mathrm{P}\{U < u\}$ for $0 < u < l$. For the erazed part to be smaller than u, the centre O_1 of the pulse train must be further than $(b + l)/2 - u = (b + l - 2u)/2$ from the centre of the message:

$$F(u) = \mathrm{P}\left\{|X| > \frac{b+l-2u}{2}\right\} = 1-\mathrm{P}\left\{|X| < \frac{b+l-2u}{2}\right\}$$
$$= 1-2\Phi\left(\frac{b+l-2u}{l}\right).$$

Thus, on the interval from 0 to l, we have $F(u) = 1-2\Phi\left(\frac{b+l-2u}{l}\right)$ (Fig. 5.38*b*).

To find the mean value and variance of the random variable U, we must find $F'(u)$ on the interval $(0, l)$. Taking into account that

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt, \quad \Phi\left(\frac{b+l-2u}{l}\right) = \frac{1}{\sqrt{2\pi}} \int_0^{\frac{b+l-2u}{l}} e^{-\frac{t^2}{2}} dt$$

and differentiating $F(u)$ with respect to the variable u, which appears in the upper limit, we obtain

$$F'(u)\,du = \frac{4}{l\sqrt{2\pi}}\, e^{-\frac{(b+l-2u)^2}{2l^2}}\, du \quad (0<u<l),$$

$$\mathrm{M}[U] = 0\cdot p_0 + l p_l + \int_0^l u F'(u)\,du,$$

$$\alpha_2[U] = 0\cdot p_0 + l^2 p_l + \int_0^l u^2 F'(u)\,du,$$

$$\mathrm{Var}[U] = \alpha_2[U] - (\mathrm{M}[U])^2, \quad \sigma_u = \sqrt{\mathrm{Var}[U]}.$$

5.39. The random variable X may have a distribution density $f_1(x)$ with probability p_1, a density $f_2(x)$ with probability p_2, . . ., a density $f_i(x)$ $(i = 1, \ldots, n)$ with probability p_i. Find the total (average) density of the random variable X.

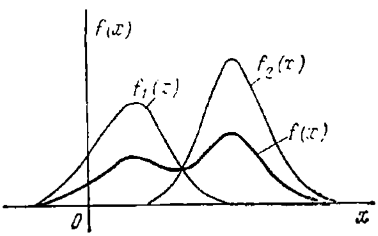


Fig. 5.39

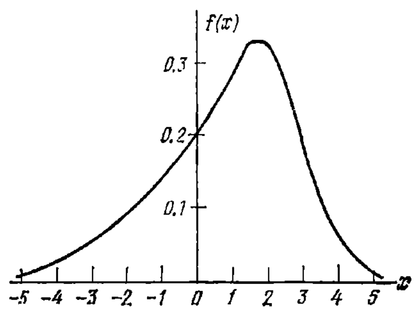


Fig. 5.40

Solution. We find the element $f(x)\,dx$ of the probability by the total probability formula on the hypotheses H_i = {the density of the random variable is $f_i(x)$} $(i = 1, \ldots, n)$. By the total probability formula

$$f(x)\,dx = \sum_{i=1}^{n} p_i f_i(x)\,dx$$

whence

$$f(x) = \sum_{i=1}^{n} p_i f_i(x)$$

In particular, if there are two hypotheses and their probabilities are $p_1 = p_2 = 1/2$, then $f(x)$ is half the sum of the densities $f_1(x)$ and $f_2(x)$ (Fig. 5.39). Two "humps" on a distribution curve always imply that the distribution has been obtained by averaging two distributions of different types.

5.40. The random variable X may have a normal distribution with parameters $m = 0$ and $\sigma = 2$ with probability 0.4 or a normal distribution with parameters $m = 2$ and $\sigma = 1$ with probability 0.6. Find the probability density function of the random variable X.

Answer.

$$f(x) = \frac{0.4}{2\sqrt{2\pi}} e^{-\frac{x^2}{8}} + \frac{0.6}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}}.$$

The graph of $f(x)$ is shown in Fig. 5.40.

5.41. Two independent random variables, a discrete variable X with an ordered series

$$X: \left|\begin{array}{c|c|c|c} x_1 & x_2 & \dots & x_n \\ \hline p_1 & p_2 & \dots & p_n \end{array}\right|$$

and a continuous variable Y with density $f(y)$ are summed up, i.e. $Z = X + Y$. Is the resultant random variable discrete, continuous or mixed? Find its distribution.

Solution. The random variable Z is continuous. Its density $\tilde{f}(z)$ can be found by the total probability formula on the hypotheses $H_1 = \{X = x_1\}$, $H_2 = \{X = x_2\}, \dots, H_n = \{X = x_n\}$. For the ith hypothesis, the conditional element of the probability of the random variable Z is

$$f_i(z)\,dz = f(z - x_i)\,dz.$$

The total element of probability

$$\tilde{f}(z)\,dz = \sum_{i=1}^{n} p_i f(z - x_i)\,dz,$$

whence

$$\tilde{f}(z) = \sum_{i=1}^{n} p_i f(z - x_i).$$

5.42. Given a variable $Z = \min\{X, a\}$, where X is a continuous random variable with density $f(x)$ and a is a nonrandom variable, is the given variable discrete, continuous or mixed? Find the distribution of the random variable Z, its mean value and variance.

Solution. The random variable Z is defined by the formula:

$$Z = \begin{cases} X & \text{if} \quad X < a, \\ a, & \text{if} \quad X \geqslant a. \end{cases}$$

The value of the random variable $Z = a$ has a probability

$$p_a = \mathrm{P}\{X \geqslant a\} = \mathrm{P}\{X > a\} = \int_a^\infty f(x)\,dx.$$

For $p_a > 0$ the random variable Z is mixed and for $p_a = 0$ it is continuous.

For $Z < a$ the distribution function $\tilde{F}(z)$ of the random variable Z coincides with $F(x) = \int_{-\infty}^{x} f(x)\,dx$ (Fig. 5.42).

$$M[Z] = ap_a + \int_{-\infty}^{a} xf(x)\,dx,$$

$$\mathrm{Var}[Z] = \alpha_2[Z] - (M[Z])^2, \quad \alpha_2[Z] = a^2p_a + \int_{-\infty}^{a} x^2 f(x)\,dx.$$

If $P\{X \geqslant a\} = 0$, then $Z = \min\{X, a\} = X$, $\tilde{F}(z) = F(z)$.

5.43. Periodic signals of the same duration l and with the same intervals L between them are sent from two sources A and B (Fig. 5.43*a*); $l < L$. The moments the signals are sent from the sources A and B are not timed. When the messages, one from each source, overlap, they are distorted. Find (1) the probability R that at least one message will be distorted (completely or partially); (2) the probability that no more than 10 per cent of each message will be distorted; (3) the distribution function of the random variable Z which is the "length of the distorted text"; (4) the average length of the distorted text $\bar{z}$.

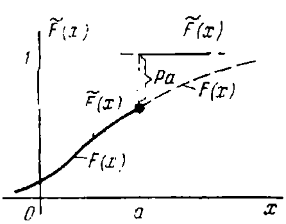


Fig. 5.42

Solution. Assume that the B-axis (see Fig. 5.43*a*) overlaps the A-axis at random. Since the beginnings and the ends of the messages are in functional relations, it is sufficient to consider only one pair of adjacen. segments on the A-axis, i.e. l, the message, and L, the interval (Fig. 5.43*b*)

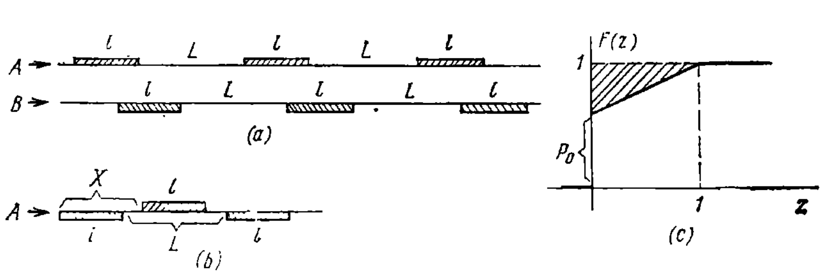


Fig. 5.43

We take the beginning of the interval l on the A-axis as the reference point and denote the abscissa of the beginning of the nearest interval l on the B-axis by X. The variable X is distributed with constant density on the interval $L + l$. The messages will not evidently overlap when

$l < X < L$; the probability of this event is

$$p_0 = (L - l)/(L + l), \quad R = 1 - p_0 = 1 - (L - l)/(L + l).$$

To calculate the probability p that the messages will overlap by no more than 10 per cent, we must increase the interval $L - l$, which is favourable to nonoverlapping, by two intervals of length $0.1l$. Hence $L - l + 2 \times 0.1l = L - 0.8l$; $p = (L - 0.8l)/(L + l)$. The random variable Z, the fraction of the distorted messages, is a mixed quantity. For $z = 0$ it has a nonzero probability $p_0 = (L - l)/(L + l)$; for $0 < z < 1$ we have $F(z) = [L - l(1 - 2z)]/(L + l)$; for $z = 1$ this expression becomes unity: $F(1) = 1$, and between $z = 0$ and $z = 1$ it increases linearly (Fig. 5.43c).

The mean value of the random variable Z is equal to the area hatched in Fig. 5.43c, i.e. $\bar{z} = [1 - (L - l)/(L + l)]/2 = l/(L + l)$.

5.44. There is a continuous random variable X with density $f(x)$ (Fig. 5.44a). The observed value of the random variable is retained if

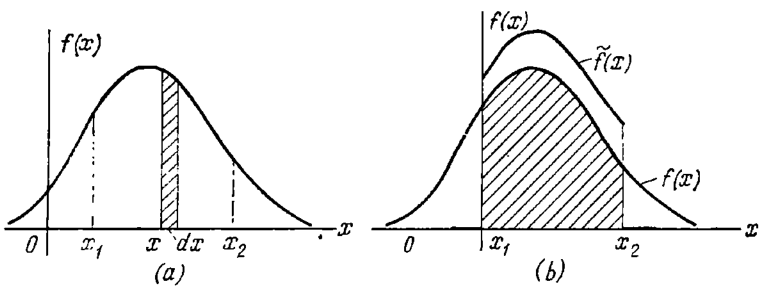


Fig. 5.44

it falls on the interval (x_1, x_2) and is rejected if it falls outside of the interval (x_1, x_2). A new random variable $\tilde{X}$ (a "shortened" random variable) results with a range of values from x_1 to x_2. Find the probability density function $\tilde{f}(x)$ of the random variable $\tilde{X}$.

Solution. The sought-for density $\tilde{f}(x)$ is the conditional density of the variable X provided that it falls on the interval (x_1, x_2). Let us calculate the probability element $f(x)\,dx$ for the interval $(x, x + dx) \subset (x_1, x_2)$ (see Fig. 5.44a). By the multiplication rule of probabilities

$$f(x)\,dx = \mathrm{P}\,\{X \in (x_1, x_2)\}\,\tilde{f}(x)\,dx,$$

where $\tilde{f}(x)\,dx$, the conditional probability element, is the probability that the variable X falls on the interval $(x, x + dx)$ under the condition that $X \in (x_1, x_2)$, and

$$\mathrm{P}\,\{X \in (x_1, x_2)\} = \int_{x_1}^{x_2} f(x)\,dx, \quad \tilde{f}(x) = f(x) \Big| \int_{x_1}^{x_2} f(x)\,dx.$$

The curve of $\tilde{f}(x)$ is similar to that of $f(x)$ and can be found from it by dividing each ordinate by the area hatched in Fig. 5.44*b* (see the thick curve in Fig. 5.44*b*). Outside of the interval (x_1, x_2) $\tilde{f}(x) = 0$.

5.45. A woman states: "My husband is of medium height, but most men are below medium height." Is the statement meaningless?

Solution. The statement is not meaningless and can even be true if the distribution density of the heights of X men is asymmetric about the mean value (expectation) as shown, for example, in Fig. 5.45. The area hatched in Fig. 5.45 is equal to the average fraction of men below the medium height m_x, and it is larger than the area left unhatched.

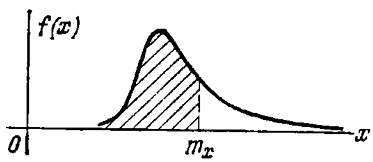


Fig. 5.45

5.46. In the theory dealing with the reliability of devices, **Weibull's distribution,** which has a distribution function

$$F(x) = 1 - e^{-\alpha x^n} \ (x > 0), \qquad (5.46)$$

where $\alpha > 0$ is a constant and n is a positive integer, is often used as the distribution of the time of failure-free performance of the device. Find (1) the probability density function $f(x)$, (2) the mean value and variance of the random variable X which has a Weibull distribution.

Solution.

(1) $f(x) = dF(x)/dx = n\alpha x^{n-1} e^{-\alpha x^n}$;

(2) $$M[X] = \int_0^\infty x n\alpha x^{n-1} e^{-\alpha x^n} \, dx$$

We make a change of the variable thus: $\alpha x^n = y$, $x = \alpha^{-1/n} y^{1/n}$

$$dx = \frac{1}{n} \alpha^{-1/n} y^{\frac{1-n}{n}} \, dy.$$

$$M[X] = \int_0^\infty n y e^{-y} \frac{1}{n} \alpha^{-\frac{1}{n}} y^{\frac{1-n}{n}} \, dy$$

$$= \alpha^{-\frac{1}{n}} \int_0^\infty y^{\frac{1}{n}} e^{-y} \, dy = \Gamma\left(1 + \frac{1}{n}\right) \alpha^{-\frac{1}{n}},$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx$ is the gamma function.

$$M[X^2] = \int_0^\infty n\alpha x^{n+1} e^{-\alpha x^n} \, dx$$

$$= \int_0^\infty n\alpha\alpha^{-\left(1+\frac{1}{n}\right)} y^{1+\frac{1}{n}} e^{-y} \frac{1}{n} \alpha^{-\frac{1}{n}} y^{\frac{1-n}{n}} dy$$

$$= \alpha^{-\frac{2}{n}} \int_0^\infty y^{\frac{2}{n}} e^{-y} dy = \Gamma\left(1+\frac{2}{n}\right) \alpha^{-\frac{2}{n}};$$

$$\mathrm{Var}\,[X] = \alpha^{-2/n} [\Gamma(1+2/n) - \{\Gamma(1+1/n)\}^2].$$

5.47. The service life of a device is a random variable T with density $f(t)$ $(t > 0)$. At a moment t_0, provided that the device has not failed, a preventive maintenance is performed, and then it operates for a time T_1 with density $f_1(t)$. If the device fails at a moment $u < t_0$, an emergency repair is immediately carried out, and then the device operates for a random time T_2 with density $f_2(t)$ (no more repairs are made). Find the mean value Θ of the time for which the device will operate, excluding the time needed for the repairs).

Solution. Let T assume a value $t < t_0$. On this hypothesis, the conditional expectation of the variable Θ is $M[\Theta \mid t] = t + m_2$, where $m_2 = \int_0^\infty t f_2(t)\, dt$ is the mean value of the operating time of the device after a repair. Now if $t \geqslant t_0$, then the device will not fail till the moment t_0, and $M[\Theta \mid t] = t_0 + m_1$, where $m_1 = \int_0^\infty t f_1(t)\, dt$. Consequently,

$$M[\Theta \mid t] = \begin{cases} t + m_2, & \text{for} \quad t < t_0, \\ t_0 + m_1 & \text{for} \quad t \geqslant t_0. \end{cases}$$

The total (unconditional) expectation of the quantity

$$M[\Theta] = \int_0^{t_0} (t + m_2) f(t)\, dt + \int_{t_0}^\infty (t_0 + m_1) f(t)\, dt$$

$$= \int_0^{t_0} t f(t)\, dt + m_2 \int_0^{t_0} f(t)\, dt + (t_0 + m_1) \int_{t_0}^\infty f(t)\, dt$$

$$= \int_0^{t_0} t f(t)\, dt + m_2 P\{T < t_0\} + (t_0 + m_1) P\{T > t_0\}$$

$$= \int_0^{t_0} t f(t)\, dt + m_2 F(t_0) + (t_0 + m_1)(1 - F(t_0)),$$

where $F(t)$ is the distribution function of the random variable T:

$$F(t) = \int_0^t f(t)\, dt.$$

5.48. A sequence of messages of the same length l are transmitted over a radio channel (Fig. 5.48*a*) with random intervals T_1, T_2 . . . between them. The intervals T_1, T_2, . . . have the same distribution. A noise interferes with a message from time to time (5.48*b*). The moments when the noise begins and ends are not connected with the sequence of the messages. The duration D of each interference period is accidental and has an exponential distribution with parameter μ; the duration of an interval between interference periods is also accidental and has an exponential distribution with parameter ν. If the interference period

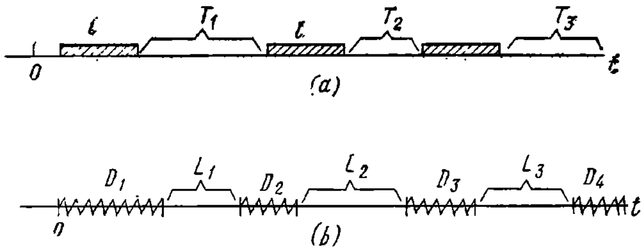


Fig. 5.48

lasts as long as a message or a part of it, then either the whole message or its corresponding part is distorted. Find the average length of the message distorted by interferences, i.e. the ratio of the average length of the distorted text to the average length of the transmitted text.

Solution. $1/\mu$ is the average length of the interference; $1/\nu$ is the average length of the interval between them; the average fraction of the time on the t-axis occupied by the interferences is

$$\frac{1/\mu}{1/\mu+1/\nu}=\frac{\nu}{\mu+\nu}$$

It is evident that the same average fraction of the messages will be distorted by the interference irrespective of the distributions of their duration and the length of the intervals between them if $\mu = 1/M[D]$, $\nu = 1/M[T]$.

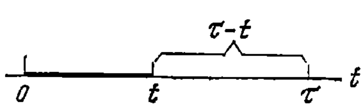


Fig. 5.49

5.49. During a time τ (an observation period) a signal arrives with probability p. The signal appears at any point of the interval τ with the same probability density. It is known that at the moment $t < \tau$ (Fig. 5.49) the signal has not yet arrived. Find the probability Q that it will arrive during the remaining time $\tau - t$.

Solution. Q is simply the conditional probability that the signal will arrive during the time $\tau - t$, if it is known that by the moment t it has not yet arrived. The total probability p of the arrival of the signal during the time τ is equal to the probability tp/τ that it will arrive during the time t plus the probability $1 - tp/\tau$ that it will not arrive

during that time, multiplied by Q. Hence

$$p = \frac{t}{\tau} p + \left(1 - \frac{t}{\tau} p\right) Q; \quad Q = \frac{p(1 - t/\tau)}{1 - tp/\tau}.$$

5.50. The moment a signal arrives is a random variable T with density $f(t)$. At a moment $t < \tau$ the signal has not yet arrived. Find the probability that it will arrive during the next time interval from t to τ (Fig. 5.50).

Solution. The problem is similar to the preceding one. We designate as p the probability that the signal will arrive during the time τ, $p = \int_0^\tau f(t)\,dt$. Reasoning as in the preceding problem, we find the total probability p of the arrival of the signal during the time τ. It is equal to the probability p_t that the signal will arrive by the moment t, i.e. $p_t = \int_0^t f(t)\,dt$, plus the probability of the complementary event $1 - p_t$ multiplied by the conditional probability Q that the signal will arrive during the remaining time $(\tau - t)$, i.e.

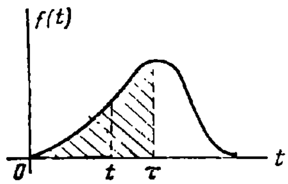


Fig. 5.50

$$\int_0^\tau f(t)\,dt = \int_0^t f(t)\,dt + \left\{1 - \int_0^t f(t)\,dt\right\} Q,$$

whence

$$Q = \frac{\int_0^\tau f(t)\,dt - \int_0^t f(t)\,dt}{1 - \int_0^t f(t)\,dt} = \frac{\int_t^\tau f(t)\,dt}{1 - F(t)} = \frac{F(\tau) - F(t)}{1 - F(t)},$$

where $F(t)$ is the distribution function of the random variable T.

If $f(t)$ is an exponential distribution with parameter λ, then $Q = 1 - e^{-\lambda(\tau - t)}$.

5.51. Under the conditions of the preceding problem $f(t)$ is a normal distribution with parameters m and σ; $t = m$ (i.e. the signal does not arrive during a time m)*). Find the probability that it will arrive during an interval of length σ starting at $t = m$.

Solution. $\tau = m + \sigma$,

$$\int_m^{m+\sigma} f(t)\,dt = \Phi\left(\frac{m + \sigma - m}{\sigma}\right) - \Phi(0) = \Phi(1) \approx 0.341,$$

$$Q \approx 0.341/0.5 = 0.682.$$

*) The problem is meaningful only if T is nonnegative, i.e. for $m - 3\sigma > 0$.

CHAPTER 6

Systems of Random Variables (Random Vectors)

6.0. A system of two random variables (X, Y) can be geometrically interpreted as a *random point* with coordinates (X, Y) on the x-y plane (Fig. 6.0.1) or as a random vector directed from the origin to the point (X, Y) whose components are random variables X and Y (Fig. 6.0.2).

A system of three random variables (X, Y, Z) is represented by a *random point* or a *random vector in a three-dimensional space*, while a system of n random variables

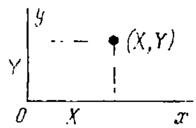


Fig. 6.0.1

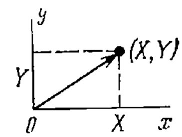


Fig. 6.0.2

$(X_1, X_2, \ldots, X_n)$ is represented by a *random point* or a *random vector in an n-dimensional space.*

The *joint probability distribution* of two random variables (X, Y) (or the probability distribution of a system of two random variables) is the probability that both inequalities $X < x$ and $Y < y$ are simultaneously satisfied, i.e.

$$F(x, y) = \mathrm{P}\{X < x, Y < y\}. \qquad (6.0.1)$$

$F(x, y)$ can be interpreted geometrically as the probability of a random point (X, Y) falling in a quadrant whose vertex is (x, y), the one hatched in Fig. 6.0.3. The prob-

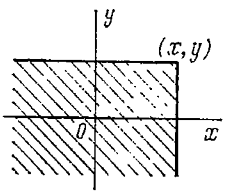


Fig. 6.0.3

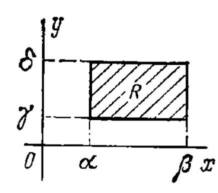


Fig. 6.0.4

ability of a random point (X, Y) falling in a rectangle R whose sides are parallel to the coordinate axes and which includes its lower and left-hand boundaries and does not include the upper and right-hand boundaries (Fig. 6.0.4), is expressed in terms of a distribution function by the formula

$$\mathrm{P}\{(X, Y) \in R\} = F(\beta, \delta) - F(\alpha, \delta) - F(\beta, \gamma) + F(\alpha, \gamma). \qquad (6.0.2)$$

The distribution function $F(x, y)$ possesses the properties (1) $F(-\infty, -\infty) = F(-\infty, y) = F(x, -\infty) = 0$, (2) $F(+\infty, +\infty) = 1$, (3) $F(x, +\infty) = F_1(x)$, $F(+\infty, y) = F_2(y)$, where $F_1(x)$ and $F_2(y)$ are the distribution functions of the random variables X and Y, (4) $F(x, y)$ is a nondecreasing function of the arguments x and y, (5) $F(x, y)$ is continuous on the left with respect to each co-

ordinate, (6) $F(\beta, \delta) - F(\alpha, \delta) - F(\beta, \gamma) + F(\alpha, \gamma) \geqslant 0$ for any $\alpha \leqslant \beta$, $\gamma \leqslant \delta$ (the last property means that the probability of falling in a rectangle is non-negative).

The *joint probability density* of two continuous random variables (or the distribution density of a system) is the limit of the ratio of the probability of a random point falling in an element of a plane Δx, Δy, adjoining the point (x, y), to the area of the element when its dimensions Δx, Δy tend to zero. The joint density can be expressed in terms of the joint probability distribution, i.e.

$$f(x, y) = \partial^2 F(x, y)/\partial x\, \partial y = F''_{xy}(x, y) \tag{6.0.3}$$

i.e. is the second mixed partial derivative of the distribution function with respect to the two arguments.

The surface representing the function $f(x, y)$ is known as the *distribution surface.*

An *element of probability* for a system of two random variables is the quantity $f(x, y)\, dx\, dy$ which is an approximate expression for the probability of the random point (X, Y) falling in an elementary rectangle with sides dx and dy adjoining the point (x, y).

The probability of a random point (X, Y) falling in an arbitrary domain D is expressed by the formula

$$\mathbf{P}\{(X, Y) \in D\} = \iint\limits_{(D)} f(x, y)\, dx\, dy. \tag{6.0.4}$$

The properties of a joint probability density:

(1) $f(x, y) \geqslant 0$ and (2) $\iint\limits_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$

The joint probability distribution function can be expressed in terms of the joint density thus:

$$F(x, y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f(x, y)\, dx\, dy. \tag{6.0.5}$$

The probability densities of the separate variables entering into a system can be expressed in terms of the joint density, i.e.

$$f_1(x) = \int\limits_{-\infty}^{\infty} f(x, y)\, dy, \quad f_2(y) = \int\limits_{-\infty}^{\infty} f(x, y)\, dx. \tag{6.0.6}$$

The *conditional distribution* of a random variable entering into a system is its distribution calculated under the condition that the other random variable has assumed a definite value.

The conditional distribution functions of two random variables X and Y in a system are designated as $F_1(x \mid y)$ and $F_2(y \mid x)$, and the conditional probability densities as $f_1(x \mid y)$ and $f_2(y \mid x)$.

There is a theorem on the multiplication of probability densities, i.e.

$$f(x, y) = f_1(x)\, f_2(y \mid x) \quad \text{or} \quad f(x, y) = f_2(y)\, f_1(x \mid y). \tag{6.0.7}$$

The expressions for conditional densities in terms of absolute densities are

$$f_2(y \mid x) = \frac{f(x, y)}{f_1(x)} \quad \text{for} \quad f_1(x) \neq 0, \qquad f_1(x, y) = \frac{f(x, y)}{f_2(y)} \quad \text{for} \quad f_2(y) \neq 0. \tag{6.0.8}$$

The random variables X, Y are said to be *mutually independent* if the conditional probability distribution of one does not depend on the value of the other, viz.

$$f_1(x \mid y) = f_1(x) \quad \text{or} \quad f_2(y \mid x) = f_2(y). \tag{6.0.9}$$

For independent random variables the multiplication theorem assumes the form

$$f_1(x, y) = f_1(x) f_2(y). \tag{6.0.10}$$

The *moment of order $k + s$ about the origin* of the system (X, Y) is a quantity

$$\alpha_{ks}[X, Y] = \mathrm{M}[X^k Y^s]. \tag{6.0.11}$$

The *central moment of order $k + s$* of system (X, Y) is a quantity

$$\mu_{ks}[X, Y] = M[\overset{\circ}{X}{}^k \overset{\circ}{Y}{}^s]. \tag{6.0.12}$$

Formulas for calculating moments:
(a) for discrete random variables

$$\alpha_{ks}[X, Y] = \sum_i \sum_j x_i^k y_j^s p_{ij}, \tag{6.0.13}$$

$$\mu_{ks}[X, Y] = \sum_i \sum_j (x_i - m_x)^k (y_j - m_y)^s p_{ij}, \tag{6.0.14}$$

where $p_{ij} = \mathrm{P}\{X = x_i,\ Y = y_j\}$;
(b) for continuous random variables

$$\alpha_{ks}[X, Y] = \int\int_{-\infty}^{\infty} x^k y^s f(x, y)\, dx\, dy, \tag{6.0.15}$$

$$\mu_{ks}[X, Y] = \int\int_{-\infty}^{\infty} (x - m_x)^k (y - m_y)^s f(x, y)\, dx\, dy, \tag{6.0.16}$$

where $f(x, y)$ is a joint probability density.

The *order* of the moment $\alpha_{ks}[X, Y]$ or $\mu_{ks}[X, Y]$ is the sum of the indices k and s.

The *covariance* of two random variables X, Y is a second-order mixed central moment, i.e. μ_{11}:

$$\mathrm{Cov}_{xy} = \mu_{11}[X, Y] = \mathrm{M}[\overset{\circ}{X}\overset{\circ}{Y}]. \tag{6.0.17}$$

It is convenient to calculate the quantity Cov_{xy} in terms of the second mixed moment about the origin:

$$\mathrm{Cov}_{xy} = \alpha_{11}[X, Y] - m_x m_y, \tag{6.0.18}$$

or, using another notation,

$$\mathrm{Cov}_{xy} = \mathrm{M}[X \cdot Y] - \mathrm{M}[X] \cdot \mathrm{M}[Y]. \tag{6.0.19}$$

For independent random variables the covariance is zero.

The *correlation coefficient* (or the normalized covariance) r_{xy} of two random variables X and Y is a dimensionless quantity

$$r(X, Y) = r_{xy} = \mathrm{Cov}_{xy}/(\sigma_x \sigma_y), \tag{6.0.20}$$

where $\sigma_x = \sqrt{\mathrm{Var}_x} = \sqrt{\mu_{20}[X, Y]}$ and $\sigma_y = \sqrt{\mathrm{Var}_y} = \sqrt{\mu_{02}[X, Y]}$.

The correlation coefficient is a measure of how closely the random variables are linearly related.

Two random variables X and Y are said to be *uncorrelated* if their covariance or the correlation coefficient, which is the same thing, is zero.

If two random variables are independent, then they are uncorrelated, but if they are uncorrelated, they are not necessarily independent.

If the random variables X and Y are linearly related, i.e. $Y = aX + b$, where a and b are nonrandom, then their correlation coefficient r_{xy} is ± 1, where the plus sign or the minus sign corresponds to the sign of the coefficient a. For any two random variables $|r_{xy}| \leqslant 1$.

The *joint distribution function* of n random variables $X_1, X_2, \ldots, X_n$ is the probability that n inequalities of the form $X_i < x_i$ are satisfied simultaneously, i.e.

$$F(x_1, x_2, \ldots, x_n) = \mathrm{P}\{X_1 < x_1, X_2 < x_2, \ldots, X_n < x_n\}. \qquad (6.0.21)$$

The *joint probability density* of n random variables is the nth mixed partial derivative of the distribution function

$$f(x_1, x_2, \ldots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2, \ldots, \partial x_n} F(x_1, x_2, \ldots, x_n). \qquad (6.0.22)$$

The distribution function $F_i(x_i)$ of one of the variables X_i, entering into the system, results from $F(x_1, x_2, \ldots, x_n)$, if all the arguments, except for x_i, are equal to $+\infty$

$$F_i(x_i) = F(+\infty, +\infty, \ldots, x_i, +\infty, \ldots, +\infty). \qquad (6.0.23)$$

The probability density of a variable X_i, entering into the system $(X_1, X_2, \ldots, X_n)$, can be expressed in terms of the joint probability density by the formula

$$f_i(x_i) = \int_{-\infty}^{\infty} \overset{(n-1)}{\cdots} \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_n)\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_n. \qquad (6.0.24)$$

The probability density of a subsystem $(X_1, X_2, \ldots, X_k)$, entering into the system $(X_1, X_2, \ldots, X_k, X_{k+1}, \ldots, X_n)$, is

$$f_{1, \ldots, k}(x_1, \ldots, x_k) = \int_{-\infty}^{\infty} \overset{(n-k)}{\cdots} \int_{-\infty}^{\infty} f(x_1, \ldots, x_k, \ldots, x_n)\, dx_{k+1} \ldots dx_n. \qquad (6.0.25)$$

The conditional probability density of the subsystem $X_1, \ldots, X_k$, when all the other random variables are fixed, is

$$f_{1, \ldots, k}(x_1, \ldots, x_k | x_{k+1}, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n)}{f_{k+1, \ldots, n}(x_{k+1}, \ldots, x_n)}. \qquad (6.0.26)$$

If the random variables $(X_1, X_2, \ldots, X_n)$ are mutually independent, then

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2), \ldots, f_n(x_n). \qquad (6.0.27)$$

The probability of a random point $(X_1, X_2 \ldots, X_n)$ falling in an n-dimensional domain D is expressed by an n-tuple integral, i.e.

$$\mathrm{P}\{(X_1, \ldots, X_n) \in D\} = \overset{n}{\int_D} \int f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n. \qquad (6.0.28)$$

The *correlation matrix* of a system of n random variables $(X_1, X_2, \ldots, X_n)$ contains the pairwise covariances of all the variables, i.e.

$$\| \mathrm{Cov}_{ij} \| = \begin{Vmatrix} \mathrm{Cov}_{11} & \mathrm{Cov}_{12} & \ldots & \mathrm{Cov}_{1n} \\ \mathrm{Cov}_{21} & \mathrm{Cov}_{22} & \ldots & \mathrm{Cov}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Cov}_{n1} & \mathrm{Cov}_{n2} & \ldots & \mathrm{Cov}_{nn} \end{Vmatrix},$$

where $\mathrm{Cov}_{ij} = \mathrm{Cov}_{x_i x_j} = \mathrm{M}[\overset{\circ}{X}_i \overset{\circ}{X}_j]$ is the covariance of the random variables X_i, X_j.

The correlation matrix is symmetric ($\mathrm{Cov}_{ij} = \mathrm{Cov}_{ji}$) and, therefore, only half the table is usually filled:

$$\begin{Vmatrix} \mathrm{Cov}_{11} & \mathrm{Cov}_{12} & \dots & \mathrm{Cov}_{1n} \\ & \mathrm{Cov}_{22} & \dots & \mathrm{Cov}_{2n} \\ & \cdots & \cdots & \cdots \\ & & & \mathrm{Cov}_{nn} \end{Vmatrix}.$$

The variances of the random variables $X_1, X_2, \ldots, X_n$ lie along the principal diagonal of the correlation matrix, i.e.

$$\mathrm{Cov}_{ii} = \mathrm{Var}[X_i]. \tag{6.0.29}$$

The *normalized correlation matrix* of a system of n random variables is compiled from the pairwise correlation coefficients of all the variables, i.e.

$$\| r_{ij} \| = \begin{Vmatrix} 1 & r_{12} & r_{13} \dots r_{1n} \\ & 1 & r_{23} \dots r_{2n} \\ & & 1 \dots r_{3n} \\ & & \cdots\cdots \\ & & 1 \end{Vmatrix},$$

where $r_{ij} = \mathrm{Cov}_{ij}/(\sigma_i \sigma_j)$ is the coefficient of correlation of two variables X_i, X_j.

The *normal probability distribution* of two random variables X, Y (the normal distribution on a plane) has a probability density of the form

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-m_x)^2}{\sigma_x^2} - \frac{2r(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\}, \tag{6.0.30}$$

where m_x, m_y are the mean values of the random variables X, Y; σ_x, σ_y are their mean square deviations, and r is their correlation coefficient.

For random variables, which have a normal distribution, uncorrelation is equivalent to independence. If the random variables X, Y are uncorrelated (independent), then $r = 0$ and

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left[\frac{(x-m_x)^2}{\sigma_x^2} + \frac{(y-m_y)^2}{\sigma_y^2}\right]\right\}. \tag{6.0.31}$$

In this case the Ox and Oy axes are called the *principal axes of scattering (dispersion)*. If, in addition $m_x = m_y = 0$, the normal distribution assumes a *canonical form*:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right\}. \tag{6.0.32}$$

The probability that a random point which has a normal distribution will fall in a rectangle R with sides parallel to the principal axes of scattering (see Fig. 6.0.4), is

$$\mathrm{P}\{(X, Y) \in R\} = \left[\Phi\left(\frac{\beta - m_x}{\sigma_x}\right) - \Phi\left(\frac{\alpha - m_x}{\sigma_x}\right)\right]\left[\Phi\left(\frac{\delta - m_y}{\sigma_y}\right) - \Phi\left(\frac{\gamma - m_y}{\sigma_y}\right)\right]. \tag{6.0.33}$$

The ***ellipse of equal probability density*** (dispersion ellipse) is an ellipse at whose all points the joint probability density of the normal distribution is constant:

$f(x, y) = \text{const}$. The semi-axes of the ellipse are proportional to σ_x and σ_y: $a = k\sigma_x$ and $b = k\sigma_y$.

The probability that a random point which has a normal distribution will fall in a domain E_k bounded by an ellipse of scattering with the semi-axes a and b is

$$\mathrm{P}\{(X, Y) \in E_k\} = 1 - e^{-k^2/2}, \tag{6.0.34}$$

where k is the size of the semi-axes of the ellipse in the mean square deviations, i.e. $k = a/\sigma_x = b/\sigma_y$.

If $\sigma_x = \sigma_y = \sigma$, then the normally distributed scattering is called *circular*. For a circular normal scattering with $m_x = m_y = 0$ the distance R from the point (X, Y) to the origin (the centre of scattering) has a **Rayleigh distribution**:

$$f(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \quad \text{for} \quad r > 0. \tag{6.0.35}$$

For the independent random variables (X, Y, Z) the normal probability distribution in a three-dimensional space is

$$f(x, y, z) = \frac{1}{(2\pi)^{3/2} \sigma_x \sigma_y \sigma_z} \times \exp\left\{-\frac{1}{2}\left[\frac{(x-m_x)^2}{\sigma_x^2} + \frac{(y-m_y)^2}{\sigma_y^2} + \frac{(z-m_z)^2}{\sigma_z^2}\right]\right\}. \tag{6.0.36}$$

The probability that a random point (X, Y, Z) will fall in a domain E_k bounded by an ellipsoid of equal probability density with semi-axes $a = k\sigma_x$, $b = k\sigma_y$, $c = k\sigma_z$ is

$$\mathrm{P}\{(X, Y, Z) \in E_k\} = 2\Phi(k) - \sqrt{2/\pi}\, k e^{-k^2/2}. \tag{6.0.37}$$

Problems and Exercises

6.1. Two messages are being transmitted, each of which independently of the other, may be distorted or not. The probability of an event $A = \{\text{a message is distorted}\}$ for the first message is p_1, for the second is p_2. We consider a system of two random variables (X, Y) defined as follows:

$$X = \begin{cases} 1, & \text{if the first message is distorted;} \\ 0, & \text{if the first message is not distorted;} \end{cases}$$

$$Y = \begin{cases} 1, & \text{if the second message is distorted;} \\ 0, & \text{if the second message is not distorted} \end{cases}$$

(X and Y are the indicators of the event A in the first and the second trial).

Find the joint probability distribution of the pair of random variables (X, Y), i.e. the set of probabilities p_{ij} for every combination of their values. Find the joint probability distribution function $F(x, y)$.

Solution. The joint probability distribution is defined by the probabilities

$$p_{00} = \mathrm{P}\{X = 0, Y = 0\} = q_1 q_2, \quad p_{10} = \mathrm{P}\{X = 1, Y = 0\} = p_1 q_2,$$
$$p_{01} = \mathrm{P}\{X = 0, Y = 1\} = q_1 p_2, \quad p_{11} = \mathrm{P}\{X = 1, Y = 1\} = p_1 p_2,$$

where $q_1 = 1 - p_1$, $q_2 = 1 - p_2$ (see the table below).

y_i \ x_i	0	1
0	q_1q_2	p_1q_2
1	q_1p_2	p_1p_2

The probability distribution on the x, y plane is concentrated at four points with coordinates (0, 0), (0, 1), (1, 0), (1, 1) (Fig. 6.1). Using the geometric interpretation of the probability distribution function as

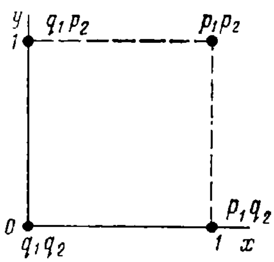


Fig. 6.1

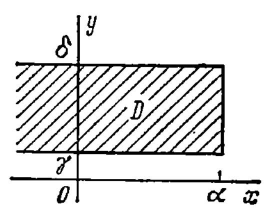


Fig. 6.2

the probability that a point will fall in a quadrant whose vertex is at the point (x, y) (see Fig. 6.0.3), we get the following table of values for $F(x, y)$:

y \ x	$x \leqslant 0$	$0 < x \leqslant 1$	$1 < x$
$y \leqslant 0$	0	0	0
$0 < y \leqslant 1$	0	q_1q_2	q_2
$1 < y$	0	q_1	1

6.2. The distribution function of a system of two random variables (X, Y) is $F(x, y)$. Find the probability that the random point (X, Y) will fall in the domain D (Fig. 6.2), which is bounded by the abscissa α on the right and by the ordinates γ, δ from above and from below.

Answer. $P\{(X, Y) \in D\} = F(\alpha, \delta) - F(\alpha, \gamma)$.

6.3. There are two independent random variables X and Y, each of which has an exponential distribution, i.e.

$$f_1(x) = \lambda e^{-\lambda x} \ (x > 0), \quad f_2(y) = \mu e^{-\mu y} \ (y > 0).$$

Write expressions (1) for the joint probability density, and (2) for the distribution function of the system (X, Y).

Answer.

$$(1)\ f(x, y) = \begin{cases} 0 & \text{for } x < 0 \text{ or } y < 0, \\ \lambda\mu e^{-(\lambda x + \mu y)} & \text{for } x > 0 \text{ and } y > 0, \end{cases}$$

$$(2)\ F(x, y) = \begin{cases} 0 & \text{for } x \leqslant 0 \text{ or } y \leqslant 0, \\ (1 - e^{-\lambda x})(1 - e^{-\mu y}) & \text{for } x > 0 \text{ and } y > 0. \end{cases}$$

6.4. A system of random variables (X, Y) is distributed with a constant density inside a square R with side a (Fig. 6.4a). Write an expression for the probability density function $f(x, y)$. Construct the

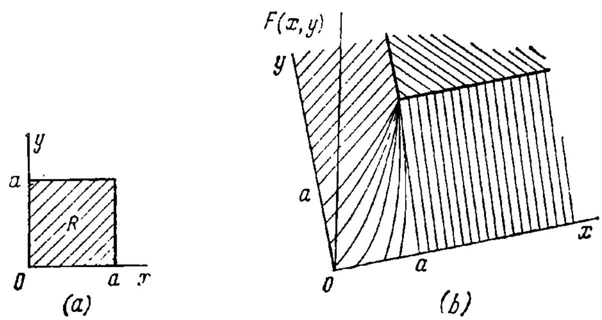


Fig. 6.4

distribution function of the system. Write expressions for $f_1(x)$ and $f_2(y)$. Find out whether the random variables X and Y are independent or not.

Answer. $$f(x, y) = \begin{cases} 1/a^2 & \text{for } (x, y) \in R, \\ 0 & \text{for } (x, y) \notin R, \end{cases}$$

$$F(x, y) = \begin{cases} 0 & \text{for } x \leqslant 0 \text{ or } y \leqslant 0, \\ xy/a^2 & \text{for } 0 < x \leqslant a \text{ and } 0 < y \leqslant a, \\ y/a & \text{for } x > a \text{ and } 0 < y \leqslant a, \\ x/a & \text{for } 0 < x \leqslant a \text{ and } y > a, \\ 1 & \text{for } x > a \text{ and } y > a. \end{cases}$$

The surface $F(x, y)$ is shown in Fig. 6.4b.

$$f_1(x) = \begin{cases} 1/a & \text{for } x \in (0, a), \\ 0, & \text{for } x \notin (0, a), \end{cases} \quad f_2(y) = \begin{cases} 1/a & \text{for } y \in (0, a), \\ 0 & \text{for } y \notin (0, a). \end{cases}$$

The random variables X and Y are independent since

$$f(x, y) = f_1(x) f_2(y).$$

6.5. A distribution surface of a system of random variables (X, Y) is a right cone (Fig. 6.5a) whose base is a circle K with centre at the origin and radius r_0. Outside the circle the joint probability density function $f(x, y)$ is zero. (1) Write the expression for $f(x, y)$; (2) find $f_1(x)$, $f_2(y)$; $f_2(y \mid x)$ and $f_1(x \mid y)$; (3) find out whether X and Y are dependent, and (4) find out whether X and Y are correlated.

Solution.

$$(1)\ f(x, y) = \begin{cases} \frac{3}{r_0^3\pi}\left(r_0 - \sqrt{x^2+y^2}\right) & \text{for} \quad x^2+y^2 < r_0^2, \\ 0 & \text{for} \quad x^2+y^2 > r_0^2, \end{cases}$$

$$(2)\ f_1(x) = \begin{cases} \frac{3}{r_0^3\pi}\left[r_0\sqrt{r_0^2-x^2} - x^2 \ln\left(\frac{r_0+\sqrt{r_0^2-x^2}}{|x|}\right)\right] & \text{for} \quad |x| < r_0, \\ 0 & \text{for} \quad |x| > r_0, \end{cases}$$

$$f_2(y) = \begin{cases} \frac{3}{r_0^3\pi}\left[r_0\sqrt{r_0^2-y^2} - y^2 \ln\left(\frac{r_0+\sqrt{r_0^2-y^2}}{|y|}\right)\right] & \text{for} \quad |y| < r_0, \\ 0 & \text{for} \quad |y| > r_0. \end{cases}$$

Furthermore, for $|x| < r_0$

$$f_2(y \mid x) = \begin{cases} \dfrac{r_0 - \sqrt{x^2+y^2}}{r_0\sqrt{r_0^2-x^2} - x^2 \ln\left(\frac{r_0+\sqrt{r_0^2-x^2}}{|x|}\right)} & \text{for} \quad |y| < \sqrt{r_0^2-x^2}, \\ 0 & \text{for} \quad |y| > \sqrt{r_0^2-x^2}. \end{cases}$$

and for $|y| < r_0$

$$f_1(x \mid y) = \begin{cases} \dfrac{r_0 - \sqrt{x^2+y^2}}{r_0\sqrt{r_0^2-y^2} - y^2 \ln\left(\frac{r_0+\sqrt{r_0^2-y^2}}{|y|}\right)} & \text{for} \quad |x| < \sqrt{r_0^2-y^2}, \\ 0 & \text{for} \quad |x| > \sqrt{r_0^2-y^2}, \end{cases}$$

(3) since $f_1(x \mid y) \neq f_1(x)$, the variables X and Y are dependent,
(4) we find the covariance Cov_{xy}. Since $m_x = m_y = 0$, we have

$$\mathrm{Cov}_{xy} = \iint_{(K)} xy f(x, y)\, dx\, dy$$

$$= \iint_{(K_1)} xy f(x, y)\, dx\, dy + \iint_{(K_2)} xy f(x, y)\, dx\, dy,$$

where K_1 is the right half of the circle, K_2 is the left half (Fig. 6.5b), the function $xy\, f(x, y)$ is odd with respect to the argument x and, there-

fore, the integrals with respect to K_1 and K_2 differ only in sign. Hence, when the integrals are summed, they cancel each other, and $\mathrm{Cov}_{xy} = 0$. The random variables X and Y are not correlated therefore.

6.6. A pair of random variables (X, Y) has a joint probability density

$$f(x, y) = a/(1 + x^2 + x^2y^2 + y^2).$$

(1) Find the coefficient a, (2) find out whether the random variables X and Y are dependent; find $f_1(x)$ and $f_2(y)$, (3) find the probability that

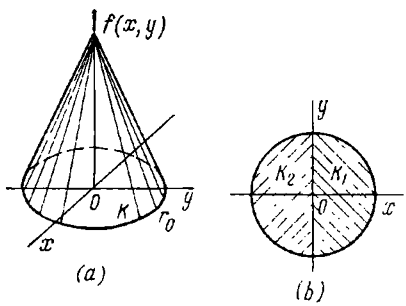


Fig. 6.5

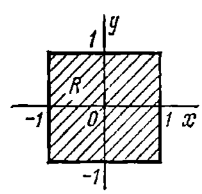


Fig. 6.6

the random point (X, Y) will fall in the square R whose centre coincides with the origin and whose sides are parallel to the coordinate axes and are $b = 2$ in length (Fig. 6.6).

Solution. (1) From the condition

$$\int_{-\infty}^{\infty}\int f(x, y)\,dx\,dy = 1$$

we find that $a = 1/\pi^2$.

(2) The random variables X and Y are independent:

$$f_1(x) = \frac{1}{\pi(1+x^2)}, \quad f_2(y) = \frac{1}{\pi(1+y^2)}, \quad f(x, y) = f_1(x)\,f_2(y),$$

$$(3)\ \mathrm{P}\{(X, Y) \in R\} = \int_{-1}^{1}\int_{-1}^{1} \frac{dx\,dy}{\pi^2(1+x^2)(1+y^2)} = \frac{1}{4}.$$

6.7. There are two independent random variables X and Y. The random variable X has a normal distribution with parameters $m_x = 0$ and $\sigma_x = 1/\sqrt{2}$. The random variable Y has a uniform distribution on the interval $(0, 1)$. Write the expressions for the joint probability density function $f(x, y)$ and the distribution function $F(x, y)$ of the system (X, Y).

Answer.

$$f(x,\ y)=\frac{1}{\sqrt{\pi}}e^{-x^2}\quad \text{for}\quad y\in(0,\ 1),$$

$$F(x,\ y)=\begin{cases}0 & \text{for}\quad y\leqslant 0,\\ y\,[\Phi(x\sqrt{2})+0.5] & \text{for}\quad 0<y\leqslant 1,\\ \Phi(x\sqrt{2})+0.5 & \text{for}\quad y>1.\end{cases}$$

6.8. The probability distribution surface $f(x,\ y)$ of a system of two random variables (X, Y) is a right circular cylinder, the centre of whose base coincides with the origin (Fig. 6.8*a*) and whose altitude is h. Find

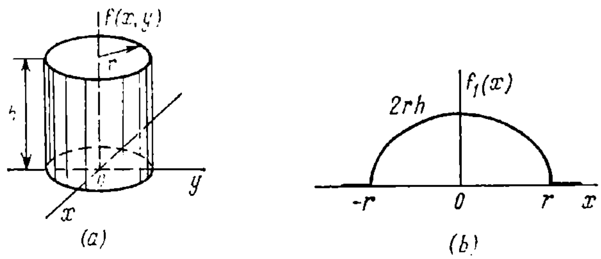


Fig. 6.8

the radius r of the cylinder, as well as $f_1(x)$, $f_2(y)$, $f_1(x \mid y)$, $f_2(y \mid x)$, m_x, Var_x and Cov_{xy}.

Solution. The radius r of the cylinder can be found from the condition that the volume of the cylinder is unity, whence $r=\sqrt{1/(\pi h)}$. The joint probability density

$$f(x,\ y)=\begin{cases}h, & \text{if}\quad x^2+y^2<r^2,\\ 0, & \text{if}\quad x^2+y^2>r^2.\end{cases}$$

Consequently

$$f_1(x)=\int_{-\infty}^{\infty} f(x,\ y)\,dy=\begin{cases}2\sqrt{r^2-x^2}\,h & \text{for}\quad |x|<r,\\ 0 & \text{for}\quad |x|>r.\end{cases}$$

Similarly

$$f_2(y)=\begin{cases}2\sqrt{r^2-y^2}\,h & \text{for}\quad |y|<r,\\ 0 & \text{for}\quad |y|>r.\end{cases}$$

The graph of the function $f_1(x)$ is shown in Fig. 6.8*b*. For $|y|<r$ we have

$$f_1(x\mid y)=\frac{f(x,\ y)}{f_2(y)}=\begin{cases}\dfrac{1}{2\sqrt{r^2-y^2}} & \text{for}\quad |x|<\sqrt{r^2-y^2},\\ 0 & \text{for}\quad |x|>\sqrt{r^2-y^2}.\end{cases}$$

Similarly, for $|x| < r$, we have

$$f_2(y|x) = \begin{cases} 1/(2\sqrt{r^2 - x^2}) & \text{for} \quad |y| < \sqrt{r^2 - x^2}, \\ 0 & \text{for} \quad |y| > \sqrt{r^2 - x^2}. \end{cases}$$

The mean values are zero, i.e. $m_x = m_y = 0$, since the function $f(x, y)$ is even with respect to both x and y, viz.

$$\text{Var}_x = 2h \int_{-r}^{r} x^2 \sqrt{r^2 - x^2}\, dx = hr^4 \frac{\pi}{4} = \frac{r^2}{4}, \quad \sigma_x = \frac{r}{2}, \quad \text{Cov}_{xy} = 0.$$

6.9. A random point (X, Y) has a constant probability distribution density inside the square R hatched in Fig. 6.9a. Write an expression

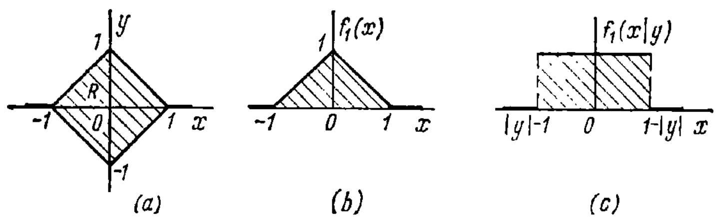


Fig. 6.9

for the joint probability density function $f(x, y)$. Find the expressions for the distribution densities $f_1(x)$ and $f_2(y)$ of individual variables X and Y entering into the system. Write the expressions for the conditional probability densities $f_1(x \mid y)$ and $f_2(y \mid x)$. Are the random variables X and Y dependent? Are they correlated?

Solution. The area of the square is equal to two, and therefore

$$f(x, y) = \begin{cases} 1/2 & \text{for} \quad (x, y) \in R, \\ 0 & \text{for} \quad (x, y) \notin R. \end{cases}$$

$$f_1(x) = \begin{cases} \frac{1}{2} \int_{-(1-x)}^{1-x} dy = 1 - x & \text{for} \quad 0 < x < 1. \\ \frac{1}{2} \int_{-(1+x)}^{1+x} dy = 1 + x & \text{for} \quad -1 < x < 0, \\ 0 & \text{for} \quad x < -1 \text{ or } x > 1, \end{cases}$$

or, briefly,

$$f_1(x) = \begin{cases} 1 - |x| & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad |x| > 1. \end{cases}$$

A graph of the probability density function $f_1(x)$ is shown in Fig. 6.9b (Simpson's distribution). Similarly,

$$f_2(y) = \begin{cases} 1 - |y| & \text{for} \quad |y| < 1, \\ 0 & \text{for} \quad |y| > 1. \end{cases}$$

Furthermore, for $|y|<1$, we have

$$f_1(x|y)=\frac{f(x,y)}{f_2(y)}=\begin{cases}\frac{1}{2(1-|y|)} & \text{for } |x|<1-|y|,\\ 0 & \text{for } |x|>1-|y|.\end{cases}$$

A graph of the probability density function $f_1(x \mid y)$ is shown in Fig. 6.9*c*. Similarly, for $|x|<1$, we have

$$f_2(y|x)=\begin{cases}\frac{1}{2(1-|x|)} & \text{for } |y|<1-|x|,\\ 0 & \text{for } |y|>1-|x|.\end{cases}$$

The random variables X and Y are dependent but not correlated.

6.10. The joint probability density of two random variables X and Y is given by the formula

$$f(x,y)=\frac{1}{1.6\pi}\exp\left\{-\frac{1}{1.28}[(x-2)^2-1.2(x-2)(y+3)+(y+3)^2]\right\}.$$

Find the correlation coefficient of the variables X and Y.

Answer. $r_{xy}=0.6$.

6.11. A system of random variables (X, Y) has a distribution with a probability density $f(x, y)$. Express the probabilities of the following events in terms of the probability density $f(x, y)$: (1) $\{X>Y\}$; (2) $\{X>|Y|\}$; (3) $\{|X|>Y\}$ and (4) $\{Y-X>1\}$.

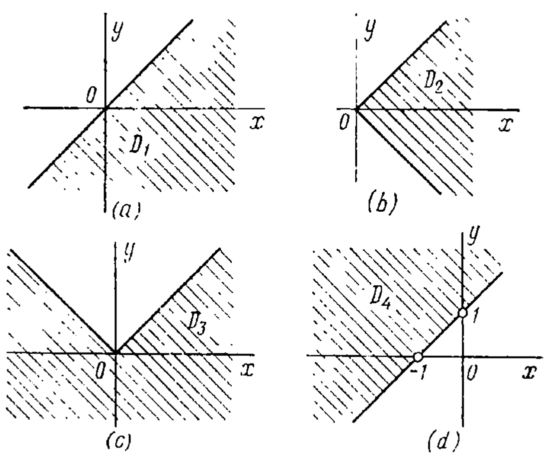


Fig. 6.11

Solution. The domains D_1, D_2, D_3, D_4 corresponding to the occurrence of events 1-4 are hatched in Fig. 6.11*a-d*. The probabilities of points falling in these domains are

$$(1)\ \mathrm{P}\{X>Y\}=\int_{-\infty}^{\infty}\int_{-\infty}^{x} f(x,y)\,dx\,dy,$$

$$(2)\ \mathrm{P}\{X > |Y|\} = \int_{0}^{\infty} \int_{-x}^{x} f(x,\ y)\, dx\, dy,$$

$$(3)\ \mathrm{P}\{|X| > Y\} = \int_{-\infty}^{\infty} \int_{-\infty}^{|x|} f(x,\ y)\, dx\, dy,$$

$$(4)\ \mathrm{P}\{Y - X > 1\} = \int_{-\infty}^{\infty} \int_{x+1}^{\infty} f(x,\ y)\, dx\, dy.$$

6.12. A system of two random variables X and Y has a normal distribution with parameters $m_x = m_y = 0$; $\sigma_x = \sigma_y = \sigma$ and $r_{xy} = 0$.

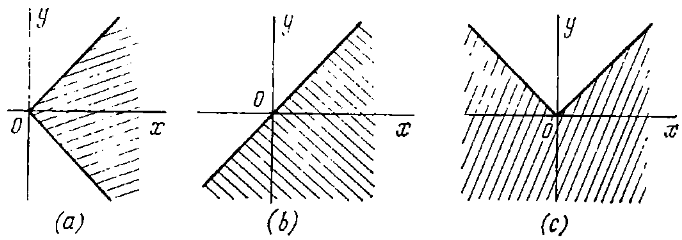


Fig. 6.12

Find the probabilities of

$$A = \{|Y| < X\}, \quad B = \{Y < X\} \quad \text{and} \quad C = \{Y < |X|\}.$$

Solution. Fig. 6.12*a* shows the domains corresponding to the events A, B and C. For circular scattering the probabilities of the events are $\mathrm{P}(A) = 0.25$, $\mathrm{P}(B) = 0.5$, and $\mathrm{P}(C) = 0.75$.

6.13. A random variable X has a probability density function $f(x)$, and a random variable Y is in a functional relation with it: $Y = X^2$. Find the distribution function $F(x,\ y)$ of the system $(X,\ Y)$.

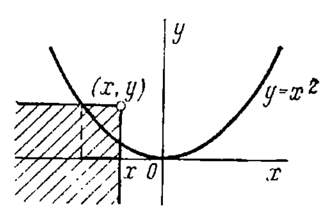


Fig. 6.13

Solution. Since the value of Y is completely defined by the value of X, the random point $(X,\ Y)$ can only lie on the curve $y = x^2$. The probability that it will fall in a quadrant with vertex at the point $(x,\ y)$ is equal to the probability that the random point will fall on the projection onto the x-axis of a section of the curve $y = x^2$ falling in the quadrant (Fig. 6.13). Using this

interpretation, we have

$$F(x, y) = \begin{cases} 0 & \text{for } y \leqslant 0 \text{ or } y > 0 \text{ and } x \leqslant -\sqrt{y}, \\ \int_{-\sqrt{y}}^{\sqrt{y}} f(x)\,dx & \text{for } y > 0 \text{ and } x > \sqrt{y}, \\ \int_{-\sqrt{y}}^{x} f(x)\,dx & \text{for } y > 0 \text{ and } -\sqrt{y} < x \leqslant \sqrt{y}. \end{cases}$$

6.14. A random point (X, Y) has a normal distribution on a plane with parameters $m_x = 1$, $m_y = -1$, $\sigma_x = 1$, $\sigma_y = 2$ and $r_{xy} = 0$. Find the probability that the random point will fall in the domain D bounded by the ellipse $(x-1)^2 + (y+1)^2/4 = 1$.

Solution. The domain D is bounded by the ellipse of scattering E_1 with the semi-axes $a = \sigma_x = 1$ and $b = \sigma_y = 2$, the probability of the point falling in this domain $p = 1 - e^{1/2} \approx 0.393$.

6.15. Shells are fired at a point target and destroy everything within a circle of radius r. The scattering of the points where the shell lands is circular with parameters $m_x = m_y = 0$ and $\sigma_x = \sigma_y = 2r$ (the centre of scattering coincides with the target). How many shells must be fired in order to destroy the target with probability $\mathrm{P} = 0.9$?

Solution. The probability of destroying the target with one shell $p = 1 - e^{-(0.5)^2/2} \approx 0.118$. The number of shots required

$$n \geqslant \log(1 - \mathrm{P})/\log(1 - p) = \log 0.1/\log 0.882 \approx 18.4 \text{ i.e.}$$
$$n = 19.$$

6.16. A system of three random variables (X, Y, Z) has a joint probability density $f(x, y, z)$. Write expressions (1) for the probability density $f_1(x)$ of the random variable X; (2) for the joint probability density $f_{2,3}(y, z)$ of the random variables (Y, Z); (3) for the conditional probability density $f_{2,3}(y, z \mid x)$; (4) for the conditional probability density $f_2(y \mid x, z)$; (5) for the distribution function $F(x, y, z)$, (6) for the distribution function $F_1(x)$ of the random variable X, (7) for the distribution function $F_{1,2}(x, y)$ of the subsystem (X, Y).

Answer.

$$(1) \qquad f_1(x) = \int_{-\infty}^{\infty}\int f(x, y, z)\,dy\,dz,$$

$$(2) \quad f_{2,3}(y, z) = \int_{-\infty}^{\infty} f(x, y, z)\,dx,$$

$$(3)\ f_{2,3}(y, z \mid x) = \frac{f(x, y, z)}{\int_{-\infty}^{\infty}\int f(x, y, z)\,dy\,dz}, \quad (4)\ f_2(y \mid x, z) = \frac{f(x, y, z)}{\int_{-\infty}^{\infty} f(x, y, z)\,dy},$$

(5) $$F(x, y, z) = \int_{-\infty}^{x} \int_{-\infty}^{y} \int_{-\infty}^{z} f(x, y, z)\, dx\, dy\, dz,$$

(6) $$F_1(x) = F(x, \infty, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\, dx\, dy\, dz,$$

(7) $$F_{1,2}(x, y) = F(x, y, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{y} \int_{-\infty}^{\infty} f(x, y, z)\, dx\, dy\, dz.$$

6.17. A shell is fired at a point airborne target. The point where the shell explodes is normally distributed with the centre of scattering at the target. The mean square deviations $\sigma_x = \sigma_y = \sigma_z = \sigma$. The target is destroyed if the distance from it to the point where the shell explodes does not exceed $r_0 = 2\sigma$. Find the probability p that the target will be destroyed with one shell.

Solution. Using formula (6.0.37) for the probability that the shell falls in an ellipsoid of equal distribution density, we have

$$p = \mathrm{P}\{(X, Y, Z) \in E_2\} = 2\Phi(2) - \frac{\sqrt{2}}{\sqrt{\pi}} 2e^{-2} \approx 0.739.$$

6.18. A system of three random variables (X, Y, Z) is distributed with a constant density in the interior of a ball of radius r. Find the probability that the random point (X, Y, Z) will fall inside a concentric ball of radius $r/2$.

Answer. $p = 1/8$.

6.19. A ball is drawn from an urn which contains a white, b black and c red balls. The random variables X, Y, Z are defined by the following conditions:

$$X = \begin{cases} 1 & \text{if a white ball is drawn,} \\ 0, & \text{if a black or a red ball is drawn,} \end{cases}$$

$$Y = \begin{cases} 1, & \text{if a black ball is drawn,} \\ 0, & \text{if a white or a red ball is drawn,} \end{cases}$$

$$Z = \begin{cases} 1, & \text{if a red ball is drawn,} \\ 0, & \text{if a white or a black ball is drawn} \end{cases}$$

(X, Y, Z are the indicators of the events {a white ball}, {a black ball}, {a red ball}). Construct the correlation matrix and normalized correlation matrix for the system of random variables X, Y, Z.

Solution. We find the covariances from the table of probabilities of separate values of X, Y and Z. Using the notation

$$\mathrm{P}_{x_i y_j z_k} = \mathrm{P}\{X = x_i, Y = y_j, Z = z_k\},$$

we have

$$P_{000} = P\{X = 0,\ Y = 0,\ Z = 0\} = 0,$$
$$P_{100} = P\{X = 1,\ Y = 0,\ Z = 0\} = a/(a + b + c),$$
$$P_{010} = P\{X = 0,\ Y = 1,\ Z = 0\} = b/(a + b + c),$$
$$P_{001} = P\{X = 0,\ Y = 0,\ Z = 1\} = c/(a + b + c),$$
$$P_{110} = P_{101} = P_{011} = P_{111} = 0;$$
$$m_x = \frac{a}{a+b+c},\quad m_y = \frac{b}{a+b+c},\quad m_z = \frac{c}{a+b+c},$$

$$\begin{aligned}\mathrm{Cov}_{xy} &= \sum_{i,j,k} (x_i - m_x)(y_j - m_y) P_{x_i y_j z_k}\\ &= \left(1 - \frac{a}{a+b+c}\right)\left(0 - \frac{b}{a+b+c}\right)\frac{a}{a+b+c}\\ &+ \left(0 - \frac{a}{a+b+c}\right)\left(1 - \frac{b}{a+b+c}\right)\frac{b}{a+b+c}\\ &+ \left(0 - \frac{a}{a+b+c}\right)\left(0 - \frac{b}{a+b+c}\right)\frac{c}{a+b+c} = \frac{-ab}{(a+b+c)^2}.\end{aligned}$$

Similarly

$$\mathrm{Cov}_{xz} = \frac{-ac}{(a+b+c)^2},\quad \mathrm{Cov}_{yz} = \frac{-bc}{(a+b+c)^2}.$$

Next we find the variance

$$\mathrm{Var}_x = \alpha_2[X] - m_x^2 = \frac{a}{a+b+c} - \frac{a^2}{(a+b+c)^2} = \frac{a(b+c)}{(a+b+c)^2};$$

similarly

$\mathrm{Var}_y = b(a + c)/(a + b + c)^2$ and $\mathrm{Var}_z = c(a + b)/(a + b + c)^2$.

The correlation matrix is

$$\|\mathrm{Cov}\| = \begin{Vmatrix} \mathrm{Var}_x & \mathrm{Cov}_{xy} & \mathrm{Cov}_{xz} \\ & \mathrm{Var}_y & \mathrm{Cov}_{yz} \\ & & \mathrm{Var}_z \end{Vmatrix}.$$

We find the correlation coefficients:

$$r_{xy} = \frac{\mathrm{Cov}_{xy}}{\sqrt{\mathrm{Var}_x \mathrm{Var}_y}} = \frac{-ab}{\sqrt{ab(a+c)(b+c)}} = -\sqrt{\frac{ab}{(a+c)(b+c)}}.$$

Similarly

$$r_{xz} = -\sqrt{\frac{ac}{(a+b)(c+b)}},\quad r_{yz} = -\sqrt{\frac{bc}{(b+a)(c+a)}}.$$

The normalized correlation matrix is

$$\|r\| = \begin{Vmatrix} 1 & r_{xy} & r_{xz} \\ & 1 & r_{yz} \\ & & 1 \end{Vmatrix}.$$

6.20. We have a system of random variables X and Y. The random variable X has an exponential distribution with parameter λ, i.e. $f_1(x) = \lambda e^{-\lambda x}$ for $x > 0$. For a given value $X = x > 0$ the random variable Y also has an exponential distribution, but with parameter x, i.e. $f_2(y \mid x) = xe^{-xy}$ for $y > 0$. Write the joint probability density function $f(x, y)$ for X and Y, find the probability density $f_2(y)$ of Y, and find the conditional probability density $f_1(x \mid y)$.

Solution.

$$f(x, y) = \begin{cases} \lambda x e^{-(\lambda+y)x} & \text{for } x > 0 \text{ and } y > 0, \\ 0 & \text{for } x < 0 \text{ or } y < 0, \end{cases}$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x \quad y)\,dx = \begin{cases} \dfrac{\lambda}{(\lambda+y)^2} & \text{for } y > 0 \\ 0 & \text{for } y < 0 \end{cases}$$

Furthermore, for $y > 0$, we have

$$f_1(x \mid y) = \frac{f(x, y)}{f_2(y)} = \begin{cases} x(\lambda+y)^2 e^{-(\lambda+y)x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

6.21. Given two independent random variables: a continuous random variable X with distribution density $f_1(x)$ and a discrete random variable Y with the values $y_1, y_2, \ldots, y_n$ which have the probabilities $p_1, p_2, \ldots, p_n$. Find the probability distribution function of the system of variables X, Y.

Answer. $F(x, y) = F_1(x)F_2(y)$, where $F_1(x) = \int_0^x f_1(x)\,dx$.

$$F_2(y) = \begin{cases} 0 & \text{for } y \leqslant y_1, \\ p_1 & \text{for } y_1 < y \leqslant y_2, \\ \ldots & \\ \sum_{i=1}^{k-1} p_i & \text{for } y_{k-1} < y < y_k \quad (k = 2, 3, \ldots, n), \\ \ldots\ldots & \\ 1 & \text{for } y > y_n \end{cases}$$

6.22. X is a discrete random variable with two values x_1 and x_2 ($x_2 > x_1$) which have probabilities p_1 and p_2. A random variable Y is continuous, its conditional distribution for $X = x_i$ being normal with mean value x_i and mean square deviation σ ($i = 1, 2$).

Find the joint probability distribution function $F(x, y)$ of the random variables X and Y. Find the probability density function $f_2(y)$ of the random variable Y.

Solution. $F(x, y) = \mathrm{P}\{X < x\}\ \mathrm{P}\{Y < y \mid X < x\}$. If $x \leqslant x_1$, then $\mathrm{P}\{X < x\} = 0$ and $F(x, y) = 0$, assuming $x_1 < x \leqslant x_2$, we have $\mathrm{P}\{X < x\} = p_1$ and $F(x, y) = p_1 \cdot \mathrm{P}\{Y < \mid X = x_1\} = p_1\left[\Phi\left(\frac{y - x_1}{\sigma}\right)\right.$

$+0.5\Big]$. Using the total probability formula, we have for $x > x_2$

$$F(x, y) = p_1\left[\Phi\left(\frac{y-x_1}{\sigma}\right)+0.5\right]+p_2\left[\Phi\left(\frac{y-x_2}{\sigma}\right)+0.5\right].$$

Consequently

$$F(x, y) = \begin{cases} 0 & \text{for } x \leqslant x_1, \\ p_1\left[\Phi\left(\frac{y-x_1}{\sigma}\right)+0.5\right] & \text{for } x_1 < x \leqslant x_2, \\ p_1\left[\Phi\left(\frac{y-x_1}{\sigma}\right)+0.5\right]+p_2\left[\varphi\left(\frac{y-x_2}{\sigma}\right)+0.5\right] & \text{for } x > x_2. \end{cases}$$

Furthermore, setting $x = \infty$ and differentiating with respect to y, we obtain

$$f_2(y) = \frac{d}{dy} F(\infty, y) = \frac{1}{\sigma\sqrt{2\pi}}\left[p_1 e^{-\frac{(y-x_1)^2}{2\sigma^2}} + p_2 e^{-\frac{(y-x_2)^2}{2\sigma^2}}\right].$$

6.23*. The stars in the sky can be regarded as a Poisson field of points. The number of stars covered by a telescope objective lens is a random variable which has a Poisson distribution with parameter λs, where s is the area of the portion of the surface of a unit sphere that can be observed through the telescope (Fig. 6.23*a*). The view field of the telescope has a coordinate grid (Fig. 6.23*b*). Show that for an arbitrary alignment of the telescope the coordinates (X, Y) of the star nearest to the cross-hairs have a normal distribution with parameters $m_x = m_y = 0$ and $\sigma_x = \sigma_y = 1/\sqrt{\pi 2\lambda}$.

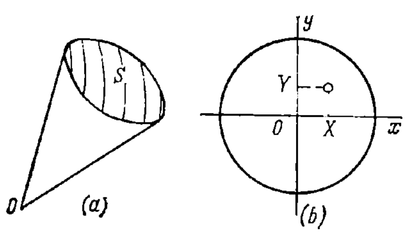


Fig. 6.23

Solution. We showed in Problem 5.19 that the distance R from the centre of the cross-hairs to the nearest point in the Poisson field had a Rayleigh distribution. But $R = \sqrt{X^2 + Y^2}$ and, consequently, the probability that the point (X, Y) will fall in a circle D of radius r, i.e. $P\{X^2 + Y^2 < r^2\}$, can be written in two forms, either

$$P\{R < r\} = \int_0^r 2\pi\lambda r e^{-\pi\lambda r^2}\, dr$$

or $\qquad\qquad$ (6.23.1)

$$P\{(X, Y) \in D\} = \iint_{(D)} f(x, y)\, dx\, dy,$$

where $f(x, y)$ is the joint probability density of the variables X, Y. By virtue of symmetry, we must assume that $f(x, y)$ depends only on the distance, i.e. $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. If we use polar coordinates (r, φ), we obtain

$$P\{(X, Y) \in D\} = \int_0^{2\pi} d\varphi \int_0^r rg(r)\, dr = 2\pi \int_0^r rg(r)\, dr. \qquad (6.23.2)$$

Comparing expressions (6.23.1) and (6.23.2), we find that $g(r) = \lambda e^{-\pi\lambda r^2}$ and, hence, $f(x, y) = \lambda e^{-\pi\lambda(x^2 + y^2)}$, and that is what we wished to prove.

6.24. A source of α-particles is at the origin of a spherical system of coordinates (r, φ, ϑ) (Fig. 6.24), where $0 \leqslant \varphi \leqslant 2\pi$, $-\pi/2 \leqslant \vartheta \leqslant \pi/2$. The particles scatter uniformly in all directions. We consider a particle which moves in a random direction defined by the angles Φ and Θ. Write the joint probability density function $f(\varphi, \vartheta)$ of the random variables Φ and Θ.

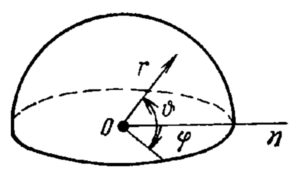


Fig. 6.24

Solution. If the α-particles are scattered uniformly in all directions, then a unit vector **e** from the origin defines the direction of the particle and all its possible endpoints on the sphere C of unit radius will have the same probability density. Consequently, the element of probability $f(\varphi, \vartheta)\, d\varphi\, d\vartheta$ must be proportional to the surface element ds on the sphere C. The surface element is $dS = d\varphi\, d\vartheta \cos\vartheta$ whence

$$f(\varphi, \vartheta)\, d\varphi\, d\vartheta = A \cos\vartheta\, d\varphi\, d\vartheta; \; f(\varphi, \vartheta) = A \cos\vartheta,$$

where A is the proportionality factor, which can be determined from the condition

$$\int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} f(\varphi, \vartheta)\, d\vartheta = 1,$$

and hence $A = 1/(4\pi)$. Thus

$$f(\varphi, \vartheta) = \frac{1}{4\pi} \cos\vartheta \;\text{ for }\; \begin{cases} 0 \leqslant \varphi \leqslant 2\pi \\ -\pi/2 \leqslant \vartheta \leqslant \pi/2. \end{cases}$$

6.25*. On the hypothesis of the preceding problem we consider a plane P which is parallel to the equatorial plane (in which the angle φ is read off) and passes through a point O' which has spherical coordinates $r = 1$, $\varphi = 0$, $\vartheta = \pi/2$ (Fig. 6.25). The plane has a system cl Cartesian coordinates $xO'y$. All the α-particles which fly in the upper hemisphere of the scatter get onto the plane P. We consider one of these particles and the corresponding random variables which are the coordinates X, Y of the point at which the α-particle falls on the plane P. Find the joint

probability density function $f(x, y)$ of these random variables. Are random variables X and Y mutually dependent?

Solution. We seek the probability element $f(x, y)\,dx\,dy$, which is approximately equal to the probability that the particle will fall in the surface element $dx\,dy$ adjoining the point (x, y). We calculate this probability as we did in the preceding problem. We find the area ds of the portion of the unit sphere C such that the particles passing it fall in the surface element $dx\,dy$. This portion is the central projection of the surface element $dx\,dy$ onto the sphere C. The projection is found by multiplying the surface element by the cosine of the angle ϑ between the direction of projecting and the plane P. In addition, the surface decreases in inverse proportion to the square of the distance R from the centre of projection. The surface element ds on the sphere C

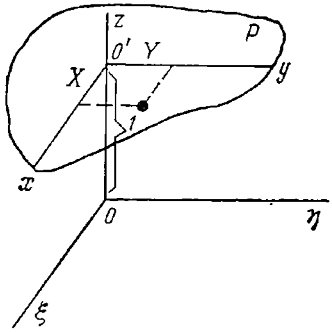


Fig. 6.25

$$ds = \frac{dx\,dy\cos\vartheta}{R^2} = \frac{dx\,dy\,\frac{1}{\sqrt{1+x^2+y^2}}}{1+x^2+y^2} = \frac{dx\,dy}{(1+x^2+y^2)^{3/2}}.$$

To obtain the probability element, we must divide ds by the area of the whole upper hemisphere, which is 2π. We obtain

$$f(x, y)\,dx\,dy = \frac{dx\,dy}{2\pi(1+x^2+y^2)^{3/2}} \quad \text{or} \quad f(x, y) = \frac{1}{2\pi(1+x^2+y^2)^{3/2}}$$

The random variables X and Y are mutually dependent since their joint probability density function does not decompose into the product of two functions, one of which depends only on x and the other only on y.

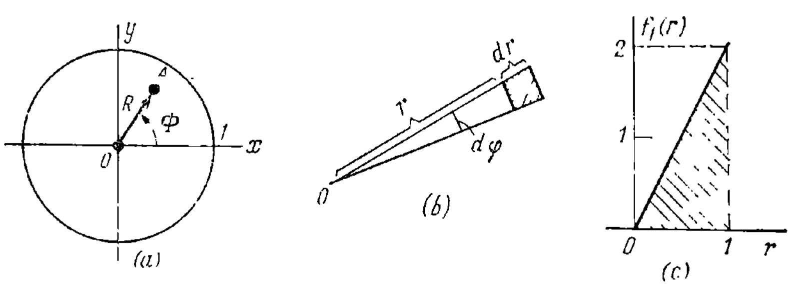


Fig. 6.26

6.26. A random point A represents an object on a circular radar screen of unit radius (Fig. 6.26*a*) and has a uniform distribution within that circle. Find the joint probability density function $f(r, \varphi)$ of the polar coordinates (R, Φ) of the point A. Are the random variables R and Φ dependent?

Solution. Let us consider, in polar coordinates, an elementary "rectangle", which corresponds to infinitesimal increments dr, $d\varphi$ of the polar coordinates r, φ of the point inside the circle (Fig. 6.26*b*). Its area (to within higher-order infinitesimals) is $r\,dr\,d\varphi$. Dividing this by the area of the circle, which is equal to π, we get the probability element

$$f(r, \varphi)\,dr\,d\varphi = r\,dr\,d\varphi/\pi,$$

whence

$$f(r, \varphi) = r/\pi \quad \text{for} \quad 0 < r < 1, \quad 0 < \varphi < 2\pi. \qquad (6.26.1)$$

We find the probability density function $f_1(r)$ by integrating (6.26.1) in the whole range of variation of φ, i.e.

$$f_1(r) = \int_0^{2\pi} \frac{r}{\pi}\,d\varphi = 2r \quad \text{for} \quad 0 < r < 1.$$

Similarly

$$f_2(\varphi) = \int_0^1 \frac{r}{\pi}\,dr = \frac{1}{2\pi} \quad \text{for} \quad 0 < \varphi < 2\pi.$$

The graph of the distribution $f_1(r)$ is a right triangle (Fig.6. 26*c*), the graph of the distribution $f_2(\varphi)$ is uniform over $(0, 2\pi)$.

Multiplying $f_1(r)$ by $f_2(\varphi)$, we get the joint probability density $f(r, \varphi)$ of the system (R, Φ), and consequently, R and Φ are independent. Note that for the same uniform distribution, the Cartesian coordinates X, Y of a point in the interior of a circle are mutually dependent. We proved this while solving Problem 6.8.

6.27. On the hypothesis of the preceding problem, the radius of the screen is not unity but a. Write the expression for the joint probability density of the polar coordinates of the point A and their separate densities.

Solution. Dividing the area of the elementary "rectangle" by that of the screen πa^2 and cancelling by $dr\,d\varphi$, we obtain

$$f(r, \varphi) = \frac{r}{\pi a^2} \quad \text{for} \quad 0 < r < a, \quad 0 < \varphi < 2\pi$$

$$f_1(r) = \frac{2}{a^2} \quad \text{for} \quad 0 < r < a,$$

$$f_2(\varphi) = \frac{1}{2\pi} \quad \text{for} \quad 0 < \varphi < 2\pi.$$

CHAPTER 7

Numerical Characteristics of Functions of Random Variables

7.0. One of the most efficient means of solving probability problems is to use numerical characteristics. The characteristics of random variables of interest can then be found without involving their distributions. In particular, it is not necessary to know the distributions of the functions of random variables in order to find their characteristics. It is sufficient to know the distributions of the argument.

If X is a discrete random variable with an ordered series

$$X: \left|\begin{array}{c|c|c|c} x_1 & x_2 & \dots & x_n \\ \hline p_1 & p_2 & \dots & p_n \end{array}\right| \left(\sum_{i=1}^{n} p_i = 1\right)$$

and the variables Y and X are in a functional relationship $Y = \varphi(X)$, then the mean value of the variable Y is

$$m_y = \mathrm{M}[\varphi(X)] = \sum_{i=1}^{n} \varphi(x_i) p_i, \tag{7.0.1}$$

and the variance can be expressed by one of two formulas, either

$$\mathrm{Var}_y = \mathrm{Var}[\varphi(X)] = \sum_{i=1}^{n} [\varphi(x_i) - m_y]^2 p_i, \tag{7.0.2}$$

or

$$\mathrm{Var}_y = \sum_{i=1}^{n} [\varphi(x_i)]^2 p_i - m_y^2. \tag{7.0.3}$$

If (X, Y) is a system of discrete random variables whose distribution is characterized by the probabilities

$$p_{ij} = \mathrm{P}\{X = x_i,\ Y = y_j\},$$

and $Z = \varphi(X, Y)$, then the mean value of Z is

$$m_z = \mathrm{M}[\varphi(X, Y)] = \sum_i \sum_j \varphi(x_i, y_j) p_{ij}, \tag{7.0.4}$$

and the variance can be expressed by one of the two formulas, either

$$\mathrm{Var}_y = \mathrm{Var}[\varphi(X, Y)] = \sum_i \sum_j [\varphi(x_i, y_j) - m_z]^2 p_{ij}, \tag{7.0.5}$$

or

$$\mathrm{Var}_y = \sum_i \sum_j [\varphi(x_i, y_j)]^2 p_{ij} - m_z^2. \tag{7.0.6}$$

If X is a continuous random variable with probability density $f(x)$, and $Y = \varphi(X)$, then the mean value of Y is

$$m_y = \mathrm{M}[\varphi(X)] = \int_{-\infty}^{\infty} \varphi(x) f(x)\, dx, \tag{7.0.7}$$

and the variance can be expressed by one of the two formulas, either

$$\mathrm{Var}_y = \mathrm{Var}\,[\varphi(X)] = \int_{-\infty}^{\infty} [\varphi(x) - m_y]^2 f(x)\,dx, \tag{7.0.8}$$

or

$$\mathrm{Var}_y = \int_{-\infty}^{\infty} [\varphi(x)]^2 f(x)\,dx - m_y^2. \tag{7.0.9}$$

If (X, Y) is a system of continuous random variables with the joint probability density $f(x, y)$, and $Z = \varphi(X, Y)$, then the mean value of Z is

$$m_z = \mathrm{M}\,[\varphi(X, Y)] = \int_{-\infty}^{\infty}\int \varphi(x, y) f(x, y)\,dx\,dy \tag{7.0.10}$$

and the variance can be expressed by one of the two formulas, either

$$\mathrm{Var}_z = \mathrm{Var}\,[\varphi(X, Y)] = \int_{-\infty}^{\infty}\int [\varphi(x, y) - m_z]^2 f(x, y)\,dx\,dy, \tag{7.0.11}$$

or

$$\mathrm{Var}_z = \int_{-\infty}^{\infty}\int [\varphi(x, y)]^2 f(x, y)\,dx\,dy - m_z^2. \tag{7.0.12}$$

If $(X_1, \ldots, X_n)$ is a system of n continuous random variables with density $f(x_1, \ldots, x_n)$, and $Y = \varphi(X_1, \ldots, X_n)$, then the mean value of Y is

$$\begin{aligned} m_y &= \mathrm{M}\,[\varphi(X_1, \ldots, X_n)] \\ &= \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} \varphi(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\,dx_1 \quad dx_n, \end{aligned} \tag{7.0.13}$$

and the variance can be expressed by one of the two formulas, either

$$\begin{aligned} \mathrm{Var}_y &= \mathrm{Var}\,[\varphi(X_1, \ldots, X_n)] \\ &= \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} [\varphi(x_1, \ldots, x_n) - m_y]^2 f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n, \end{aligned} \tag{7.0.14}$$

or

$$\mathrm{Var}_y = \int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} [\varphi(x_1, \ldots, x_n)]^2 f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n - m_y^2. \tag{7.0.15}$$

In some cases, we do not even need to know the distributions of the arguments but only their characteristics in order to get the characteristics of functions. Here are the fundamental theorems on characteristics.

1. If c is a nonrandom variable, then

$$\mathrm{M}\,[c] = c, \quad \mathrm{Var}\,[c] = 0. \tag{7.0.16}$$

2. If c is a nonrandom variable and X is a random variable, then

$$\mathrm{M}\,[cX] = c\mathrm{M}\,[X] \quad \mathrm{Var}\,[cX] = c^2\,\mathrm{Var}\,[X]. \tag{7.0.17}$$

3. The addition theorem for mean values. *The mean value of a sum of random variables is equal to the sum of their mean values.*

$$\mathrm{M}[X+Y]=\mathrm{M}[X]+\mathrm{M}[Y], \tag{7.0.18}$$

and, in general,

$$\mathrm{M}\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n}\mathrm{M}[X_i]. \tag{7.0.19}$$

4 The mean value of a linear function of several random variables

$$Y=\sum_{i=1}^{n} a_i X_i+b,$$

where a_i and b are nonrandom coefficients, is equal to the same linear function of their mean values:

$$m_y=\mathrm{M}\left[\sum_{i=1}^{n} a_i X_i+b\right]=\sum_{i=1}^{n} a_i m_{x_i}+b \tag{7.0.20}$$

where $m_{x_i}=\mathrm{M}[X_i]$ This rule can be written in a concise form viz.

$$\mathrm{M}[L(X_1, X_2, \ldots, X_n)]=L(m_{x_1}, m_{x_2}, \ldots m_{x_n}), \tag{7.0.21}$$

where L is a linear function.

5. The mean value of the product of two random variables X and Y is expressed by the formula

$$\mathrm{M}[XY]=\mathrm{M}[X]\,\mathrm{M}[Y]+\mathrm{Cov}_{xy}, \tag{7.0.22}$$

where Cov_{xy} is the covariance of the variables X and Y This formula can be rewritten as follows:

$$\mathrm{Cov}_{xy}=\mathrm{M}[XY]-m_x m_y, \tag{7.0.23}$$

or bearing in mind that $\mathrm{M}[XY]=\alpha_{11}[X, Y]$, as

$$\mathrm{Cov}_{xy}=\alpha_{11}[X, Y]-m_x m_y. \tag{7.0.24}$$

6. The multiplication theorem for mean values. *The mean value of the product of two uncorrelated random variables X, Y is equal to the product of their mean values*:

$$\mathrm{M}[XY]=\mathrm{M}[X]\,\mathrm{M}[Y]. \tag{7.0.25}$$

7. If $X_1, X_2, \ldots, X_n$ are independent random variables, then the mean value of their product is equal to the product of their mean values:

$$\mathrm{M}\left[\prod_{i=1}^{n} X_i\right]=\prod_{i=1}^{n}\mathrm{M}[X_i] \tag{7.0.26}$$

8. The variance of the sum of two random variables is expressed by the formula

$$\mathrm{Var}[X+Y]=\mathrm{Var}[X]+\mathrm{Var}[Y]+2\mathrm{Cov}_{xy}. \tag{7.0.27}$$

9. The variance of the sum of several random variables is expressed by the formula

$$\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n}\mathrm{Var}[X_i]+2\sum_{i<j}\mathrm{Cov}\, x_i x_j, \tag{7.0.28}$$

where $\mathrm{Cov}_{x_i x_j}$ is the covariance of the random variables X_i and X_j.

10. The addition theorem for variances. *The variance of the sum of two uncorrelated random variables X and Y is equal to the sum of their variances*:

$$\operatorname{Var}[X+Y]=\operatorname{Var}[X]+\operatorname{Var}[Y], \tag{7.0.29}$$

and, in general, for uncorrelated random variables

$$\operatorname{Var}\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n} \operatorname{Var}[X_i]. \tag{7.0.30}$$

11. *The variance of a linear function of several randon variables*, i.e.

$$Y=\sum_{i=1}^{n} a_i X_i+b,$$

where a_i and b are nonrandom variables, is expressed by the formula

$$\operatorname{Var}_y=\operatorname{Var}\left[\sum_{i=1}^{n} a_i X_i+b\right]=\sum_{i=1}^{n} a_i^2 \operatorname{Var}[X_i]+2\sum_{i<j} a_i a_j \operatorname{Cov}_{x_i x_j}. \tag{7.0.31}$$

When the variables $X_1, X_2, \ldots, X_n$ are uncorrelated, we have

$$\operatorname{Var}_y=\operatorname{Var}\left[\sum_{i=1}^{n} a_i X_i+b\right]=\sum_{i=1}^{n} a_i^2 \operatorname{Var}[X_i], \tag{7.0.32}$$

12. *When several uncorrelated random vectors are added, their covariances are added*, e.g. if

$$X=X_1+X_2,\quad Y=Y_1+Y_2;\quad \operatorname{Cov}_{x_i x_j}=\operatorname{Cov}_{x_1 y_2}=\operatorname{Cov}_{y_1 y_2}=\operatorname{Cov}_{y_1 x_2}=0,$$

then

$$\operatorname{Cov}_{xy}=\operatorname{Cov}_{x_1 y_1}+\operatorname{Cov}_{x_2 y_2}. \tag{7.0.33}$$

Linearization of functions. The function $\varphi(X_1, X_2, \ldots, X_n)$ of several random arguments $X_1, X_2, \ldots, X_n$ is said to be *almost linear* if it can be linearized (approximated by a linear function) with an accuracy sufficient for applications in the whole range of the practically possible values of the arguments. This means that

$$\varphi(X_1, X_2, \ldots, X_n)\approx\varphi(m_{x_1}, m_{x_2}, \ldots, m_{x_n})+\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i}\right)_m (X_i-m_{x_i}), \tag{7.0.34}$$

where

$$\left(\frac{\partial\varphi}{\partial x_i}\right)_m=\frac{\partial\varphi(m_{x_1}, m_{x_2}, \ldots, m_{x_n})}{\partial x_i}$$

is the partial derivative of the function $\varphi(x_1, x_2, \ldots, x_n)$ with respect to the argument x_i in which each argument is replaced by its mean value.

The *mean value of the almost linear function* $Y=\varphi(X_1, X_2, \ldots, X_n)$ can be approximated by

$$m_y\approx\varphi(m_{x_1}, m_{x_2}, \ldots, m_{x_n}). \tag{7.0.35}$$

The *variance of an almost linear function* can be approximated by

$$\operatorname{Var}_y=\sum_{i=1}^{n}\left(\frac{\partial\varphi}{\partial x_i}\right)_m^2 \operatorname{Var}_{x_i}+2\sum_{i<j}\left(\frac{\partial\varphi}{\partial x_i}\right)_m\left(\frac{\partial\varphi}{\partial x_j}\right)_m \operatorname{Cov}_{x_i x_j} \tag{7.0.36}$$

where Var_{x_i} is the variance of the random variable X_i and $\mathrm{Cov}_{x_i x_j}$ is the covariance of the random variables X_i and X_j.

When the random arguments $X_1, X_2, \ldots, X_n$ are uncorrelated, then

$$\mathrm{Var}_y = \sum_{i=1}^{n} \left(\frac{\partial \varphi}{\partial x_i}\right)_m^2 \mathrm{Var}_{x_i}. \tag{7.0.37}$$

Problems and Exercises

7.1. The discrete random variable X has an ordered series

$$X: \left| \begin{array}{c|c|c|c} -1 & 0 & 1 & 2 \\ \hline 0.2 & 0.1 & 0.3 & 0.4 \end{array} \right|.$$

Find the mean value and variance of the random variable $Y = 2^X$.

Solution. $m_y = 2^{-1} \times 0.2 + 2^0 \times 0.1 + 2^1 \times 0.3 + 2^2 \times 0.4 = 2.4$ and $\mathrm{Var}_y = \alpha_2[Y] - m_y^2 = (2^{-1})^2 \times 0.2 + (2^0)^2 \times 0.1 + (2^1)^2 \times 0.3 + (2^2)^2 \times 0.4 - 2.4^2 = 1.99$.

7.2. A continuous random variable X is distributed in the interval $(0, 1)$ with density $f(x) = 2x$ for $x \in (0, 1)$ (Fig. 7.2). Find the mean value and variance of the square of this random variable $Y = X^2$.

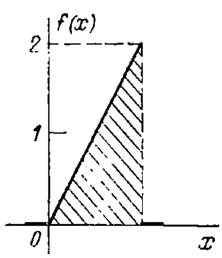


Fig. 7.2

Solution.

$$m_y = \alpha_2[X] = \int_0^1 x^2\, 2x\, dx = \frac{1}{2},$$

$$\mathrm{Var}_y = \alpha_2[Y] - m_y^2 = \int_0^1 (x^2)^2\, 2x\, dx - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.$$

7.3. A random variable X has an exponential distribution with probability density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ $(\lambda > 0)$. Find the mean value and variance of the random variable $Y = e^{-X}$.

Solution.

$$m_y = \int_0^\infty e^{-x} \lambda e^{-\lambda x}\, dx = \frac{\lambda}{\lambda+1}.$$

$$\mathrm{Var}_y = \alpha_2[Y] - m_y^2 = \int_0^\infty e^{-2x} \lambda e^{-\lambda x} dx - \left(\frac{\lambda}{\lambda+1}\right)^2 = \frac{\lambda}{(\lambda+2)(\lambda+1)}.$$

7.4. A random variable X has an exponential distribution with probability density $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Determine the conditions under which the mean value and variance of the random variable $Y = e^X$ exist and find what they are equal to.

Solution.

$$m_y = \int_0^\infty e^x \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda-1)x}\,dx, \text{ for } \lambda - 1 > 0,$$

i.e. for $\lambda > 1$ this integral exists and is $m_y = \lambda/(\lambda-1)$, for $\lambda \leqslant 1$ it diverges, i.e.

$$\alpha_2[Y] = \int_0^\infty e^{2x}\lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{-(\lambda-2)x}\,dx$$

For $\lambda > 2$ this integral exists and is $\lambda/(\lambda-2)$ and the variance $\mathrm{Var}_y = \lambda/(\lambda-2) - [\lambda/(\lambda-1)]^2$; for $\lambda \leqslant 2$ the integral diverges and the variance Var_y does not exist.

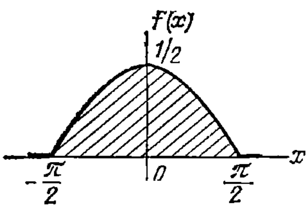


Fig. 7.5

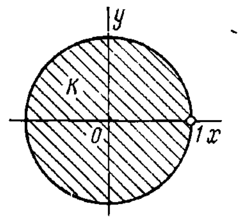


Fig. 7.7

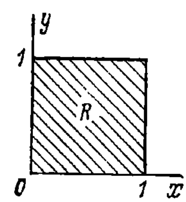


Fig. 7.8

7.5. A random variable X is distributed with probability density $f(x) = 0.5 \cos x$ for $x \in (-\pi/2, \pi/2)$ (Fig. 7.5). Find the mean value and variance of the random variable $Y = \sin X$.

Solution.

$$m_y = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = 0$$

$$\mathrm{Var}_y = \alpha_2[Y] = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx = \frac{1}{3}$$

7.6. A random variable X has the same distribution as in the preceding problem. Find the mean value and variance of the random variable $Y = |\sin X|$.

Solution.

$$m_y = \frac{1}{2} \int_{-\pi/2}^{\pi/2} |\sin x| \cos x\,dx = \int_0^{\pi/2} \sin x \cos x\,dx = \frac{1}{2},$$

$$\alpha_2[Y] = \frac{1}{2} \int_{-\pi/2}^{\pi/2} |\sin x|^2 \cos x\,dx = \int_0^{\pi/2} \sin^2 x \cos x\,dx = \frac{1}{3},$$

$$\mathrm{Var}_y = \alpha_2[Y] - m_y^2 = \frac{1}{12}.$$

7.7. A random point (X, Y) is uniformly distributed in the interior of a circle K of radius $r = 1$ (Fig. 7.7). Find the mean value and variance of the random variable $Z = XY$

Solution.

$$f(x, y) = 1/\pi \quad \text{for } (x, y) \in K,$$

$$m_z = \frac{1}{\pi} \iint_{(K)} xy\, dx\, dy = 0$$

$$\mathrm{Var}_z = \frac{1}{\pi} \iint_{(K)} x^2 y^2\, dx\, dy$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\varphi \int_0^1 r^5 \cos^2\varphi \sin^2\varphi\, dr = \frac{1}{24}.$$

7.8. A random point (X, Y) is uniformly distributed in the interior of a square R (Fig. 7.8). Find the mean value and variance of the random variable $Z = XY$.

Solution. Since the random variables X and Y are mutually independent, we have

$$m_z = m_x m_y = (1/2)(1/2) = 1/4,$$

$$\mathrm{Var}_z = \alpha_2 [Z] - m_z^2 = \mathrm{M}[(XY)^2] - m_z^2 = \mathrm{M}[X^2]\,\mathrm{M}[Y^2] - m_z^2,$$

$$\mathrm{M}[X^2] = \alpha_2 [X] = 1/3, \quad \mathrm{M}[Y^2] = 1/3, \quad \mathrm{Var}_z = 7/144.$$

7.9. Two random variables X and Y are related as $Y = 2 - 3X$. The numerical characteristics of the variable X are given, i.e. $m_x = -1$ and $\mathrm{Var}_x = 4$. Find: (1) the mean value and variance of the variable Y, and (2) the covariance and correlation coefficient of the variables X and Y.

Solution.

(1) $\quad m_y = 2 - 3m_x = 5, \quad \mathrm{Var}_y = (-3)^2 \times 4 = 36$

(2) $\mathrm{Cov}_{xy} = \mathrm{M}[XY] - m_x m_y = \mathrm{M}[X(2 - 3X)] + 1 \times 5$

$$= 2\mathrm{M}[X] - 3\mathrm{M}[X^2] + 5,$$

$$\mathrm{M}[X^2] = \alpha_2 [X] = \mathrm{Var}_x + m_x^2 = 4 + 1 = 5,$$

whence $\mathrm{Cov}_{xy} = -2 - 3 \times 5 + 5 = -12$;

$$r_{xy} = -12/(\sigma_x \sigma_y) = -12/\sqrt{4 \times 36} = -1,$$

which is quite natural since X and Y are in a linear functional relationship with a negative coefficient in X.

7.10. A system of random variables (X, Y, Z) has the mean values m_x, m_y and m_z and the correlation matrix

$$\begin{Vmatrix} \mathrm{Var}_x & \mathrm{Cov}_{xy} & \mathrm{Cov}_{xz} \\ & \mathrm{Var}_y & \mathrm{Cov}_{yz} \\ & & \mathrm{Var}_z \end{Vmatrix}.$$

Find the mean value and variance of the random variable $U = aX - bY + cZ - d$.

Answer. $m_u = am_x - bm_y + cm_z - d$, $\mathrm{Var}_u = a^2 \mathrm{Var}_x + b^2 \mathrm{Var}_y + c^2 \mathrm{Var}_z - 2ab \mathrm{Cov}_{xy} + 2ac \mathrm{Cov}_{xz} - 2bc \mathrm{Cov}_{yz}$.

7.11. An n-dimensional random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is made up of n random variables X_i with mean values m_{xi} $(i = 1, \ldots, n)$ and variances Var_{x_i} $(i = 1, \ldots, n)$, and has a normalized correlation matrix $\| r_{x_i x_j} \|$ $(i = 1, 2, \ldots, n,\ j > i)$. The random vector $\mathbf{X}$ is transformed into an m-dimensional random vector $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_m)$, the components of the vector $\mathbf{Y}$ resulting from the components of the vector $\mathbf{X}$ upon linear transformations:

$$Y_k = \sum_{i=1}^{n} a_{ik} X_i + b_k \quad (k = 1, 2, \ldots, m).$$

Find the characteristics of the random vector $\mathbf{Y}$, i.e. the mean value m_{y_k} $(k = 1, \ldots, m)$, the variance Var_{y_k} $(k = 1, \ldots, m)$ and the elements of the normalized cor·elation matrix $\| r_{y_k y_l} \|$ $(l = 1, 2, \ldots, m;\ k < l)$.

Answer.

$$m_{y_k} = \sum_{i=1}^{n} a_{ik} m_{x_i} + b_k \quad (k = 1, \ldots, m),$$

$$\mathrm{Var}_{y_k} = \sum_{i=1}^{n} a_{ik}^2 \mathrm{Var}_{x_i} + 2 \sum_{i<j} r_{x_i x_j} \sqrt{\mathrm{Var}_{x_i} \mathrm{Var}_{x_j}},$$

$$r_{y_k y_l} = \mathrm{Cov}_{y_k y_l} / \sqrt{\mathrm{Var}_{y_k} \mathrm{Var}_{y_l}}$$

where

$$\mathrm{Cov}_{y_k y_l} = \sum_{i=1}^{n} a_{ik} a_{il} \mathrm{Var}_{x_i} + \sum_{i<j} (a_{ik} a_l + a_{jk} a_{il}) r_{x_i x_j} \sqrt{\mathrm{Var}_{x_i} \mathrm{Var}_{x_j}}.$$

7.12. There are two mutually independent random variables X and Y. The variable X has a normal distribution $f_1(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-1)^2}{8}}$ and the variable Y has a uniform distribution in the interval $(0, 2)$. Find (1) $M[X + Y]$, (2) $M[XY]$, (3) $M[X^2]$, (4) $M[X - Y^2]$, (5) $\mathrm{Var}[X + Y]$ and (6) $\mathrm{Var}[X - Y]$.

Solution

(1) $M[X + Y] = M[X] + M[Y] = 1 + 1 = 2,$

(2) $M[XY] = M[X] M[Y] = 1 \times 1 = 1,$

(3) $M[X^2] = \alpha_2[X] = \mathrm{Var}[X] + m_x^2 = 4 + 1 = 5,$

(4) $M[X - Y^2] = M[X] - M[Y^2] = 1 - \alpha_2[Y] = -1/3,$

(5) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 4 + 1/3 = 13/3,$

(6) $\mathrm{Var}[X - Y] = \mathrm{Var}[X] + (-1)^2 \mathrm{Var}[Y] = 13/3.$

7.13. A random variable X has a normal distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$. Find the mean value of the random variable $Y = 1 - 3X^2 + 4X^3$.

Solution. $m_y = \mathrm{M}[1 - 3X^2 + 4X^3] = 1 - 3\mathrm{M}[X^2] + 4\mathrm{M}[X^3]$. Since $m_x = 0$, it follows that $\mathrm{M}[X^2] = \sigma^2$; for our normal distribution $\mathrm{M}[X^3] = 0$, hence $m_y = 1 - 3\sigma^2$.

7.14. The independent random variables X and Y have distributions $f_1(x)$ and $f_2(y)$ the graphs of whose probability density functions are shown in Fig. 7.14*a*, *b*. Find (1) $\mathrm{M}[X+Y]$, (2) $\mathrm{Var}[3X - 6Y + 1]$, (3) $\mathrm{M}[XY]$, (4) $\mathrm{M}[2XY - 3X^2 + Y^2 - 1]$.

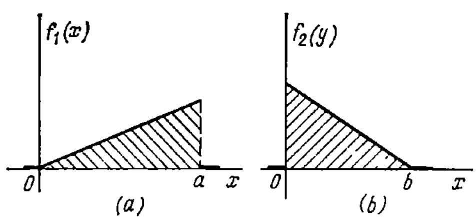


Fig. 7.14

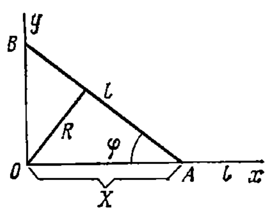


Fig. 7.16

Solution. $\mathrm{M}[X] = 2a/3$, $\mathrm{M}[Y] = b/3$, $\mathrm{Var}[X] = a^2/18$, $\mathrm{Var}[Y] = b^2/18$.

$$\begin{aligned}
&(1)\ \mathrm{M}[X+Y] &&= (2a+b)/3,\\
&(2)\ \mathrm{Var}[3X - 6Y + 1] &&= 9\,\mathrm{Var}_x + 36\,\mathrm{Var}_y = a^2/2 + 2b^2,\\
&(3)\ \mathrm{M}[XY] &&= 2ab/9,\\
&(4)\ \mathrm{M}[2XY - 3X^2 + Y^2 - 1] &&= 2\mathrm{M}[XY] - 3\alpha_2[X] + \alpha_2[Y] - 1\\
& &&= 4ab/9 - 3a^2/2 + b^2/6 - 1.
\end{aligned}$$

7.15. Answer questions 1-3 of the preceding problem given that the variables X and Y are mutually dependent and their correlation coefficient $r_{xy} = -0.9$.

Solution. (1) $\mathrm{M}[X + Y] = (2a + b)/3$, (2) $\mathrm{Var}[3X - 6Y + 1] = a^2/2 + 2b^2 + (36ab/\sqrt{18\times18})\,0.9 = a^2/2 + 2b^2 + 1.8ab$, (3) $\mathrm{M}[XY] = 2ab/9 - 0.9ab/18 = 31ab/180$.

7.16. The ends of a ruler AB of length l slide along the sides of a right angle xOy. The ruler occupies a random position shown in Fig. 7.16 with all the values of the abscissa X of its end A on the x-axis in the interval from 0 to l being equally probable. Find the mean value of the distance R from the origin to the ruler.

Solution. The random variable X is uniformly distributed in the interval $(0, l)$, $f(x) = 1/l$ for $x \in (0, l)$. The random variable R can be expressed in terms of X (see Fig. 7.16); $R = X\sqrt{1 - (X/l)^2}$. Its

mean value

$$m_r = \frac{1}{l}\int_0^l x\sqrt{1-\left(\frac{x}{l}\right)^2}\,dx = \frac{1}{3}.$$

7.17. The random variables V and U are in a linear relationship with the random variables X and Y, i.e. $V = aX + bY + c$ and $U = dX + fY + g$. The numerical characteristics m_x, m_y, Var_x, Var_y, Cov_{xy} of the system of the random variables X and Y are known. Find the characteristics of the system of random variables V, U, i.e. m_v, m_u, Var_v, Var_u, Cov_{vu}, r_{vu}.

Solution.

$$m_v = am_x + bm_y + c,\ \mathrm{Var}_v = a^2\,\mathrm{Var}_x + b^2\,\mathrm{Var}_y + 2ab\,\mathrm{Cov}_{xy},$$
$$m_u = dm_x + fm_y + g,\ \mathrm{Var}_u = d^2\,\mathrm{Var}_x + f^2\,\mathrm{Var}_y + 2df\,\mathrm{Cov}_{xy}.$$

Furthermore,

$$\mathring{V} = a\mathring{X} + b\mathring{Y},\quad \mathring{U} = d\mathring{X} + f\mathring{Y},$$

$$\mathrm{Cov}_{vu} = \mathrm{M}\,[\mathring{V}\mathring{U}] = ad\,\mathrm{Var}_x + bf\,\mathrm{Var}_y + (af + bd)\,\mathrm{Cov}_{xy},$$

$$r_{vu} = \mathrm{Cov}_{vu}/\sqrt{\mathrm{Var}_v\,\mathrm{Var}_u}.$$

7.18. An analytical balance is used to weigh an object. The actual (unknown) mass of the object is a. Because of errors, the result of each weighing is accidental and has a normal distribution with parameters a and σ. To reduce the errors, the object is weighed n times, and the arithmetic mean of the results of n weighings is taken as an approximate value of the mass, i.e. $Y(n) = \frac{1}{n}\sum_{i=1}^{n} X_i$. (1) Find the mean value and the mean square deviation. (2) How many times must the object be weighed to reduce the mean square error of the mass ten times?

Solution. (1) $\mathrm{M}\,[Y(n)] = \frac{1}{n}\sum_{i=1}^{n}\mathrm{M}\,[X_i]$. Since the object is weighed under the same conditions every time, it follows that $\mathrm{M}\,[X_i] = a$ for any i, and then

$$\mathrm{M}\,[Y(n)] = \frac{1}{n}\sum_{i=1}^{n} a = \frac{na}{n} = a.$$

Assuming the errors in each weighing to be independent, we find the variance of $Y(n)$:

$$\mathrm{Var}\,[Y(n)] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\,[X_i] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

(2) The number of weighings n can be found from the condition $\sigma\,[Y(n)] = \sqrt{\sigma^2/n} = \sigma/\sqrt{n} = \sigma/10$. Hence $n = 100$

7.19. A luminous spot representing a target being tracked on a circular radar screen can accidentally occupy any place on the screen (the probability density is constant). The diameter of the screen is D. Find the mean value of the distance R from the luminous spot to the centre of the screen.

Solution. $R=\sqrt{X^2+Y^2}$, where (X, Y) is a system of random variables uniformly distributed in the circle K_D of diameter D:

$$f(x, y)=4/(\pi D^2) \text{ for } (x, y)\in K_D$$

$$m_r=\mathrm{M}[R]\iint\limits_{(K_D)}\sqrt{x^2+y^2}\,\frac{4}{\pi D^2}\,dx\,dy,$$

or, passing to polar coordinates (r, φ)

$$m_r=\frac{4}{\pi D^2}\int\limits_0^{2\pi}d\varphi\int\limits_0^{D/2}r^2\,dr=\frac{D}{3}$$

7.20. Each of two points X and Y may independently occupy an accidental position on the interval (0, 1) of the abscissa axis (Fig. 7.20*a*),

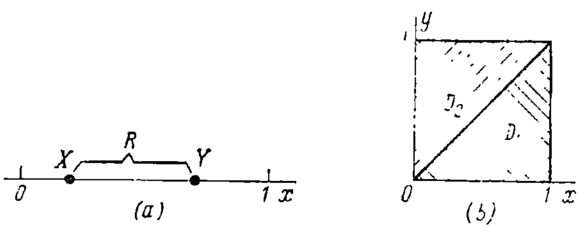


Fig. 7.20

the probability density on the interval being constant for the two variables. Find the mean value of the distance R between the points and the square of the distance between them.

Solution. We have $R=|Y-X|$, $m_r=\mathrm{M}[|Y-X|]$. We represent the system of random variables (X, Y) as a random point on the x, y-plane (Fig. 7.20*b*) distributed with constant density $f(x, y)=1$ in a square with a side equal to unity. In the domain D_1: $X>Y$; $|Y-X|=X-Y$. In the domain D_2: $Y>X$; $|Y-X|=Y-X$.

$$m_r=\iint\limits_{(D_1)}(x-y)\,dx\,dy+\iint\limits_{(D_2)}(y-x)\,dx\,dy$$

$$=\int\limits_0^1 dx\int\limits_0^x(x-y)\,dy+\int\limits_0^1 dy\int\limits_0^y(y-x)\,dx=\frac{1}{3}$$

$$\mathrm{M}[R^2]=\mathrm{M}[|Y-X|^2]=\alpha_2[Y]+\alpha_2[X]-2m_xm_y$$
$$=2(\mathrm{Var}_x+m_x^2)-2m_x^2=1/6.$$

7.21. There is a unit square K (Fig. 7.21). Points X and Y fall at random and independently of each other on the adjacent sides of the square. Each point is uniformly distributed within the limits of the respective side. Find the mean value of the square of the distance between them.

Solution. $R^2 = X^2 + Y^2$; $\mathrm{M}[R^2] = \alpha_2[X] + \alpha_2[Y] = 2/3$.

7.22. The conditions of the preceding problem are changed so that the points X and Y fall not on the adjacent but on the opposite sides

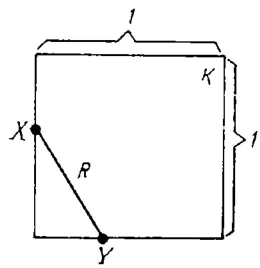


Fig. 7.21

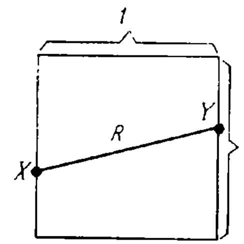


Fig. 7.22

of the square (Fig. 7.22). Find the mean value of the square of the distance between X and Y.

Solution. $R^2 = 1 + (Y - X)^2$; $\mathrm{M}[R^2] = 1 + \alpha_2[Y] + \alpha_2[X] - 2\mathrm{M}[X]\,\mathrm{M}[Y] = 1 + 2/3 - 1/2 = 7/6$.

7.23. The conditions of the preceding problems (7.21 and 7.22) are changed so that the points X and Y may accidentally and independently of each other occupy, with constant probability density, any positions on the perimeter of the square K. Find the mean value of the square of the distance between them.

Solution. Let us choose three hypotheses:

H_1 = {the points X and Y fall on the same side of the square};

H_2 = {the points X and Y fall on the adjacent sides of the square};

H_3 = {the points X and Y fall on the opposite sides of the square}.

The mean value of the variable R^2 can be found by the complete mathematical expectation formula

$$\mathrm{M}[R^2] = \mathrm{P}(H_1)\,\mathrm{M}[R^2 \mid H_1] + \mathrm{P}(H_2)\,\mathrm{M}[R^2 \mid H_2] + \mathrm{P}(H_3)\,\mathrm{M}[R^2 \mid H_3],$$

where $\mathrm{M}[R^2 \mid H_1]$, $\mathrm{M}[R^2 \mid H_2]$, $\mathrm{M}[R^2 \mid H_3]$ are the conditional expectations of the variable R^2 on the corresponding hypotheses.

From the solutions of Problems 7.20, 7.21 and 7.22 we have

$$\mathrm{M}[R^2 \mid H_1] = 1/6,\quad \mathrm{M}[R^2 \mid H_2] = 2/3,\quad \mathrm{M}[R^2 \mid H_3] = 7/6.$$

We can find the probabilities of the hypotheses, i.e. $P(H_1) = 1/4$, $P(H_2) = 1/2$ and $P(H_3) = 1/4$. Hence

$$M[R^2] = \frac{1}{4} \times \frac{1}{6} + \frac{1}{2} \times \frac{2}{3} + \frac{1}{4} \times \frac{7}{6} = \frac{2}{3}.$$

7.24. A random variable X has a probability density $f(x)$. We consider the function $Y = \min\{X, a\}$, where a is a nonrandom variable. Find the mean value and variance of the random variable Y without resorting to its distribution.

Solution. By the general formula (7.0.7)

$$M[Y] = \int_{-\infty}^{\infty} \min\{x, a\} f(x)\, dx.$$

For $x < a$, we get $\min\{x, a\} = x$; for $x \geqslant a$, we get $\min\{x, a\} = a$. Hence

$$M[Y] = \int_{-\infty}^{a} xf(x)\, dx + a \int_{a}^{\infty} f(x)\, dx = \int_{-\infty}^{a} xf(x)\, dx + aP\{X > a\}. \tag{7.24.1}$$

By analogy we find the second moment about the origin

$$\alpha_2[Y] = \int_{-\infty}^{a} x^2 f(x)\, dx + a^2 \int_{a}^{\infty} f(x)\, dx = \int_{-\infty}^{a} x^2 f(x)\, dx + a^2 P\{X > a\} \tag{7.24.2}$$

and the variance

$$\mathrm{Var}[Y] = \alpha_2[Y] - (M[Y])^2.$$

7.25. The same question as in the preceding problem, but X is a discrete random variable assuming positive integral values with the probabilities defined by the ordered series

$$X: \left|\begin{array}{c|c|c|c|c|c} 1 & 2 & \dots & k & \dots & n \\ \hline p_1 & p_2 & \dots & p_k & \dots & p_n \end{array}\right|, \quad Y = \min\{X, a\},$$

a is a nonrandom positive integer included between 1 and n $(1 < a < n)$.

Solution. $M[Y] = \sum_{k=1}^{n} \min\{k, a\} p_k$; for $k < a$, we get $\min\{k, a\} = k$; for $k \geqslant a$, we get $\min\{k, a\} = a$. Then

$$M[Y] = \sum_{k=1}^{a-1} kp_k + a \sum_{k=a}^{n} p_k.$$

By analogy, we find the second moment about the origin

$$\alpha_2[Y] = \sum_{k=1}^{a-1} k^2 p_k + a^2 \sum_{k=a}^{n} p_k$$

and the variance

$$\mathrm{Var}\,[Y] = \alpha_2\,[Y] - (\mathrm{M}\,[Y])^2.$$

7.26. A number of independent trials are carried out in each of which an event A may occur with probability p. The trials are terminated as soon as the event A occurs and the total number of trials must not exceed N. Find the mean value and variance of the number Y of trials that will be made.

Solution. We first consider a random variable X which is the number of trials if the restriction N concerning the total number of trials is removed. The random variable X has a geometric distribution beginning with unity and its ordered series is

$$X: \left|\begin{array}{c|c|c|c|c} 1 & 2 & \dots & k & \dots \\ \hline p & pq & \dots & pq^{k-1} & \dots \end{array}\right. \quad (q = 1-p).$$

The random variable Y is the minimum of X and N: $Y = \min\,\{X,\ N\}$. On the basis of the results of Problem 7.25 we have (setting $n = \infty$)

$$\mathrm{M}[Y] = \sum_{k=1}^{N-1} kp_k + N\sum_{k=N}^{\infty} p_k = \sum_{k=1}^{N-1} kpq^{k-1} + N\sum_{k=N}^{\infty} pq^{k-1}$$
$$= p\left\{\sum_{k=1}^{N-1} kq^{k-1} + N\sum_{k=N}^{\infty} q^{k-1}\right\}.$$

We calculate the first sum

$$\sum_{k=1}^{N-1} kq^{k-1} = \sum_{k=1}^{N-1} \frac{d}{dq}\,q^k = \frac{d}{dq}\,\frac{q-q^N}{1-q} = \frac{1-Nq^{N-1}+(N-1)\,q^N}{(1-q)^2}.$$

The second sum is equal to $(Nq^{N-1})/(1-q)$, whence

$$\mathrm{M}[Y] = p\,\frac{1-Nq^{N-1}+(N-1)\,q^N+Nq^{N-1}-Nq^N}{(1-q)^2} = \frac{1-q^N}{p}.$$

7.27. In Chapter 1 we considered Buffon's needle problem: a needle of length l is thrown at random on a plane which is ruled with parallel straight lines $L > l$ apart. The probability of the needle hitting one of the lines is $p = 2l/(L\pi)$. Since for $l < L$ the number of hits can only be 0 or 1, the mean value of the number of hits is p. What will be the mean value of hits if the restriction $l < L$ is removed?

Solution. We divide the length l of the needle into n elementary sections $\Delta l = l/n < L$. The random variable X is the number of times the needle hits the lines. It can be represented as a sum $X = \sum_{i=1}^{n} X_i$, where X_i is the number of intersections of the needle and the lines of the ith elementary section. The random variable X_i is the indicator of the intersection of the ith section and one of the lines and has a mean

value equal to the probability of a hit:

$$\mathrm{M}[X_i] = p_i = 2\Delta l/(L\pi),$$

whence

$$\mathrm{M}[X] = \sum_{i=1}^{n} p_i = \frac{2n\,\Delta l}{L\pi} = \frac{2l}{L\pi}.$$

7.28. Any contour (convex or nonconvex, open or closed) of length l is thrown at random on a plane ruled with parallel straight lines appearing in the preceding problem. Find the mean value of the number of times the contour and the lines intersect.

Solution. As in the preceding problem, $\mathrm{M}[Y] = 2l/(L\pi)$. To prove this, we must divide the contour into n elementary, practically straight,

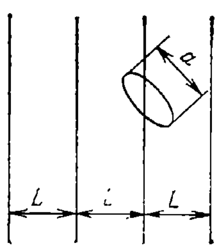


Fig. 7.29

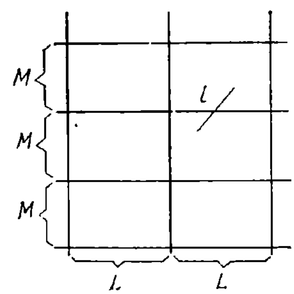


Fig. 7.30

sections of length Δl. For each of them the expectation of the number of hits is $2\Delta l/(L\pi)$, and for the whole contour it is $2l/(L\pi)$.

7.29. A convex closed contour of length l whose largest dimension a does not exceed L is thrown at random on a plane ruled with parallel straight lines L apart (Fig. 7.29). Find the probability that it will intersect a line.

Solution. We designate the required probability as p, Y being the number of points of intersection of the contour and a line. Since the contour is convex and closed, and its largest dimension is smaller than L, it may either intersect the lines twice or not at all. The ordered series of the random variable Y is

$$Y: \left|\begin{array}{c|c} 0 & 2 \\ \hline 1-p & p \end{array}\right|.$$

From Problem 7.28, we get $\mathrm{M}[Y] = 0\cdot(1-p) + 2p = 2p = 2l/(L\pi)$, whence $p = l/(L\pi)$.

7.30. A plane is ruled in rectangles with sides L and M (Fig. 7.30). A needle of length l ($l < L$, $l < M$) is thrown at random on the plane. Find (a) the probability that the needle cuts at least one line, (b) the mean value of the number of intersections between the needle and a line with the restrictions $l < L$, $l < M$ removed.

Solution. (a) Let us consider the straight lines which bound the rectangles to be two systems of lines, horizontal and vertical. We consider the following events:

$$A = \{\text{the needle cuts a vertical line}\},$$

$$B = \{\text{the needle cuts a horizontal line}\}.$$

Since the position of the needle with respect to the vertical lines does not affect its position with respect to the horizontal lines, the events A and B are disjoint; therefore, the required probability is

$$\text{P}(A + B) = \text{P}(A) + \text{P}(B) - \text{P}(A)\,\text{P}(B).$$

On the basis of Buffon's problem, $\text{P}(A) = 2l/(\pi L)$, $\text{P}(B) = 2l/(\pi M)$, whence

$$\text{P}(A + B) = 2l/(\pi L) + 2l/(\pi M) - 4l^2/(\pi^2 LM).$$

(b) The random variable $X = X_1 + X_2$, where X_1 and X_2 are the numbers of intersections between the needle and a horizontal and a vertical line respectively.

Dividing the needle of an arbitrary length l into elementary sections, as before, we have

$$\text{M}[X] = \text{M}[X_1] + \text{M}[X_2] = 2l/(\pi L) + 2l/(\pi M).$$

7.31. A rectangle of size $l_1 \times l_2$ is thrown on a plane at random (Fig. 7.31); all the values of the angle Θ are equiprobable. Find the mean value of the length X of its projection onto the x-axis.

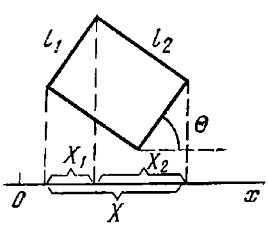


Fig. 7.31

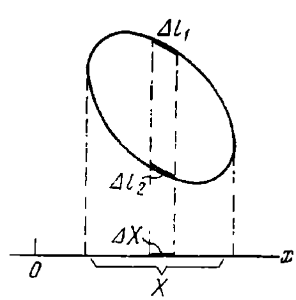


Fig. 7.32

Solution. We represent X as a sum $X = X_1 + X_2$, where X_1 is the projection of the section l_1 and X_2 is the projection of the section l_2. The required mean value

$$\text{M}[X] = \text{M}[X_1] + \text{M}[X_2] = 2l_1/\pi + 2l_2/\pi = 2(l_1 + l_2)/\pi,$$

i.e. it is equal to the perimeter of the rectangle divided by π.

7.32. A convex closed contour of length l is thrown at random on a plane, all the orientations being equiprobable (Fig. 7.32). Find the mean value of the length X of its projection onto the x-axis.

Solution. Since the contour is convex, each element Δx of the projection results from the projection of two and only two opposite elements of the contour: Δl_1 and Δl_2 (see Fig. 7.32); hence the average length of the contour projection is half the sum of the average lengths of the projections of the elementary sections Δl into which the contour can be divided:

$$\mathrm{M}[X]=\frac{1}{2}\sum\frac{2\Delta l}{\pi}=\frac{l}{\pi}.$$

7.33. There is a random variable X with probability density $f(x)$. Find the mean value and variance of the random variable $Y = |X|$.

Solution. The notation $Y = |X|$ means that

$$Y=\begin{cases}-X & \text{for} \quad X<0\\ X & \text{for} \quad X>0.\end{cases}$$

$$m_y=\mathrm{M}\,|Y|=\int_{-\infty}^{\infty}|x|\,f(x)\,dx=-\int_{-\infty}^{0}xf(x)\,dx\int_{0}^{\infty}xf(x)\,dx$$

$$=\int_{0}^{\infty}x\,[f(x)+f(-x)\,dx,$$

$$\mathrm{Var}_y=\alpha_2[Y]-m_y^2=\int_{-\infty}^{\infty}|x|^2 f(x)\,dx-m_y^2$$

$$=\alpha_2[X]-m_y^2=\mathrm{Var}_x+m_x^2-m_y^2.$$

7.34. Find the mean value and variance of the modulus of the random variable X, which is normally distributed with parameters m, σ.

Solution. It follows from the preceding problem that

$$m_y=-\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{0}xe^{-\frac{(x-m)}{2\sigma^2}}\,dx+\frac{1}{\sigma\sqrt{2\pi}}\int_{0}^{\infty}xe^{-\frac{(x-m)^2}{2\sigma^2}}\,dx.$$

Changing the variables $(x-m)/\sigma = t$, we get

$$m_y=-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{m}{\sigma}}(t\sigma+m)\,e^{-\frac{t^2}{2}}\,dt+\frac{1}{\sqrt{2\pi}}\int_{-\frac{m}{\sigma}}^{\infty}(t\sigma+m)\,e^{-\frac{t^2}{2}}\,dt$$

$$=\frac{2\sigma}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{m}{\sigma}\right)^2}+m\Phi\left(\frac{m}{\sigma}\right),$$

where Φ is the error function

$$\mathrm{Var}_y=\sigma^2+m^2-m_y^2.$$

In particular, for $m = 0$,

$$m_y=\sqrt{\frac{2}{\pi}}\,\sigma\approx 0.80\sigma;\quad \mathrm{Var}_y=\sigma^2-\frac{2}{\pi}\sigma^2=\left(1-\frac{2}{\pi}\right)\sigma^2\approx 0.36\sigma^2.$$

7.35*. Independent random variables X and Y have probability densities $f_1(x)$ and $f_2(y)$. Find the mean value and variance of the modulus of their difference $Z = |X - Y|$.

Solution. We have

$$m_z = \int\int_{-\infty}^{\infty} |x-y| f_1(x) f_2(y)\,dx\,dy.$$

A straight line $y = x$ divides the x, y-plane into two domains I and II (Fig. 7.35). In domain I $x > y$, $|x - y| = x - y$. In domain II $y > x$, $|x - y| = y - x$. Hence

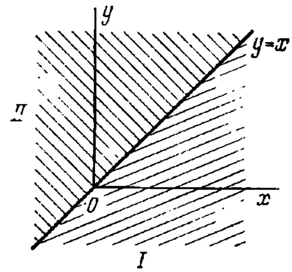


Fig. 7.35

$$m_z = \int\int_{(I)} (x-y) f_1(x) f_2(y)\,dx\,dy + \int\int_{(II)} (y-x) f_1(x) f_2(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x f_1(x) \left\{ \int_{-\infty}^{x} f_2(y)\,dy \right\} dx - \int_{-\infty}^{\infty} y f_2(y) \left\{ \int_{y}^{\infty} f_1(x)\,dx \right\} dy$$

$$+ \int_{-\infty}^{\infty} y f_2(y) \left\{ \int_{-\infty}^{y} f_1(x)\,dx \right\} dy - \int_{-\infty}^{\infty} x f_1(x) \left\{ \int_{x}^{\infty} f_2(y)\,dy \right\} dx.$$

We introduce the distribution functions

$$F_1(x) = \int_{-\infty}^{x} f_1(x)\,dx, \quad F_2(y) = \int_{-\infty}^{y} f_2(y)\,dy.$$

Then we have

$$m_z = \int_{-\infty}^{\infty} x f_1(x) F_2(x)\,dx - \int_{-\infty}^{\infty} y f_2(y)[1 - F_1(y)]\,dy$$

$$+ \int_{-\infty}^{\infty} y f_2(y) F_1(y)\,dy - \int_{-\infty}^{\infty} x f_1(x)[1 - F_2(x)]\,dx.$$

Combining the first integral with the fourth and the second with the third, we obtain

$$m_z = \int_{-\infty}^{\infty} [2x f_1(x) F_2(x) - x f_1(x)]\,dx + \int_{-\infty}^{\infty} [2y f_2(y) F_1(y) - y f_2(y)]\,dy$$

$$= 2\int_{-\infty}^{\infty} x f_1(x) F_2(x)\,dx - m_x + 2\int_{-\infty}^{\infty} y f_2(y) F_1(y)\,dy - m_y.$$

Since X and Y are independent, it follows that

$$\alpha_2[Z] = \mathrm{M}[|X-Y|^2] = \mathrm{M}[(X-Y)^2] = \mathrm{M}[X^2] + \mathrm{M}[Y^2] - 2\mathrm{M}[X]\,\mathrm{M}[Y]$$
$$= \alpha_2[X] + \alpha_2[Y] - 2m_x m_y = \mathrm{Var}_x + \mathrm{Var}_y + (m_x - m_y)^2.$$

From this we find that $\mathrm{Var}_z = \alpha_2[Z] - m_z^2$.

7.36. Two independent random variables X and Y have probability densities $f_1(x)$ and $f_2(y)$. Find the mean value and variance of the minimum value of two variables, i.e. $Z = \min\{X, Y\}$.

Solution.

$$Z = \begin{cases} X, & \text{if} \quad X \leqslant Y, \\ Y, & \text{if} \quad X > Y. \end{cases}$$

A straight line $y = x$ divides the x, y-plane into two domains (see Fig. 7.35): I, where $Z = Y$, and II, where $Z = X$ (the case $X = Y$ is not considered since it has a zero probability).

$$m_z = \mathrm{M}[Z] = \iint_{(\mathrm{II})} x f_1(x) f_2(y)\,dx\,dy + \iint_{(\mathrm{I})} y f_1(x) f_2(y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty} x f_1(x)[1 - F_2(x)]\,dx + \int_{-\infty}^{\infty} y f_2(y)[1 - F_1(y)]\,dy,$$

where F_1 and F_2 are the distribution functions of the random variables X and Y.

$$\alpha_2[Z] = \int_{-\infty}^{\infty} x^2 f_1(x)[1 - F_2(x)]\,dx + \int_{-\infty}^{\infty} y^2 f_2(y)[1 - F_1(y)\,dy],$$
$$\mathrm{Var}_z = \alpha_2[Z] - m_z^2.$$

7.37. A random voltage V has a normal distribution $f(v)$ with parameters m_v and σ_v. The voltage V arrives at a limiter which leaves it equal to V if $V \leqslant v_0$ and makes it equal to v_0 if $V > v_0$:

$$Z = \min\{V, v_0\} = \begin{cases} V & \text{for} \quad V \leqslant v_0, \\ v_0 & \text{for} \quad V > v_0. \end{cases}$$

Find the mean value and variance of the random variable Z, which is the voltage taken off by the limiter.

Solution. Proceeding from Problem 7.24 we have

$$m_z = \mathrm{M}[Z] = \int_{-\infty}^{\infty} \min\{v, v_0\} f(v)\,dv = \int_{-\infty}^{v_0} v f(v)\,dv + \int_{v_0}^{\infty} v_0 f(v)\,dv$$
$$= \int_{-\infty}^{v_0} \frac{v}{\sqrt{2\pi}\,\sigma_v} \exp\left[-\frac{(v-m_v)^2}{2\sigma_v^2}\right] dv + v_0 \int_{v_0}^{\infty} f(v)\,dv$$
$$= m_v\left[\Phi\left(\frac{v_0 - m_v}{\sigma_v}\right) + 0.5\right] - \frac{\sigma_v}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{v_0 - m_v}{\sigma_v}\right)^2\right]$$

$$+v_0\left[0.5-\Phi\left(\frac{v_0-m_v}{\sigma_v}\right)\right]=v_0-\sigma_v\left[t_0\left(\Phi(t_0)+0.5\right)\right.$$

$$\left.+\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{t_0^2}{2}\right]\right], \quad \text{where } t_0=\frac{v_0-m_v}{\sigma_v},$$

$$\alpha_2[Z]=\int_{-\infty}^{v_0} v^2 f(v)\,dv+\int_{v_0}^{\infty} v_0^2 f(v)\,dv=(m_v^2+\sigma_v^2)\ \left(\Phi(t_0)+0.5\right)$$

$$+v_0\left[0.5-\Phi(t_0)\right]-\frac{2\sigma_v m_v+\sigma_v^2 t_0}{\sqrt{2\pi}}\exp\left[-\frac{t_0^2}{2}\right]$$

$$\mathrm{Var}_z=\alpha_2[Z]-m_z^2=\sigma_v^2\left\{(1+t_0^2)\left(\Phi(t_0)+0.5\right)\right.$$

$$+\frac{t_0}{\sqrt{2\pi}}\exp\left[-\frac{t_0^2}{2}\right]-\left[t_0\left(\Phi(t_0)+0.5\right)\right.$$

$$\left.\left.+\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{t_0^2}{2}\right]\right]^2\right\}.$$

Note that if $v_0=m_v$ and $t_0=0$, then $m_z=m_v-\sigma_v\left(\sqrt{2\pi}\right)^{-1}$, $\mathrm{Var}_z=\sigma_v^2(\pi-1)(2\pi)^{-1}$.

7.38. N messages are being sent over a communication channel. The lengths of the messages are accidental, have the same mean value m and variance Var, and are independent. Find the mean value and variance of the total time T during which all N messages will be transmitted. Find $T_{\max}$, which is the maximum practically possible time during which the messages can be transmitted.

Solution. $T=\sum_{i=1}^{n} T_i$, where T_i is the duration of the ith message $(i=1, 2, \ldots, N)$. By the addition theorem for expectations we have

$$\mathrm{M}[T]=\mathrm{M}\left[\sum_{i=1}^{N} T_i\right]=\sum_{i=1}^{N}\mathrm{M}(T_i)=Nm.$$

By the addition theorem for variances, we have

$$\mathrm{Var}[T]=\mathrm{Var}\left[\sum_{i=1}^{N} T_i\right]=\sum_{i=1}^{N}\mathrm{Var}[T_i]=N\,\mathrm{Var},$$

$$\sigma_t=\sqrt{\mathrm{Var}[T]}=\sqrt{N\,\mathrm{Var}}.$$

By the three-sigma rule $T_{\max}=Nm+3\sqrt{N\mathrm{Var}}$.

7.39. Solve the preceding problem for the case when the lengths T_i of the messages are dependent and the correlation coefficient of the random variables T_i and T_j is r_{ij}.

Solution. The mean value is $\mathrm{M}[T]=Nm$ as before. To calculate the variance, we find the covariance of the variables T_i and T_j: $\mathrm{Cov}_{ij}=$

$r_{ij}\sigma^2 = r_{ij}$ Var. By formula (7.0.28) for the variance of the sum

$$\text{Var}[T] = N\,\text{Var} + 2\sum_{i<j} r_{ij}\,\text{Var} = \text{Var}\left(N + 2\sum_{i<j} r_{ij}\right),$$

$$T_{\max} = Nm + 3\sqrt{\text{Var}[T]}.$$

7.40. A system of random variables (X, Y) is uniformly distributed in a rectangle R (Fig. 7.40). Find (1) M $[X + Y]$, (2) M $[X - Y]$,

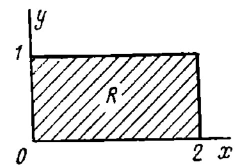


Fig. 7.40

(3) M $[XY]$, (4) Var $[X + Y]$, (5) Var $[X - Y]$, (6) M $[(X - Y)^2]$ and (7) M $[2X^3 + 3Y^2 + 1]$.

Answer. (1) 3/2, (2) 1/2, (3) 1/2, (4) 5/12, (5) 5/12, (6) 2/3 and (7) 6.

7.41. Random faults occur when an electronic device is operating. The average number of faults occurring per unit operation time is λ, and the number of faults occurring during the operation time τ of the device is a random variable having a Poisson distribution with parameter $a = \lambda\tau$. To clear a fault (to repair the device) a random time T_{rep} is required, which has an exponential distribution $f(t) = \mu e^{-\mu t}$ for $t > 0$. The times required to clear separate faults are independent. Find (1) the average fraction of time during which the device will operate fault-free and the average fraction of time the repairs will last, and (2) the average time interval between two faults.

Solution. (1) The average time of fault-free performance of the device (the mean value of the time from the moment the device is put into operation until the moment it is stopped for repairs) $\bar{t}_{\text{sound}} = 1/\lambda$. The average time needed for repairs $\bar{t}_{\text{rep}} = 1/\mu$. The average fraction of time during which the device will operate fault-free

$$\alpha = \frac{\bar{t}_{\text{sound}}}{\bar{t}_{\text{sound}} + \bar{t}_{\text{rep}}} = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{\mu}{\lambda + \mu}.$$

Similarly, the average fraction of time during which the device will be repaired

$$\beta = 1 - \alpha = \lambda/(\lambda + \mu).$$

(2) The average interval of time $\bar{T}_t$ between two successive faults

$$\bar{T}_t = \bar{t}_{\text{sound}} + \bar{t}_{\text{rep}} = 1/\lambda + 1/\mu = (\lambda + \mu)/(\lambda\mu).$$

7.42. A random point (X, Y) has a normal distribution on a plane with circular scattering, i.e. $m_x = m_y = 0$ and $\sigma_x = \sigma_y = \sigma$. A ran-

dom variable R is the distance from the point (X, Y) to the centre of scattering. Find the mean value and variance of the variable R.

Solution.

$$R=\sqrt{X^2+Y^2}.$$

$$m_r=\mathrm{M}[R]=\int\int_{-\infty}^{\infty}\sqrt{x^2+y^2}\,\frac{1}{2\sigma^2\pi}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dx\,dy.$$

Using polar coordinates r and φ, we get

$$m_r=\int_0^{\infty} r^2\frac{1}{2\sigma^2\pi}\exp\left(-\frac{r^2}{2\sigma^2}\right)dr\int_0^{2\pi}d\varphi=\sigma\sqrt{\frac{\pi}{2}}\approx 1.25\sigma.$$

$$\mathrm{Var}_r=\mathrm{Var}[R]=\alpha_2[R]-m_r^2=\int\int_{-\infty}^{\infty}(x^2+y^2)\frac{1}{2\sigma^2\pi}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dx$$

$$dy-m_r^2=\int_0^{\infty}r^3\frac{1}{2\sigma^2\pi}\exp\left(-\frac{r^2}{2\sigma^2}\right)dr\int_0^{2\pi}d\varphi-\sigma^2\frac{\pi}{2}$$

$$=2\sigma^2\int_0^{\infty}te^{-t}\,dt-\sigma^2\frac{\pi}{2}=\sigma^2\frac{4-\pi}{2}.$$

7.43. Prove that if a random variable X has a binomial distribution with parameters n and p, then its mean value $\mathrm{M}[X]=np$, and its variance $\mathrm{Var}[X]=npq$ $(q=1-p)$.

Solution. X is the number of occurrences of an event A in n independent trials. It can be represented in the form $X=\sum_{i=1}^{n}X_i$, where X_i is the indicator of the event A in the ith trial, i.e.

$$X_i=\begin{cases}1 & \text{if the event occurs on the } i\text{th trial;}\\ 0 & \text{if the event does not occur on the } i\text{th trial,}\end{cases}$$

$$\mathrm{M}[X_i]=p,\quad \mathrm{Var}[X_i]=pq,$$

$$\mathrm{M}[X]=\sum_{i=1}^{n}\mathrm{M}[X_i]=np,\quad \mathrm{Var}[X]=\sum_{i=1}^{n}pq=npq.$$

7.44. Prove that for n independent trials, in which an event A occurs with probabilities $p_1, p_2, \ldots, p_n$, the mean value and variance of the number X of occurrences of the event are

$$\mathrm{M}[X]=\sum_{i=1}^{n}p_i;\quad \mathrm{Var}[X]=\sum_{i=1}^{n}p_iq_i,\ \text{where } q_i=1-p_i.$$

Solution. By analogy with the preceding problem $X = \sum_{i=1}^{n} X_i$, where X_i is the indicator of the event A in the ith trial:

$$M[X] = \sum_{i=1}^{n} M[X_i] = \sum_{i=1}^{n} p_i, \quad \text{Var}[X] = \sum_{i=1}^{n} \text{Var}[X_i] \sum_{i=1}^{n} p_i q_i.$$

7.45. We consider n independent trials in each of which an event A may occur with probability p. Find the mean value and variance of the frequency Y of the event A. Find the range of practically possible values of the frequency.

Solution. $Y = X/n$, where X is the number of occurrences of the event A.

$$M[Y] = M[X]/n = \frac{1}{n} np = p, \quad \text{Var}[Y] = npq/n^2 = pq/n,$$

$$\sigma_y = \sqrt{\text{Var}[Y]} = \sqrt{pq/n},$$

where $q = 1 - p$. The range of practically possible values of Y is $m_y \pm 3\ \sigma_y = p \pm 3\sqrt{pq/n}.$

7.46. Prove that the mean value and variance of a random variable X, which has a hypergeometric distribution (see 4.0.34) with parameters a, b, n, are

$$M[X] = \frac{na}{a+b}, \quad \text{Var}[X] = \frac{nab}{(a+b)^2}$$

$$+ n(n-1)\left[\frac{a}{a+b}\frac{a-1}{a+b-1} - \left(\frac{a}{a+b}\right)^2\right] \qquad (7.46.1)$$

respectively.

Solution. Let us consider a physical model of a situation in which a hypergeometric distribution occurs, say, drawing n balls from an urn which contains a white and b black balls, X is the number of the white selected balls. We can represent n selections of a ball as n trials in each of which an event $A = \{\text{a white ball}\}$ may occur. We represent the variable X as the sum of the variables X_i which are the indicators of the event A in the ith trial; $X = \sum_{i=1}^{n} X_i$. The random variables X_i are mutually dependent, but the addition theorem for expectations is applicable, viz.

$$M[X] = \sum_{i=1}^{n} M[X_i], \quad M[X_i] = \frac{a}{a+b},$$

$$M[X] = \frac{na}{a+b}.$$

The variance of the sum of the random variables X_i can be found from formula (7.0.28):

$$\operatorname{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \operatorname{Var}[X_i] + 2\sum_{i<j} \operatorname{Cov}_{x_i x_j}. \qquad (7.46.2)$$

$$\operatorname{Var}[X_i] = pq = \frac{a}{a+b}\,\frac{b}{a+b} = \frac{ab}{(a+b)^2},$$

$$\sum_{i=1}^{n} \operatorname{Var}[X_i] = \frac{nab}{(a+b)^2}.$$

Let us find $\operatorname{Cov}_{x_i x_j} = \mathrm{M}[X_i X_j] - \mathrm{M}[X_i]\,\mathrm{M}[X_j]$. The product of the indicators $X_i X_j$ of the event A in the ith and the jth trial is equal to unity when $X_i = 1$ and $X_j = 1$, i.e. the event A occurs both on the ith and on the jth trial, the probability being equal to $\frac{a}{a+b} \times \frac{(a-1)}{(a+b-1)}$; the expectation is equal to the same value. Hence

$$\mathrm{M}[X_i X_j] = \frac{a(a-1)}{(a+b)(a+b-1)}.$$

Hence

$$\operatorname{Cov}_{x_i x_j} = \frac{a(a-1)}{(a+b)(a+b-1)} - \left(\frac{a}{a+b}\right)^2. \qquad (7.46.3)$$

The number of terms in the sum $\sum_{i<j} \operatorname{Cov}_{x_i x_j}$ is $C_n^2 = n(n-1)/2$. Substituting (7.46.3) into (7.46.2), we get formula (7.46.1).

7.47. There are five white and seven black balls in an urn. Six balls are drawn at a time. A random variable X is the number of black balls drawn. Find the mean value and variance of the variable X.

Answer. $\mathrm{M}[X] = 3.5$ and $\operatorname{Var}[X] = 35/44 \approx 0.795$.

7.48. A radar installation scans a region of space where there are N targets. During one surveillance cycle the ith target may be located with probability p_i $(i = 1, \ldots, N)$ independently of the other targets. During a period n cycles are realized. Find the mean value and variance of the number of targets X which will be located.

Solution. $X = \sum_{i=1}^{N} X_i$, where X_i is the indicator of an event A_i = {the ith target is located}.

$$\mathrm{P}(A_i) = 1 - (1-p_i)^n,$$

$$\mathrm{M}[X] = \sum_{i=1}^{N} [1-(1-p_i)^n] = N - \sum_{i=1}^{N} (1-p_i)^n,$$

$$\operatorname{Var}(X) = \sum_{i=1}^{N} [1-(1-p_i)^n](1-p_i)^n.$$

As the number of cycles n tends to infinity

$$\lim_{n\to\infty} \mathrm{M}[X] = N \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}[X] = 0,$$

i.e. in the limit, as $n \to \infty$, all the targets will be located.

7.49*. We take an arbitrary point A inside a circle of radius a (Fig. 7.49) and draw a chord BC through it at an arbitrary angle Θ to the radius which passes through the point A. Find the mean length of the chord.

Solution. We take the point A at random inside the circle and, therefore, its polar coordinates R and Φ are independent and have distributions

$$f_1(r) = \frac{2}{a^2} r \quad \text{for} \quad 0 < r < a, \quad f_2(\varphi) = \frac{1}{2\pi} \quad \text{for } 0 < \varphi < 2\pi.$$

Since the mean value of the chord is evidently independent of the angle φ, we take the point A on an arbitrary radius (at a distance R from

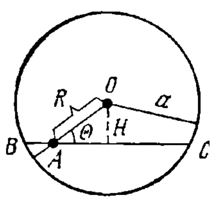


Fig. 7.49

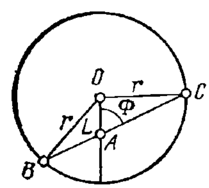


Fig. 7.50

the centre) and draw the chord BC through it at a random angle Θ to the radius. It follows from the hypothesis that the random variable Θ has a probability density $f_3(\vartheta) = 1/2\pi$ for $0 < \vartheta < 2\pi$. We express the length D of the chord BC in terms of the random variables R and Θ by drawing a perpendicular from the centre of the circle to the chord and designate its length as H. It is evident that

$$D = 2\sqrt{a^2 - H^2}, \quad H = R \sin\Theta,$$
$$D = 2\sqrt{a^2 - R^2 \sin^2\Theta}.$$

The mean value of the random variable D can be found as

$$\mathrm{M}[D] = \int_0^{2\pi} d\vartheta \int_0^a 2\sqrt{a^2 - r^2 \sin^2\vartheta}\, \frac{2r}{a^2} \frac{1}{2\pi}\, dr$$
$$= \frac{8}{\pi a^2} \int_0^{\pi/2} \frac{a^3(1-\cos^3\vartheta)}{3\sin^2\vartheta}\, d\vartheta = \frac{16a}{3\pi} \approx 1\ 70a$$

7.50*. Find the mean value of the length of the chord BC (Fig. 7.50), drawn through a point A inside the circle, which is at the distance L

from the centre of the circle of radius r, all the directions of the chord being equiprobable.

Solution. The chord $\overline{BC}$ can be expressed in terms of the quantities L, Φ and r as follows:

$$\overline{BC}=2r\sqrt{1-\frac{L^2}{r^2}\sin^2\Phi}.$$

If we consider the length of the chord $\overline{BC}$ to be a random variable X, then

$$m_x=\int_0^{\pi/2}\frac{2}{\pi}2r\sqrt{1-\left(\frac{L}{r}\right)^2\sin^2\varphi\,d\varphi}=\frac{4r}{\pi}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\,d\varphi,$$

where $k=L/r$. The integral obtained is a complete elliptic integral $E(k, \pi/2)$ with modulus k; its values can be found in handbooks. For instance, for $k=1/2$ the integral $E(1/2, \pi/2)=1.4675$ and $m_x\approx 1.87r$.

Since the complete elliptic integral $E(k, \pi/2)$ is measured from $\pi/2$ (for $k=0$) to 1 (for $k=1$), the mean length of the chord assumes the values from $2r$ (for $k=0$, i.e. for the point A at the centre of the circle) to $4r/\pi$ (for $L=r$, i.e. for points A on the circle).

7.51*. A device consists of n units, each of which may fail independently of the others. The time of the failure-free operation of the ith unit has an exponential distribution with parameter λ_i:

$$f_i(t)=\lambda_i e^{-\lambda_i t}\quad t>0).$$

A unit that fails is immediately replaced by a new one and is repaired. The repairs of the ith unit last a random time distributed according to the exponential law with parameter μ_i: $\varphi_i(t)=\mu_i e^{-\mu_i t}$ $(t>0)$. The device operates for a time τ.

Find (1) the mean value and variance of the number of units which will be replaced, and (2) the mean value of the total time T which will be taken by the repairs of the units that failed.

Solution. (1) We designate as X_i the number of units of the ith type that fail during the time τ. This random variable has a Poisson distribution. Its mean value $m_{x_i}=\lambda_i\tau$ and variance $\mathrm{Var}_{x_i}=\lambda_i\tau$. We designate as X the total number of units that failed during the time τ. We have

$$X=\sum_{i=1}^{n}X_i,\quad m_x=\sum_{i=1}^{n}m_{x_i}=\tau\sum_{i=1}^{n}\lambda_i.$$

The variables X_i being mutually independent, we have

$$\mathrm{Var}_x=\sum_{i=1}^{n}\mathrm{Var}_{x_i}=\tau\sum_{i=1}^{n}\lambda_i.$$

(2) We designate as T_i the total time needed to repair all the units of type i that failed during the time τ. It is the sum of the times needed

to repair all the units. Since the number of units is X_i, we have

$$T^i = T_i^{(1)} + T_i^{(2)} + \ldots + T_i^{(X_i)} = \sum_{k=1}^{X_i} T_i^{(k)},$$

where T_i^k is a random variable which has an exponential distribution with parameter μ_i and the quantities $T_i^{(1)}$, $T_i^{(2)}$, . . . are disjoint.

Let us find the mean value of the random variable T_i using the integral formula for the complete expectation. We assume that the random variable X_i assumes a definite value m, and then the mean value of the variable T_i

$$\mathrm{M}[T_i \mid m] = \sum_{k=1}^{m} \mathrm{M}[T_i^{(k)}] = \sum_{k=1}^{m} \frac{1}{\mu_i} = \frac{m}{\mu_i}.$$

We multiply this conditional expectation by the probability P_m that the random variable X_i assumes the value m, sum the products and find the complete (absolute) expectation of the variable T_i:

$$\mathrm{M}[T_i] = \sum_{m=1}^{\infty} \mathrm{P}_m \frac{m}{\mu} = \frac{1}{\mu_i} \sum_{m=1}^{\infty} m\mathrm{P}_m = \frac{1}{\mu_i} \mathrm{M}[X_i] = \frac{\lambda_i \tau}{\mu_i}.$$

Using then the theorem for the addition of expectations, we obtain

$$\mathrm{M}[T] = \tau \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}.$$

Note that the same result can be obtained by a less strict argument. The average number of failures of the units of the ith type during the time τ is $\lambda_i\tau$; the average time taken to repair one unit of that type is $1/\mu_i$ and the average time which will be spent to repair all the units of type i that fail during the time τ is $\lambda_i\tau/\mu_i$. The average time that will be spent to repair all the units of all types is $\tau \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}$.

7.52*. The conditions of Problem 7.52 are changed so that each unit that fails is sent to a repairshop and the device is stopped for the time needed for the repair. When the device does not operate, other units cannot fail. Find (1) the mean value of the number of stops of the device during the time τ, (2) the mean value of the part of the time τ during which the device will be idle (it is also the mean time spent on repairs).

Solution. (1) We designate the number of stops during the time τ as X and find its mean value m_x. We shall use the following not very strict (but true) arguments to solve the problem. We represent the

operating process of the device, infinite in time, as a sequence of "cycles" (Fig. 7.52), each of which consists of a period of operation of the system (marked by a thick line) and a period of repair. The duration of each cycle is the sum of two random variables: T_{oper} (the duration of operation of the device) and T_{rep} (the time taken to repair it). The average

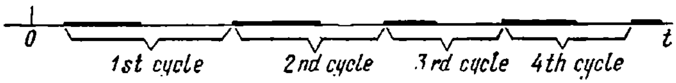


Fig. 7.52

duration of the operation of the device $m_{t_{\text{oper}}}$ can be calculated as the mean time between two consecutive failures in the flow of failures of intensity $\lambda = \sum_{i=1}^{n} \lambda_i$; it is

$$m_{t_{\text{oper}}} = 1/\lambda = 1\Big/ \sum_{i=1}^{n} \lambda_i .$$

We seek the mean time of repairs $m_{t_{\text{rep}}}$. We find it using the complete expectation formula on the hypotheses H_i = {a unit of type i is being repaired} ($i = 1, 2, \ldots, n$).

The probability of each hypothesis is proportional to the parameter λ_i:

$$\mathrm{P}(H_i) = \lambda_i \Big/ \sum_{i=1}^{n} \lambda_i = \lambda_i/\lambda .$$

The conditional expectation of the time taken by the repair on this hypothesis is $1/\mu_i$; hence

$$m_{t_{\text{rep}}} = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda\mu_i} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i} .$$

The average time of a cycle

$$m_{t_c} = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i} = \frac{1}{\lambda}\left(1 + \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}\right).$$

Let us now represent the sequence of stops of the device as a sequence of random points on the t-axis partitioned by intervals equal to $m_{t_{\text{rep}}}$. The average number of stops during the time τ is equal to the average number of random points on the interval τ long;

$$m_x = \tau/m_{t_{\text{rep}}} = \lambda\tau \Big/ \left(1 + \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}\right).$$

(2) During each cycle the device will be idle (will be repaired) for time $m_{t_{\text{rep}}} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}$ on the average; consequently, the average idle time

$$m_x m_{t_{\text{rep}}} = \frac{\lambda\tau}{1+\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}} \frac{1}{\lambda} \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i} = \tau \sum_{i=1}^{n} \frac{\lambda_i}{\mu_i} \Big/ \left(1+\sum_{i=1}^{n} \frac{\lambda_i}{\mu_i}\right).$$

7.53. A random variable X is normally distributed with characteristics m_x and σ_x. The random variables Y and Z are related to X, as $Y = X^2$ and $Z = X^3$. Find the covariances Cov_{xy}, Cov_{xz} and Cov_{yz}.

Solution. To simplify the calculations, we pass to the standardized variables and use the fact that for a standardized normal variable $\overset{\circ}{X} = X - m_x$ all the central moments of odd orders are equal to zero and $\mathrm{M}[\overset{\circ}{X}{}^2] = \sigma_x^2$, $\mathrm{M}[\overset{\circ}{X}{}^4] = 3\sigma_x^4$ (see Problem 5.26). Since

$$\overset{\circ}{Y} = (\overset{\circ}{X} + m_x)^2 - \mathrm{M}[X^2] = \overset{\circ}{X}{}^2 + 2\overset{\circ}{X} m_x + m_x^2 - \mathrm{Var}_x - m_x^2$$
$$= \overset{\circ}{X}{}^2 + 2\overset{\circ}{X} m_x - \sigma_x^2,$$

it follows that

$$\mathrm{Cov}_{xy} = \mathrm{M}[\overset{\circ}{X}\overset{\circ}{Y}] = M[\overset{\circ}{X}(\overset{\circ}{X}{}^2 + 2\overset{\circ}{X} m_x - \sigma_x^2)] = 2\sigma_x^2 m_x.$$

Furthermore

$$\overset{\circ}{Z} = (\overset{\circ}{X} + m_x)^3 - \mathrm{M}[X^3] = \overset{\circ}{X}{}^3 + 3\overset{\circ}{X}{}^2 m_x + 3\overset{\circ}{X} m_x^2 + m_x^3 - (3m_x\sigma_x^2 + m_x^3)$$
$$= X^3 + 3\overset{\circ}{X}{}^2 m_x + 3\overset{\circ}{X} m_x^2 - 3m_x\sigma_x^2,$$

and therefore

$$\mathrm{Cov}_{xz} = M[\overset{\circ}{X}\overset{\circ}{Z}] = M[\overset{\circ}{X}{}^4] + 3m_x\mathrm{M}[\overset{\circ}{X}{}^3] + 3m_x^2\,\mathrm{M}[\overset{\circ}{X}{}^2]$$
$$- 3m_x\sigma_x^2\mathrm{M}[\overset{\circ}{X}] = 3\sigma_x^4 + 3m_x^2\sigma_x^2,$$

and finally

$$\mathrm{Cov}_{yz} = \mathrm{M}[(\overset{\circ}{X}{}^2 + 2\overset{\circ}{X} m_x - \sigma_x^2)(\overset{\circ}{X}{}^3 + 3\overset{\circ}{X}{}^2 m_x + 3\overset{\circ}{X} m_x^2 - 3m_x\sigma_x^2)]$$
$$= 5m_x\mathrm{M}[\overset{\circ}{X}{}^4] + 6m_x(m_x^2 - \sigma_x^2)\,\mathrm{M}[\overset{\circ}{X}{}^2] + 3m_x\sigma_x^4 = 12m_x\sigma_x^4 + 6m_x^3\sigma_x^2.$$

7.54. An object whose mass is a g is weighed four times using an analytical balance: the results obtained are X_1, X_2, X_3, X_4. Their arithmetic mean is taken as the measured value of the mass: $Y = (X_1 + X_2 + X_3 + X_4)/4$. The results of the weighings are mutually independent. The balance gives a systematic error $m_x = +0.001$ g.

The mean square deviation of each weighing $\sigma_x = 0.002$ g. Find the mean value and the mean square deviation of the random variable Y.

Answer: $m_y = a + 0.001$ g; $\sigma_y = \sigma_x/2 = 0.001$ g.

7.55. Four independent measurements are made of the same variable X. Each measurement is characterized by the same expectation m_x and the same mean square deviation σ_x. The results of the measurements are X_1, X_2, X_3, X_4. We consider the differences between the consecutive measurements: $Y_1 = X_2 - X_1$, $Y_2 = X_3 - X_2$, $Y_3 = X_4 - X_3$. Find the mean values m_{y_1}, m_{y_2} and m_{y_3}, the mean square deviations σ_{y_1}, σ_{y_2}, σ_{y_3} and the standardized correlation matrix $\| r_{ij} \|$.

Solution.

$$m_{y_1} = m_{y_2} = m_{y_3} = 0, \quad \sigma_{y_1}^2 = \sigma_{y_2}^2 = \sigma_{y_3}^2 = 2\sigma_x^2,$$
$$\sigma_{y_1} = \sigma_{y_2} = \sigma_{y_3} = \sigma_x \sqrt{2}.$$

Since the variables $\overset{\circ}{X}_1, \overset{\circ}{X}_2, \overset{\circ}{X}_3, \overset{\circ}{X}_4$ are mutually independent,

$$\mathrm{Cov}_{y_1y_2} = \mathrm{M}[(\overset{\circ}{X}_2 - \overset{\circ}{X}_1)(\overset{\circ}{X}_3 - \overset{\circ}{X}_2)] = -\mathrm{M}[\overset{\circ}{X}{}_2^2] = -\sigma_x^2,$$
$$\mathrm{Cov}_{y_2y_3} = \mathrm{M}[(\overset{\circ}{X}_3 - \overset{\circ}{X}_2)(\overset{\circ}{X}_4 - \overset{\circ}{X}_3)] = -\mathrm{M}[\overset{\circ}{X}{}_3^2] = -\sigma_x^2,$$
$$\mathrm{Cov}_{y_1y_3} = \mathrm{M}[(\overset{\circ}{X}_2 - \overset{\circ}{X}_1)(\overset{\circ}{X}_4 - \overset{\circ}{X}_3)] = 0,$$
$$r_{y_1y_2} = r_{y_2y_3} = \frac{-\sigma_x^2}{2\sigma_x^2} = -\frac{1}{2}, \quad r_{y_1y_3} = 0,$$
$$\| r_{ij} \| = \begin{Vmatrix} 1 & -1/2 & 0 \\ & 1 & -1/2 \\ & & 1 \end{Vmatrix}.$$

7.56. A hole may occur with equal probability in one of the six walls and at any point on each wall of a cubical tank full of fuel. All the fuel above the hole leaks out of the tank. When sound, the tank is kept 3/4 full. Find the average amount of fuel which will remain in the tank after a hole appears.

Solution. We shall assume for simplicity that an edge of the tank is equal to unity. We designate the height of the wall as X and the amount of the remaining fuel as Y. Since the base area is equal to unity, we have

$$Y = \begin{cases} X & \text{for} \quad X < 0.75, \\ 0.75 & \text{for} \quad 0.75 < X < 1. \end{cases}$$

If the hole occurs higher than 0.75 of the way from the bottom of the tank ($X > 0.75$), then no fuel will leak out and the same amount of fuel, $Y = 0.75$, will remain in the tank. The probability of the latter case is equal to the fraction of the surface area of the tank which is above the 0.75 level:

$$\mathrm{P}\{Y = 0.75\} = \mathrm{P}\{X > 0.75\} = 1/6 + (4/6) \times 0.25 = 1/3.$$

If a hole appears in the bottom of the tank ($X = 0$), then all the fuel will run out, and the probability of this is equal to the fraction of the bottom area of the tank:

$$P\{Y = 0\} = P\{X = 0\} = 1/6.$$

If a hole appears in one of the walls of the tank $X < 0.75$ distant from the bottom, then the amount $Y = X$ of fuel will remain in the tank. The probability density in the interval (0, 0.75) is constant and equal to $(1 - 1/3 - 1/6)/0.75 = 2/3$. The average amount of fuel remaining in the tank

$$m_y = 0.75 \times \frac{1}{3} + 0 \times \frac{1}{6} + \int_0^{0.75} x \cdot \frac{2}{3}\, dx = 0.44.$$

7.57. A point a is fixed in the interval (0, 1) (Fig. 7.57). A random point X is uniformly distributed in that interval. Find the coefficient

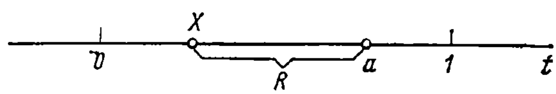


Fig. 7.57

of correlation between the random variable X and the distance R from the point a to X (the distance R is always assumed to be positive). Find the value of a for which the variables X and R are uncorrelated.

Solution. We find Cov_{xr} from the formula $\mathrm{Cov}_{xr} = M[XR] - m_x m_r$.

$$M[XR] = M[X \mid a - X \mid] = \int_0^1 x \mid a - x \mid f(x)\, dx = \int_0^1 x \mid a - x \mid dx$$

$$= \int_0^a x(a-x)\, dx - \int_a^1 x(a-x)\, dx = \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3} \quad m_x = \frac{1}{2};$$

$$m_r = \int_0^1 \mid a - x \mid dx = \int_0^a (a-x)\, dx - \int_0^1 (a-x)\, dx = a^2 - a + \frac{1}{2}.$$

Hence

$$\mathrm{Cov}_{xr} = \frac{a^3}{3} - \frac{a}{2} + \frac{1}{3} - \frac{1}{2}\left(a^2 - a + \frac{1}{2}\right) = \frac{a^3}{3} - \frac{a^2}{2} + \frac{1}{12}.$$

We find

$$\mathrm{Var}_x = \frac{1}{12}, \quad \sigma_x = \frac{1}{2\sqrt{3}}, \quad \mathrm{Var}_r = \alpha_2[R] - m_r^2,$$

$$\alpha_2[R] = \int_0^1 (a - x)^2\, dx = a^2 - a + \frac{1}{3},$$

$$\mathrm{Var}_r = 2a^3 - a^4 - a^2 + \frac{1}{12}, \quad \sigma_r = \sqrt{\mathrm{Var}_r},$$

Hence

$$r_{xr} = \left(\frac{a^3}{3} - \frac{a^2}{2} + \frac{1}{12}\right) \Big/ \left(\sqrt{2a^3 - a^4 - a^2 + \frac{1}{12}}\, \frac{1}{2\sqrt{3}}\right).$$

The equation $a^3/3 - a^2/2 + 1/12 = 0$ has only one root in the interval (0, 1), viz. $a = 1/2$. Therefore, the random variables X and R will be uncorrelated only for $a = 1/2$.

7.58. A car can travel along a highway with an arbitrary speed v $(0 \leqslant v \leqslant v_{\max})$. The higher the speed of the car, the higher the probability that it will be stopped by an inspector. Each time it is detained for an average time t_0. Inspectors are stationed at random along the route. The random number of traffic cops per unit length along the route has a Poisson distribution with parameter λ. There is a linear relationship between the probability of being stopped and the speed of the car: $p(v) = kv$ $(0 \leqslant v \leqslant v_{\max})$, where $k = 1/v_{\max}$. Find a rational speed for the car v_r at which it will cover the whole route s in a minimum time on the average.

Solution. The average time of covering the distance s is

$$t = s/v + \lambda sp(v)\, t_0 = s/v + \lambda skvt_0.$$

If the minimum of this function lies in the interval $(0, v_{\max})$, then it can be found from the equation

$$\frac{\partial t}{\partial v} = -\frac{s}{v^2} + \lambda skt_0 = 0,$$

whence

$$v = v_r = \sqrt{1/(\lambda kt_0)} = \sqrt{v_{\max}/(\lambda t_0)}.$$

This formula is valid for $v_r < v_{\max}$, i.e. for $v_{\max} > 1/(\lambda t_0)$.

Thus, for instance, for $v_{\max} = 100\,\text{km/h}$, $\lambda = 1/20\,\text{km}^{-1}$ and $t_0 = 20\,\text{min}$

$$v_r = \sqrt{100\Big/\left(\frac{1}{20} \times \frac{1}{3}\right)} \approx 77.5\,\text{km/h}.$$

If $v_{\max} < 1/(\lambda t_0)$, then the minimum of the function $t = s/v + \lambda sp(v)\, t_0$ lies outside of the interval $(0, v_{\max})$, and the speed $v_r = v_{\max}$ is the most advantageous. For example, if for the data given above the time the car is delayed by a cop decreases to 10 min, then $v_r = v_{\max} = 100$ km/h.

7.59. A number of independent trials are made in each of which an event A may occur with probability p. The trials continue until the event A occurs k times, and then they are terminated. A random variable X is the number of trials which must be made. Find its mean value, variance and mean square deviation.

Solution. We represent the random variable X as a sum

$$X = X_1 + X_2 + \ldots + X_k = \sum_{i=1}^{k} X_i,$$

where X_1 is the number of trials made till the first occurrence of the event A; X_2 is the number of trials made from the first to the second occurrence of the event A, . . .; X_k is the number of trials from the $(k-1)$th to the kth occurrence of the event A.

Each random variable X_i has a geometric distribution beginning with unity [see (4.0.32)], and its characteristics

$$\mathrm{M}[X_i] = 1/p; \quad \mathrm{Var}[X_i] = q/p^2 \quad (q = 1 - p).$$

Using the theorems for the addition of expectations and variances, we obtain

$$\mathrm{M}[X] = \sum_{i=1}^{k} \mathrm{M}[X_i] = kp, \quad \mathrm{Var}[X] = \sum_{i=1}^{k} \mathrm{Var}[X_i] = kq/p^2,$$

$$\sigma_x = \sqrt{\mathrm{Var}[X]} = \sqrt{kq}/p.$$

7.60. To assemble a reliable device, k high-quality homogeneous parts are needed. Each part is subjected to various tests and (independently of the other parts) may be found to be of high quality with probability p. When k high quality parts are selected, the tests are terminated. The supply of parts is practically unlimited. Find the mean value m_x and variance Var_x of the random variable X (the number of parts that have been tested). Find the maximum practically possible number of parts which will need to be tested, $X_{\max}$.

Answer. From the solution of the preceding problem we have

$$m_x = kp, \ \mathrm{Var}_x = kq/p^2, \ \sigma_x = \sqrt{kq}/p \ \text{and}$$

$$X_{\max} = kp + 3\sqrt{kq}/p.$$

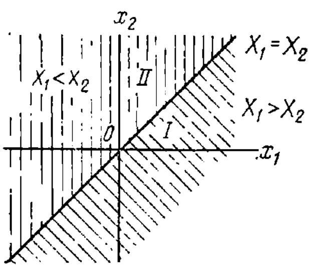


Fig. 7.61

7.61. According to a network of management planning the moment Y an operation begins is the maximum time two supporting operations X_1 and X_2 are finished. The random variables X_1 and X_2 are mutually independent and have probability densities $f_1(x_1)$ and $f_2(x_2)$. Find the mean value and variance of the random variable Y, as well as the range of its practically possible values.

Solution.

$$Y = \max\{X_1, X_2\} = \begin{cases} X_1 & \text{for } X_1 > X_2, \\ X_2 & \text{for } X_1 < X_2. \end{cases}$$

The domain I, where $X_1 > X_2$, and the domain II, where $X_1 < X_2$, are shown in Fig. 7.61.

$$m_y = \mathrm{M}[Y] = \int_{-\infty}^{\infty} x_1 f_1(x_1) \left\{ \int_{-\infty}^{x_1} f_2(x_2)\, dx_2 \right\} dx_1$$

$$+\int_{-\infty}^{\infty} x_2 f_2(x_2)\left\{\int_{-\infty}^{x_2} f_1(x_1)\,dx_1\right\}dx_2$$

$$=\int_{-\infty}^{\infty} x_1 f_1(x_1) F_2(x_1)\,dx_1+\int_{-\infty}^{\infty} x_2 f_2(x_2) F_1(x_2)\,dx_2,$$

where $F_1(x)$ and $F_2(x)$ are the distribution functions of the random variables X_1 and X_2 respectively.

$$\mathrm{M}[Y^2]=\int_{-\infty}^{\infty} x_1^2 f_1(x_1) F_2(x_1)\,dx_1+\int_{-\infty}^{\infty} x_2^2 f_2(x_2) F_1(x_2)\,dx_2,$$

$$\mathrm{Var}[Y]=\mathrm{M}[Y^2]-m_y^2,\quad \sigma_x=\sqrt{\mathrm{Var}[Y]}.$$

The range of the practically possible values of Y is $m_y \pm 3\sigma_y$.

7.62. Under the conditions of the preceding problem, the n supporting operations must be finished before the planned operation is begun (moment Y). The moments $X_1, X_2, \ldots, X_n$ when they are finished are mutually independent and have probability densities $f_1(x_1)$, $f_2(x_2)$, $\ldots$, $f_n(x_n)$. Answer the same questions.

Solution.

$$Y=\max\{X_1, X_2, \ldots, X_n\}=X_i \text{ for } X_i>X_j \quad (i=1, 2, \ldots, n, i\neq j),$$

$$m_y=\mathrm{M}[Y]=\int_{-\infty}^{\infty} x_1 f_1(x_1)\left\{\int_{-\infty}^{x_1}(n-1)\int_{-\infty}^{x_1} f_2(x_2) f_3(x_3)\right.$$

$$\left.\ldots f_n(x_n)\,dx_2\,dx_3\ldots dx_n\right\}dx_1$$

$$+\int_{-\infty}^{\infty} x_2 f_2(x_2)\left\{\int_{-\infty}^{x_2}(n-1)\int_{-\infty}^{x_2} f_1(x_1) f_3(x_3)\right.$$

$$\left.\ldots f_n(x_n)\,dx_1 dx_3\ldots dx_n\right\}dx_2$$

$$+\ldots+\int_{-\infty}^{\infty} x_n f_n(x_n)\left\{\int_{\infty}^{x_n}(n-1)\int_{-\infty}^{x_n} f_1(x_1) f_2(x_2)\ldots f_{n-1}(x_{n-1})\,dx_1\,dx_2\right.$$

$$\left.\ldots dx_{n-1}\right\}dx_n.$$

In this case the $(n-1)$-tuple integral decomposes into the product of $n-1$ simple integrals, and, therefore,

$$m_y=\sum_{i=1}^{n}\int_{-\infty}^{\infty} x_i f_i(x_i)\prod_{\substack{1\leqslant j\leqslant n\\ i\neq j}} F_j(x_i)\,dx_i$$

where $F_j(x_i)$ is the distribution function of the random variable X_j ($j = 1, 2, \ldots, n$) of the argument x_i. By analogy we find that

$$\mathrm{M}[Y^2] = \sum_{i=1}^{n} \int_{-\infty}^{\infty} x_i^2 f_i(x_i) \prod_{\substack{1\leqslant j\leqslant n \\ i\neq j}} E_j(x_i)\,dx_i; \quad \mathrm{Var}[Y] = \mathrm{M}[Y^2] - m_y^2.$$

If the random variables X_j ($j = 1, 2, \ldots, n$) have the same distribution with density $f(x)$ and distribution function $F(x)$, then

$$m_y = n \int_{-\infty}^{\infty} x f(x)\,[F(x)]^{n-1}\,dx.$$

$\mathrm{M}[Y^2]$ and $\mathrm{Var}[Y]$ can be calculated by analogy:

$$\mathrm{M}[Y^2] = n \int_{-\infty}^{\infty} x^2 f(x)\,[F(x)]^{n-1}\,dx \quad \text{and} \quad \mathrm{Var}[Y] = [Y^2] - m_y^2.$$

7.63. A device (designed as a maximum voltmeter) registers the larger of two voltages, V_1 and V_2. The random variables V_1 and V_2 are mutually independent and have the same density $f(v)$. Find the mean value of the voltmeter reading, i.e. of the random variable $V = \max\{V_1, V_2\}$, if $f(v) = \lambda e^{-\lambda v}$ ($v > 0$) is an exponential distribution with parameter λ.

Solution. In accordance with the solution of the preceding problem

$$\mathrm{M}[V] = 2\int_{-\infty}^{\infty} v f(v) F(v)\,dv = 2\int_{0}^{\infty} \lambda v e^{\;\lambda v}(1 - e^{-\lambda v})\,dv$$

$$= 2\int_{0}^{\infty} \lambda v e^{-\lambda v}\,dv - 2\lambda \int_{0}^{\infty} v e^{-2\lambda v}\,dv$$

$$= \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}.$$

It can be seen that $\mathrm{M}[V]$ is 1.5 times as large as the mean values of either V_1 or V_2.

7.64. *The mean value and the variance of the sum of a random number of random terms.* A random variable Z is a sum $Z = \sum_{i=1}^{Y} X_i$, where the random variables X_i are independent and have the same distribution with mean value m_x and variance Var_x; the number of terms Y is an integral random variable which does not depend on the terms X_i, has a mean value m_y and a variance Var_y. Find the mean value and variance of the random variable Z.

Solution. Assume that the discrete random variable Y has an ordered series

$$Y: \begin{array}{|c|c|c|c|c}1 & 2 & \dots & k & \dots \\ \hline p_1 & p_2 & \dots & p_k & \dots\end{array}.$$

We make a hypothesis $\{Y = k\}$. On this hypothesis

$$\mathrm{M}[Z \mid Y = k] = \sum_{i=1}^{k} \mathrm{M}[X_i] = km_x.$$

By the complete expectation formula

$$\mathrm{M}[Z] = \sum_k kp_k m_x = m_x \sum_k kp_k = m_x \mathrm{M}[Y] = m_x m_y.$$

Similarly, the conditional second moment about the origin of the variable Z

$$\mathrm{M}[Z^2 \mid Y = k] = \mathrm{M}\left[\left(\sum_{i=1}^{k} X_i\right)^2\right] = \mathrm{M}\left[\sum_{i=1}^{k} X_i^2 + 2\sum_{i<j} X_i X_j\right]$$
$$= \sum_{i=1}^{k} \alpha_2[X] + 2\sum_{i<j} m_x m_x = k\alpha_2[X] + k(k-1)m_x^2$$
$$= k[\mathrm{Var}_x + m_x^2] + k(k-1)m_x^2 = k\mathrm{Var}_x + k^2 m_x^2.$$

By the complete expectation formula

$$\alpha_2[Z[= \mathrm{Var}_x \sum_k kp_k + m_x^2 \sum_k k^2 p_k = \mathrm{Var}_x m_y + m_x^2 \alpha_2[Y]$$
$$= \mathrm{Var}_x m_y + m_x^2(\mathrm{Var}_y + m_y^2) = \mathrm{Var}_x m_y + m_x^2 \mathrm{Var}_y + m_x^2 m_y^2,$$
$$\mathrm{Var}[Z] = \alpha_2[Z] - m_x^2 m_y^2 = \mathrm{Var}_x m_y + m_x^2 \mathrm{Var}_y.$$

Thus we have

$$m_z = m_x m_y, \quad \mathrm{Var}_z = \mathrm{Var}_x m_y + m_x^2 \mathrm{Var}_y.$$

If the random variable Y has a Poisson distribution with parameter a, then

$$m_z = am_x, \quad \mathrm{Var}_z = a(\mathrm{Var}_x + m_x^2).$$

7.65. A message is sent over a communication channel in a binary code consisting of n symbols 0 or 1. Each is equally probable and independent. Find the mean value and variance of the number X of changes in symbols in the message as well as the maximum practically possible number of changes.

Solution. Let us consider the $n-1$ changes from one symbol to the next to be $n-1$ independent trials and consider, for each of them, a variable X_i, which is the indicator of a change of symbol. This value is equal to unity if a symbol is changed and to zero if there is no change.

$$X = \sum_{i=1}^{n-1} X_i, \quad \mathrm{M}[X] = \sum_{i=1}^{n-1} \mathrm{M}[X_i] = \sum_{i=1}^{n-1} p = (n-1)p,$$

where p is the probability that a symbol is changed at a given (ith) interval. Similarly,

$$\mathrm{Var}[X] = \sum_{i=1}^{n-1} \mathrm{Var}[X_i] = (n-1)\,p\,(1-p).$$

It is evident in this case that $p = 1/2$ and $\mathrm{M}[X] = (n-1)/2$, $\mathrm{Var}[X] = (n-1)/4$. By the three-sigma rule $X_{\max} = (n-1)/2 + 3\sqrt{n-1}/2$.

7.66. A group of four radar units scans a region of space in which there are three targets: S_1, S_2, S_3. The region is scanned for a time τ. During that time each unit, independently of the others, may detect each target with probability p_i, which depends on the ordinal number of the target, and transmits its coordinates to the central control station. Find the mean value of the number of targets X whose coordinates will be registered at the station.

Solution. We designate as X_i the indicator of the event A_i = {the ith target is detected}:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th target is detected,} \\ 0 & \text{if the } i\text{th target is not detected} \end{cases} \quad (i = 1, 2, 3).$$

The mean value of the random variable X_i is $\mathrm{M}[X_i] = \mathrm{P}$ {the ith target is detected} $= 1-(1-p_i)^4$.

Since $X = \sum_{i=1}^{3} X_i$, it follows that

$$\mathrm{M}[X] = \sum_{i=1}^{3} [1-(1-p_i)^4] = 3 - \sum_{i=1}^{3} (1-p_i)^4,$$

7.67. The conditions of the preceding problem are changed so that the probabilities of detecting a target by different units differ: the jth unit detects the ith target with probability p_{ij} $(i = 1, 2, 3;\ j = 1, 2, 3, 4)$.

Answer. $\mathrm{M}[X] = 3 - \sum_{i=1}^{3} \prod_{j=1}^{4} (1-p_{ij})$.

7.68. A radar installation scans a region of space in which there are four targets. Depending on the time of scanning, the probability of detecting a target is a function $p(t)$ and does not depend on the detection of the other targets. Find (1) the mean value of the time T_1 in which at least one target will be detected and (2) the mean value of the time T_4 in which all the four targets will be detected.

Solution. (1) The probability that not a single target will be detected in time t is $[1-p(t)]^4$; the probability that at least one target will be detected in that time is $1-[1-p(t)]^4$. This is simply a distribution function $F_1(t)$ of a random variable T_1. As we proved in Problem 5.28,

for a nonnegative random variable T_1

$$\mathrm{M}[T_1]=\int_0^\infty [1-F_1(t)]\,dt=\int_0^\infty [1-p(t)]^4\,dt.$$

(2) The probability that all the four targets will be detected in time t is $[p(t)]^4$. This is a distribution function $F_4(t)$ of the random variable T_4. Its mean value is

$$\mathrm{M}[T_4]=\int_0^\infty [1-F_4(t)]\,dt=\int_0^\infty \{1-[p(t)]^4\}\,dt.$$

For example, if the probability $p(t)$ is defined by the formula $p(t) = 1 - e^{-\alpha t}$, then

$$\mathrm{M}[T_1]=\int_0^\infty [1-(1-e^{-\alpha t})]^4\,dt=\int_0^\infty e^{-4\alpha t}\,dt=\frac{1}{4\alpha},$$

$$\mathrm{M}[T_4]=\int_0^\infty [1-(1-e^{-\alpha t})^4]\,dt=\frac{25}{12\alpha}.$$

In this example the mean time of detecting all the four targets is eight times as long as the mean time of detecting at least one target.

7.69. We make n trials in each of which an event A may or may not occur. A random variable X is defined as the number of occurrences of the event A in the whole series of trials. Find the mean value of the variable X.

Solution. We represent the variable X as the sum of n random variable X_i $(i = 1, \ldots, n)$, where X_i is the indicator of the event A in the ith trial:

$$X_i=\begin{cases}1 & \text{if the event } A \text{ occurs on the } i\text{th trial,}\\ 0 & \text{if the event } A \text{ does not occur on the } i\text{th trial.}\end{cases}$$

$$X=\sum_{i=1}^n X_i;\quad \mathrm{M}[X]=\sum_{i=1}^n \mathrm{M}[X_i],$$

$\mathrm{M}[X_i] = p_i$, where p_i is the probability of occurrence of the event on the ith trial. Thus we have

$$\mathrm{M}[X]=\sum_{i=1}^n p_i. \tag{7.69.1}$$

In particular, if $p_1 = p_2 = \ldots = p$, then

$$\mathrm{M}[X] = np. \tag{7.69.2}$$

It should be emphasized that for formulas (7.69.1) and (7.69.2) to be applicable, the trials need not necessarily be independent.

7.70. Considering the trials to be independent, under the conditions of Problem 7.69, find the variance of the random variable X.

Solution. By the theorem for the addition of variances,

$$\mathrm{Var}[X] = \mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{Var}[X_i].$$

The variance of the indicator of the event A in the ith trial [see formula (4.0.19)] is $p_i(1-p_i)$, then

$$\mathrm{Var}[X] = \sum_{i=1}^{n} p_i(1-p_i). \tag{7.70.1}$$

In the particular case when $p_1 = p_2 = \ldots = p_n = p$,

$$\mathrm{Var}[X] = np(1-p). \tag{7.70.2}$$

It should be emphasized that formulas (7.70.1) and (7.70.2) can be used only for independent trials.

7.71. We make n independent trials in each of which an event A may occur or not occur. A random variable X is defined as the number of occurrences of the event A. Find its variance.

Solution. $X = \sum_{i=1}^{n} X_i$, where X_i is the indicator of the event A in the ith trial. By the general formula (7.0.28) for the variance of random variables, we have

$$\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2\sum_{i<j} \mathrm{Cov}_{x_i x_j},$$

where $\mathrm{Cov}_{x_i x_j}$ is the covariance of the random variables X_i and X_j:

$$\mathrm{Cov}_{x_i x_j} = \mathrm{M}[X_i X_j] - \mathrm{M}[X_i]\,\mathrm{M}[X_j].$$

The variable $X_i X_j$ turns into unity only if $X_i = 1$ and $X_j = 1$ (i.e. the event A occurred both on the ith and on the jth trial). $\mathrm{M}[X_i X_j] = p_{ij}$, where p_{ij} is the probability that the event A occurred both on the ith and on the jth trial. $\mathrm{Cov}_{x_i x_j} = p_{ij} - p_i p_j$; hence

$$\mathrm{Var}[X] = \sum_{i=1}^{n} p_i(1-p_i) + 2\sum_{i<j} (p_{ij} - p_i p_j). \tag{7.71}$$

Thus, in order to find the variance of the number of occurrences of an event in n dependent trials, it is not sufficient to know the probability p_i of the occurrence of the event in each trial; it is also necessary to know the probability p_{ij} of the simultaneous occurrence of the event in each pair of trials.

In particular, when $p_1 = p_2 = \ldots = p_n = p$ and when p_{ij} does not depend on i or j and is equal to P, formula (7.71) assumes the form

$$\mathrm{Var}[X] = np(1-p) + n(n-1)(\mathrm{P} - p^2),$$

where $P = p_{ij}$ is the probability of the simultaneous occurrence of the event in any pair of trials.

7.72. There are a white and b black balls in an urn. We draw k balls at random. A random variable X is the number of white balls among the drawn balls. Without resorting to the distribution of the random variable X (hypergeometric, see Chapter 4), find the mean value and variance of the random variable X.

Solution. $X=\sum_{i=1}^{k} X_i$, where

$$X_i = \begin{cases} 1 & \text{if the } i\text{th drawn ball is white,} \\ 0 & \text{if it is black.} \end{cases}$$

$$\mathrm{M}[X]=\sum_{i=1}^{k} \mathrm{M}[X_i]=kp,$$

where p is the probability of drawing a white ball;

$$p = a/(a+b), \quad \mathrm{M}[X] = ka/(a+b).$$

We seek the variance of the random variable X as the variance of the number of occurrences of the event in k dependent trials [see Problem 7.71]:

$$\operatorname{Var}[X] = kp(1-p) + k(k-1)(P-p^2). \tag{7.72}$$

Let us find P, which is the probability that some two drawn balls (the ith and the jth) are white: $P=\frac{a}{a+b}\,\frac{a-1}{a+b-1}$. Substituting this expression and $p=\frac{a}{a+b}$ into (7.72), we obtain

$$\operatorname{Var}[X]=\frac{kab}{(a+b)^2}+k(k-1)\left(\frac{a}{a+b}\,\frac{a-b}{a+b-1}-\frac{a^2}{(a+b)^2}\right).$$

7.73*. A train consisting of m carriages arrives at a railway shunting yard. Out of these carriages m_1 are bound for A_1, m_2 are bound for $A_2, \ldots, m_k$ are bound for A_k $\left(\sum_{i=1}^{k} m_i = m\right)$. The carriages are located at random along the train and irrespective of their destinations. If two adjacent carriages have the same destination, they should not be uncoupled; if their destinations are different, then they are uncoupled. Find the mean value and variance of the number of uncouplings that must be made, and also estimate (using the three-sigma rule) the range of the practically possible number of uncouplings.

Solution. Let us consider the $m-1$ couplings of the carriages to be $m-1$ possible positions at each of which the carriages may be uncoupled or not. We designate as X the number of the uncouplings and the number of "nonuncouplings" as Y; $Y = m-1-X$. It is easier

to manipulate the random variable Y. We represent it as the sum $Y = \sum_{i=1}^{m-1} Y_i$, where

$$Y_i = \begin{cases} 1 & \text{if there is no uncoupling at the } i\text{th position,} \\ 0 & \text{if there is an uncoupling} \end{cases}$$

(the indicator of uncoupling at the ith position).

By the theorem for the addition of expectations, $M[Y] = \sum_{i=1}^{m-1} M[Y_i]$, $M[Y_i] = q_i$, where q_i is the probability that there is no uncoupling at the ith position. All the positions are under identical conditions: $q_1 = q_2 = \ldots = q_{m-1} = q$. The probability q that there is no uncoupling at the ith position is equal to the probability that two carriages corresponding to the ith position have the same destination. Adding up the probabilities that these carriages are bound for the 1st, the 2nd, . . ., the kth point of destination, we find

$$q = \sum_{i=1}^{k} \frac{m_i(m_i-1)}{m(m-1)} = \frac{1}{m(m-l)} \sum_{i=1}^{k} m_i(m_i-1),$$

$$M[Y] = \frac{m-1}{m(m-1)} \sum_{i-1}^{k} m_i(m_i-1) = \frac{1}{m} \sum_{i-1}^{k} m_i(m_i-1).$$

Hence

$$M[X] = m - 1 - \frac{1}{m} \sum_{i=1}^{k} m_i(m_i-1)$$

Since the variables X and Y differ by a constant, Var $[X]$ = Var $[Y]$. The variance of the random variable Y can be found from formula (7.0.28) for the variance of the sum:

$$\mathrm{Var}[Y] = \sum_{i=1}^{m-1} \mathrm{Var}[Y_i] + 2 \sum_{i<j} \mathrm{Cov}_{y_i y_j}$$

The covariance of the random variables Y_i and Y_j can be found from the formula

$$\mathrm{Cov}_{y_i y_j} = M[Y_i Y_j] - M[Y_i] M[Y_j] = M[Y_i Y_j] - q^2.$$

The product $Y_i Y_j$ is zero if there is an uncoupling at least at one position, and is equal to unity if there is no uncoupling at either position. Hence

$$\mathrm{Cov}_{y_i y_j} = P\{Y_i = 1, Y_j = 1\} - q^2.$$

We seek the probability that there is no uncoupling at either position (i or j). This probability is different when the positions are adjacent ($j - i = 1$) and not adjacent ($j - i > 1$). We designate the probability as q_{ad} when the positions are adjacent and as $q_{\text{n.ad}}$ when they are not adjacent. The probability that there is no uncoupling at two adjacent positions is equal to the probability that the ith, the $(i+1)$th and the $(i+2)$th carriages have the same destination

$$q_{\text{ad}} = \sum_{i=1}^{k} \frac{m_i(m_i-1)(m_i-2)}{m(m-1)(m-2)} = \frac{1}{m(m-1)(m-2)} \sum_{i=1}^{k} m_i(m_i-1)(m_i-2).$$

For two adjacent positions $M[Y_iY_j] = q_{\text{ad}}$, $\text{Cov}_{y_iy_j} = q_{\text{ad}} - q^2$. The probability that there are uncouplings at two non-adjacent positions $(j-i>1)$ is equal to the probability that each pair of carriages, between which this position is, has the same destination. This may happen either if all the four carriages are bound for the same place (the lth), or if the first pair is bound for the lth place and the second pair for the rth place. The probability of the first variant is $\frac{m_l(m_l-1)(m_l-2)(m_l-3)}{m(m-1)(m-2)(m-3)}$ and that of the second variant is $\frac{m_l(m_l-1)m_r(m_r-1)}{m(m-1)(m-2)(m-3)}$. The total probability that there is no uncoupling at two non-adjacent positions is

$$q_{\text{n.ad}} = \frac{1}{m(m-1)(m-2)(m-3)} \sum_{l=1}^{k} \Big[m_l(m_l-1)(m_l-2)(m_l-3) + \sum_{r \neq l} m_l(m_l-1)m_r(m_r-1) \Big].$$

The covariance $\text{Cov}_{y_iy_j}$ for two non-adjacent positions is $q_{\text{n.ad}} - q^2$. With due regard for the number of adjacent $(m-2)$ and non-adjacent $(C_{m-1}^2 - m + 2)$ pairs of positions, we obtain

$$\text{Var}[X] = \text{Var}[Y] = (m-1)q(1-q) + 2(m-2)(q_{\text{ad}} - q^2) + [(m-2)(m-3) - 2(m-2)](q_{\text{n.ad}} - q^2), \quad \sigma_x = \sigma_y = \sqrt{\text{Var}[Y]}.$$

The range of the practically possible values of X is $M[X] \pm 3\sigma_x$.

Remark. The calculations only have sense for $m > 3$. If in the formulas for q_{ad} and $q_{\text{n.ad}}$ some of the terms are negative, then the corresponding terms are assumed to be zero.

7.74*). A number of trials are made, in each of which an event A ("success") may occur or not occur. The probability that the event A occurs at least once in the first m trials is defined by a nondecreasing function $R(m)$ (Fig. 7.74)*). Find the average number of trials which will be made before a success is achieved.

*) The function $R(m)$ is defined only for an integer m, but to make the figure clearer, the points in Fig. 7.74 have been connected by lines.

Solution. Let us assume that the trials are not terminated after a success is achieved. Each of them (except for the first) can be "necessary" if a success is not yet achieved, and "superfluous" if a success is achieved. Let us connect a random variable X_i with each (ith) trial and assume it to be unity if the trial is necessary and zero if it is superfluous.

We consider the random variable Z, the number of trials which must be made before a success is achieved. It is equal to the sum of all the random variables X_i, the first of which X_1 is always equal to unity (the first trial is always necessary):

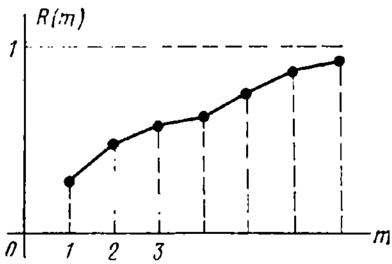


Fig. 7.74

$$Z = X_1 + \sum_{i=2}^{\infty} X_i.$$

The ordered series of the random variable X_i ($i > 1$) has the form

$$X_i: \left|\begin{array}{c|c} 0 & 1 \\ \hline R(i-1) & 1-R(i-1) \end{array}\right|$$

(if a success is achieved in the preceding $i-1$ trials, the ith trial is superfluous, if it is not, then the trial is necessary). The mean value of the random variable X_i is

$$\mathrm{M}[X_i] = 0 \cdot R(i-1) + 1 \cdot [1 - R(i-1)] = 1 - R(i-1).$$

We can evidently use the same notation for $\mathrm{M}[X_1] = \mathrm{M}[1] = 1 - R(0)$ $(R(0) = 0)$. Hence

$$\mathrm{M}[Z] = \sum_{i=1}^{\infty} [1 - R(i-1)] = \sum_{k=0}^{\infty} [1 - R(k)].$$

7.75*. For the conditions of the preceding problem find the variance of the number of trials needed to achieve a success.

Solution. The random variables $X_1, X_2, \ldots, X_i, \ldots$ are mutually dependent, and we cannot, therefore, simply add their variances together. We must find the second moment about the origin of the variable Z:

$$\mathrm{M}[Z^2] = \mathrm{M}[(\sum_{i=1}^{\infty} X_i)^2] = \mathrm{M}[\sum_{i=1}^{\infty} X_i^2] + 2\mathrm{M}[\sum_{j>i} X_i X_j],$$

$$\mathrm{M}[X_i^2] = 1^2 \cdot [1 - R(i-1)] = 1 - R(i-1).$$

The variable $X_i X_j$ is unity if X_i and X_j are both unity, i.e. both trials (the ith and the jth) are necessary. For the two trials (the ith and the jth) to be necessary, a later (jth) trial must be necessary:

$$\mathrm{M}[X_i X_j] = \mathrm{P}\{X_i = 1, X_j = 1\} = 1 - R(j-1).$$

Hence

$$\mathrm{M}[Z^2] = \sum_{i=1}^{\infty} [1 - R(i-1)] + 2 \sum_{j>i} [1 - R(j-1)].$$

The last sum can be calculated as follows: we first fix j and take a sum over i from 1 to $j-1$; then we take a sum over j from 2 to ∞. Under the summation sign over i, for a fixed j, all the terms are equal and do not depend on i; their number is equal to $j-1$. Therefore,

$$\sum_{j<i}[1-R(j-1)]=\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}[1-R(j-1)]$$

$$=\sum_{j=2}^{\infty}(j-1)[1-R(j-1)]$$

$$=\sum_{k=1}^{\infty}k[1-R(k)]=\sum_{k=0}^{\infty}k[1-R(k)].$$

Hence

$$\operatorname{Var}[Z]=\sum_{k=0}^{\infty}[1-R(k)]+2\sum_{k=0}^{\infty}k[1-R(k)]-\{\sum_{k=0}^{\infty}[1-R(k)]\}^2.$$

7.76. The probability that a target will be detected by a radar installation increases exponentially with the number of surveillance cycles n:

$$\mathrm{P}(n)=1-\alpha^n \quad (0<\alpha<1). \tag{7.76}$$

Find the mean value of the number of cycles X after which the target will be detected.

Solution. Setting $\alpha=1-p$, we rewrite formula (7.76) as follows:

$$\mathrm{P}(n)=1-(1-p)^n.$$

We can see that the cycles are mutually independent and the probability that the target will be detected in each cycle is $p=1-\alpha$. Under these conditions the variable X has a geometric distribution beginning with unity, and its mean value

$$\mathrm{M}[X]=1/p=1/(1-\alpha).$$

We can obtain the same result using the solution of Problem 7.74:

$$\mathrm{M}[X]=\sum_{k=0}^{\infty}[1-\mathrm{P}(k)]=\sum_{k=0}^{\infty}\alpha^k=\frac{1}{1-\alpha}.$$

7.77. A radar installation scans a region of space in which there are n targets. During each surveillance cycle the radar may detect a target (independently of the other targets and other cycles) with probability p. How many cycles will be necessary (1) for the probability of detecting all the targets to become no less than P, and (2) for the mean number of detected targets to become no less than a given number $m<n$?

Solution. (1) We designate the probability of detecting all n targets in k cycles as $G_n(k)$, $G_n(k) = [G(k)]^n$, where $G(k)$ is the probability of detecting one target at least once in k cycles. $G(k) = 1 - (1-p)^k$, whence $G_n(k) = [1-(1-p)^k]^n$. We set $G_n(k) \geqslant P$, $[1-(1-p)^k]^n \geqslant \mathrm{P}$, $(1-p)^k \leqslant 1 - \sqrt[n]{\mathrm{P}}$, $k \log (1-p) \leqslant \log(1-\sqrt[n]{\mathrm{P}})$. Since $\log(1-p) < 0$, the sign of the product resulting from the multiplication by this expression is reversed, hence

$$k \geqslant \log(1-\sqrt[n]{\mathrm{P}})/\log(1-p).$$

(2) By the theorem on the addition of expectations, the mean number of targets $\mathrm{M}[X]$ detected during k cycles is $n[1-(1-p)^k]$. Setting $n[1-(1-p)^k] \geqslant m$ and solving the inequality for k, we obtain

$$k \geqslant \log\left(1-\frac{m}{n}\right)\Big/\log(1-p).$$

7.78*. We make n independent trials under different conditions, the probability that an event A may occur on the first, the second, and so on, trial is $p_1, p_2, \ldots, p_n$. We must find the mean value and variance of the random variable X, which is the total number of occurrences of the event A. To simplify the calculations, we average the probabilities p_i and replace them by one, constant, probability:

$$\bar{p} = \frac{1}{n}\sum_{i=1}^{n} p_i.$$

Will $\mathrm{M}[X]$ and $\mathrm{Var}[X]$ be calculated correctly?

Solution. $\mathrm{M}[X]$ will be calculated correctly:

$$\mathrm{M}[X] = n\bar{p} = \frac{n}{n}\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} p_i.$$

As to the variance, it will be overestimated. To prove this, we compare the approximate expression for the variance

$$\widetilde{\mathrm{Var}}_x = n\overline{pq}, \text{ where } \bar{q} = 1-\bar{p} = \frac{1}{n}\sum_{i=1}^{n}(1-p_i)$$

with its exact value

$$\mathrm{Var}_x = \sum_{i=1}^{n} p_i q_i, \quad \text{where } q_i = 1-p_i.$$

We transform the sum in two ways:

$$\sum_{i=1}^{n}(p_i-\bar{p})(q_i-\bar{q}) = \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} p_i\bar{q} - \sum_{i=1}^{n}\bar{p}q_i + n\overline{pq}$$
$$= \sum_{i=1}^{n} p_i q_i - n\bar{p}\bar{q},$$

$$\sum_{i=1}^{n}(p_i-\bar{p})(q_i-\bar{q})=\sum_{i=1}^{n}(p_i-\bar{p})(1-p_i-1+\bar{p})$$
$$=-\sum_{i=1}^{n}(p_i-\bar{p})^2\leqslant 0.$$

Hence

$$\sum_{i=1}^{n}p_iq_i-n\overline{p}\overline{q}=\mathrm{Var}_x-\widetilde{\mathrm{Var}}_x\leqslant 0;\quad \widetilde{\mathrm{Var}}_x\geqslant \mathrm{Var}_x,$$

and that is what we wished to prove. Note that the equality in $\widetilde{\mathrm{Var}}_x \geqslant \mathrm{Var}_x$ is attained only for $p_1 = p_2 = \ldots = p_n = p$.

7.79*. Prove that if $X_1, X_2, \ldots, X_n$ are mutually independent, positive and similarly distributed, then

$$\mathrm{M}\left[\sum_{i=1}^{k}X_i\Big/\sum_{i=1}^{n}X_j\right]=\frac{k}{n}.$$

Solution. Since all the variables $X_1, X_2, \ldots, X_n$ are positive, the denominator never contains zero. By the theorem on the addition of expectations

$$\mathrm{M}\left[\sum_{i=0}^{k}X_i\Big/\sum_{j=1}^{n}X_j\right]=\sum_{i=1}^{k}\mathrm{M}\left[X_i\Big/\sum_{i=1}^{n}X_j\right].$$

Since all the variables $X_1, X_2, \ldots, X_n$ have the same distribution, we have

$$\mathrm{M}\left[X_i\Big/\sum_{j=1}^{n}X_j\right]=\mathrm{M}\left[X_m\Big/\sum_{j=1}^{n}X_j\right]$$

for any i and m. We designate their common value as α

$$\mathrm{M}\left[X_i\Big/\sum_{j=1}^{n}X_j\right]=\alpha\quad (i=1,\ 2,\ \ldots,\ n).$$

At the same time it is clear that the sum of all the variables of the form $X_i\Big/\sum_{j=1}^{n}X_j$ is equal to unity, and, consequently, the mean value is also unity:

$$\mathrm{M}\left[\sum_{i=1}^{n}X_i\Big/\sum_{j=1}^{n}X_j\right]=\sum_{i=1}^{n}\mathrm{M}\left[X_i\Big/\sum_{j=1}^{n}X_j\right]=1\cdot$$

Replacing the expression under the sign of the mean value by α, we get $\sum_{j=1}^{n} \alpha = n\alpha = 1$, whence we have $\alpha = 1/n$. Consequently,

$$M\left[\sum_{i=1}^{k} X_i \Big/ \sum_{j=1}^{n} X_j\right] = \sum_{i=1}^{k} M\left[X_i \Big/ \sum_{j=1}^{n} X_j\right] = \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{n},$$

and that is what we wished to prove.

7.80. There are n loads whose energy requirements per unit time are independent random variables $X_1, X_2, \ldots, X_n$ with the same (arbitrary) distribution. We consider a random variable $Z_i = X_i/\sum_{i=1}^{n} X_i$ $(i = 1, \ldots, n)$, which is the fraction of the total consumption of energy taken by the ith load. Prove that for any i the mean value of the random variable Z_i is $1/n$.

Solution. It is clear that the mean value of the variable Z_i exists since it is included between zero and unity. The distributions of all the random variables Z_i $(i = 1, \ldots, n)$ are the same (since the problem is completely symmetrical) and their mean values are equal: $M[Z_1] = M[Z_2] = \ldots = M[Z_n]$. Since the sum of all the random variables Z_i is equal to unity, the sum of their mean values is also equal to unity:

$$M[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} M[Z_i] = 1, \quad \text{i.e. } nM[Z_i] = 1,$$

$$M[Z_i] = 1/n \quad (i = 1, \ldots, n).$$

7.81. An equilateral triangle with side $a = 3$ cm can be constructed by drawing a line segment of length a from an arbitrary point 0, then drawing another line segment a at 60° to the first, and connecting the endpoint of the second line to the origin (Fig. 7.81). The segments of length a are measured with a ruler graduated in millimetres. The maximum possible error is thus 0.5 mm. The angle is constructed with a protractor with a maximum possible error of 1°. Using the linearization method, find the mean value and the mean square deviation of the length of the third side X.

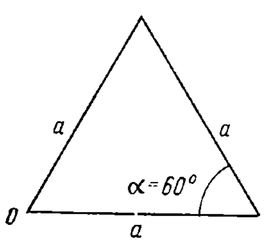


Fig. 7.81

Solution. We designate the actual length of the first side as X_1, that of the second side as X_2, and the actual angle as Θ. These random variables can be considered to be mutually independent. We have

$$X = \sqrt{X_1^2 + X_2^2 - 2X_1X_2\cos\Theta}.$$

Using the linearization method, we find

$$m_x = \sqrt{m_{x_1}^2 + m_{x_2}^2 - 2m_{x_1}m_{x_2}\cos m_\vartheta},$$

where $m_{x_1} = m_{x_2} = 30\,\text{mm}$, $\cos m_\vartheta = 0.5$, whence $m_x = \sqrt{900+900-900} = 30\,\text{mm}$. Furthermore,

$$\left(\frac{\partial x}{\partial x_1}\right)_m = \left(\frac{1}{2}\,\frac{2x_1 - 2x_2\cos\vartheta}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\vartheta}}\right)_m = \frac{1}{2},$$

$$\left(\frac{\partial x}{\partial x_2}\right)_m = \left(\frac{1}{2}\,\frac{2x_2 - 2x_1\cos\vartheta}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\vartheta}}\right)_m = \frac{1}{2},$$

$$\left(\frac{\partial x}{\partial \vartheta}\right)_m = \left(\frac{1}{2}\,\frac{2x_1x_2\sin\vartheta}{\sqrt{x_1^2+x_2^2-2x_1x_2\cos\vartheta}}\right)_m = \frac{30\sqrt{3}}{2} = 15\sqrt{3}\,\text{mm}.$$

We calculate

$$\mathrm{Var}_x \approx \left(\frac{\partial x}{\partial x_1}\right)_m^2 \mathrm{Var}_{x_1} + \left(\frac{\partial x}{\partial x_2}\right)_m^2 \mathrm{Var}_{x_2} + \left(\frac{\partial x}{\partial \vartheta}\right)_m^2 \mathrm{Var}_\vartheta.$$

We do not know the variances of the arguments, we only know the maximum practically possible deviations of their mean values: $\Delta x_1 = \Delta x_2 = 0.5$ mm; $\Delta\vartheta = 1° = 0.01745$ rad. Setting approximately $\sigma_{x_1} = \sigma_{x_2} = \Delta x_1/3 = 0.167$ mm, $\mathrm{Var}_{x_1} = \mathrm{Var}_{x_2} = 0.0278\ \text{mm}^2$, $\sigma_\vartheta = \Delta\vartheta/3 = 0.00582$ rad, $\mathrm{Var}_\vartheta = 3.39\cdot 10^{-5}$ rad, we get $\mathrm{Var}_x = 0.25\cdot 0.0278\cdot 2 + 675\cdot 3.39\cdot 10^{-5} \approx 0.0368\ \text{mm}^2$, $\sigma_x \approx 0.192$ mm.

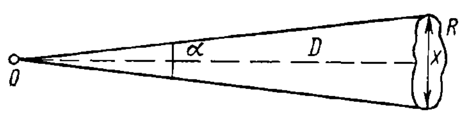


Fig. 7.82

7.82. The distance D from a point O to an object R is defined by measuring the angle α at which the object is seen from the point O (Fig. 7.82), and then, given the linear dimension of the object X, and assuming the angle α to be small, the distance is found from the approximate formula

$$D = X/[2\sin(\alpha/2)] \approx X/\alpha.$$

The linear dimension of the object X, as seen from the point O, may change, depending of its accidental rotation, in the range from 8 to 12 m. The angle α is determined to within 0.0001 rad. The distance D is large as compared to the size of the object X. Find an approximate mean square deviation σ_D of the error in determining the distance D if the measure of the angle α is equal to 0.001 rad.

Solution. Using the linearization method, we have

$$\sigma_D^2 = \left(\frac{\partial D}{\partial x}\right)_m^2 \sigma_x^2 + \left(\frac{\partial D}{\partial \alpha}\right)_m^2 \sigma_\alpha^2.$$

We assume the linear dimension X to be uniformly distributed in the interval (8, 12): $\sigma_x = (12-8)/(2\sqrt{3}) = 2/\sqrt{3}$ m, $\sigma_x^2 = 4/3\ \text{m}^2$; $m_x = 10$ m. Furthermore,

$$\sigma_\alpha \approx \frac{1}{3}\,0.0001;\quad \sigma_\alpha^2 = \frac{1}{9}\,10^{-8};\quad m_\alpha = 0.001,$$

whence

$$\sigma_D^2 = \left(\frac{1}{\alpha}\right)_m^2 \sigma_x^2 + \left(-\frac{x}{\alpha^2}\right)_m^2 \sigma_\alpha^2 = \frac{13}{9}\,10^6 \text{ m}^2,$$

$$\sigma_D = \frac{\sqrt{13}}{3}\cdot 10^3 = 1.20\cdot 10^3 \text{ m}.$$

7.83. There are two almost linear functions of n random arguments: $Y = \varphi_y(X_1, X_2, \ldots, X_n)$ and $Z = \varphi_z(X_1, X_2, \ldots, X_n)$. Given the characteristics of the system m_{x_i}, Var_{x_i} $(i = 1, 2, \ldots, n)$ and the correlation matrix $\|\mathrm{Cov}_{ij}\|$, find an approximation of the covariance Cov_{yz}.

Solution. Linearizing the functions φ_y and φ_z, we get

$$Y \approx \varphi_y(m_{x_1}, m_{x_2}, \ldots, m_{x_n}) + \sum_{i=1}^{n}\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m \mathring{X}_i,$$

$$Z \approx \varphi_z(m_{x_1}, m_{x_2}, \ldots, m_{x_n}) + \sum_{i=1}^{n}\left(\frac{\partial\varphi_z}{\partial x_i}\right)_m \mathring{X}_i,$$

whence

$$\mathring{Y} = \sum_{i=1}^{n}\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m \mathring{X}_i, \quad \mathring{Z} = \sum_{i=1}^{n}\left(\frac{\partial\varphi_z}{\partial x_i}\right)_m \mathring{X}_i,$$

$$\mathrm{Cov}_{yz} = \mathrm{M}[\mathring{Y}\mathring{Z}] = \mathrm{M}\left[\sum_{i=1}^{n}\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m \mathring{X}_i \sum_{i=1}^{n}\left(\frac{\partial\varphi_z}{\partial x_i}\right)_m \mathring{X}_j\right]$$

$$= \sum_{i=1}^{n}\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m\left(\frac{\partial\varphi_z}{\partial x_i}\right)_m \mathrm{Var}_{x_i} + \sum_{i\neq j}\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m\left(\frac{\partial\varphi_z}{\partial x_j}\right)_m \mathrm{Cov}_{ij}.$$

The last sum contains $n(n-1)$ terms, and each Cov_{ij} is associated with two of the terms of the sum:

$$\left(\frac{\partial\varphi_y}{\partial x_i}\right)_m\left(\frac{\partial\varphi_z}{\partial x_j}\right)_m \mathrm{Cov}_{ij} \quad \text{and}$$

$$\left(\frac{\partial\varphi_y}{\partial x_j}\right)_m\left(\frac{\partial\varphi_z}{\partial x_i}\right)_m \mathrm{Cov}_{ij}.$$

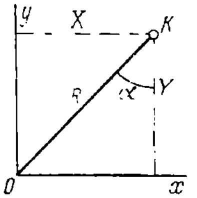


Fig. 7.84

7.84. The distance R from a point K to the origin can be found in two ways: (1) the distances X and Y to the coordinate axes can be determined and then R can be found from the formula $R_1 = \sqrt{X^2 + Y^2}$, or (2) only the distance Y to the abscissa axis and the angle α need be found (Fig. 7.84), and then R can be found from the formula $R_2 = Y/\cos\alpha$. We must determine which way will lead to the smaller error if the distances X and Y and the angle α are found with mutually independent errors, with the mean square deviations of the errors in X and Y being $\sigma_x = \sigma_y$ and that of the error in the angle being σ_α.

To do the numerical calculations, we assign values to the mean deviations of the errors $\sigma_x = \sigma_y = 1$ m and $\sigma_\alpha = 1° = 0.0174$ rad and to the mean values of the parameters $m_x = 100$ m, $m_y = 60$ m and $m_\alpha = \arctan(m_x/m_y) \approx 59° \approx 1.03$ rad.

Solution. (1) $\frac{\partial R_1}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$, $\frac{\partial R_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$,

$$\operatorname{Var}[R_1] = \left(\frac{x^2}{x^2+y^2}\right)_m \sigma_x^2 + \left(\frac{y^2}{x^2+y^2}\right)_m \sigma_y^2 = \sigma_y^2, \quad \sigma_1 = \sqrt{\operatorname{Var}[R_1]} = \sigma_y.$$

$$(2) \left(\frac{\partial R_2}{\partial y} = \frac{1}{\cos\alpha}, \quad \frac{\partial R_2}{\partial \alpha} = \frac{y \sin\alpha}{\cos^2\alpha},\right.$$

$$\operatorname{Var}[R_2] = \left(\frac{1}{\cos^2\alpha}\right)_m \sigma_y^2 + \left(\frac{y^2\tan^2\alpha}{\cos^2\alpha}\right)_m \sigma_\alpha^2 > \sigma_y^2, \; \sigma_2 = \sqrt{\operatorname{Var}[R_2]} > \sigma_1.$$

Thus the condition $\sigma_2 > \sigma_1$ is fulfilled for $\alpha > 0$. For the numerical data of the problem we have

$$\sigma_1 = \sigma_y = 1 \text{ m}, \quad \sigma_2 = \left(\left[1 + \left(\frac{100}{60}\right)^2\right]\right.$$
$$\left.\times \left[1^2 + 60^2\left(\frac{100}{60}\right)^2 0.0174^2\right]\right)^{1/2} = 3.9 \text{ m}.$$

7.85. A system of three random variables X, Y, Z has mean values $m_x = 10$, $m_y = 5$ and $m_z = 3$, mean square deviations $\sigma_x = 0.1$, $\sigma_y = 0.06$ and $\sigma_z = 0.08$, and a normalized correlation matrix

$$\|r\| = \left\|\begin{matrix} 1 & 0.7 & -0.3 \\ & 1 & 0.6 \\ & & 1 \end{matrix}\right\|.$$

Using the linearization method, find the mean value and the mean square deviation of a random variable $U = (3X^2 + 1)/(Y^2 + 2Z^2)$.

Solution. $m_u = (3\cdot 100 + 1)/(25 + 2\cdot 9) = 301/43 = 7.$

$$\frac{\partial u}{\partial x} = \frac{6x}{y^2+2z^2}, \quad \frac{\partial u}{\partial y} = -\frac{(3x^2+1)\,2y}{(y^2+2z^2)^2},$$

$$\frac{\partial u}{\partial z} = -\frac{(3x^2+1)\,4z}{(y^2+2z^2)^2}, \quad \left(\frac{\partial u}{\partial x}\right)_m = \frac{6\cdot 10}{43} = 1.4,$$

$$\left(\frac{\partial u}{\partial y}\right)_m = -\frac{301\cdot 10}{(43)^2} = -1.63 \text{ and } \left(\frac{\partial u}{\partial z}\right)_m = -\frac{301\cdot 12}{(43)^2} = -1.95,$$

$$\operatorname{Var}[U] = 1.4^2\cdot 0.1^2 + 1.63^2\cdot 0.06^2 + 1.95^2\cdot 0.08^2$$
$$+ 2[-1.4\cdot 1.63\cdot 0.7\cdot 0.1\cdot 0.06 + 1.4\cdot 1.95\cdot 0.3\cdot 0.1\cdot 0.08$$
$$+ 1.63\cdot 1.95\cdot 0.6\cdot 0.06\cdot 0.08] \approx 0.066; \quad \sigma_u \approx 0.26.$$

7.86. Two arbitrary resistors R_1 and R_2 are connected in parallel. The rated values of the resistors are the same and equal to $m_{r_1} = m_{r_2} = 900\ \Omega$. The maximum error in R which can arise when the resistors are manufactured is 1 per cent of the rated value. Use the linearization method to find the rated value of the resistance for the two resis-

tors in parallel and its mean square deviation.

Solution. $R=\frac{R_1R_2}{R_1+R_2}=\varphi(R_1,\ R_2),\ m_r=\varphi(m_{r_1},\ m_{r_2})=450\ \Omega,$

$$\sigma_{r_1}=\sigma_{r_2}=\frac{1}{3}\,\frac{900}{100}=3\ \Omega,\ \left(\frac{\partial\varphi}{\partial r_1}\right)_m=\left[\frac{r_2^2}{(r_1+r_2)^2}\right]_m=\frac{1}{4},$$

$$\left(\frac{\partial\varphi}{\partial r_2}\right)_m=\left(\frac{\partial\varphi}{\partial r_1}\right)_m=\frac{1}{4},$$

$$\mathrm{Var}\,[R]=\sum_{i=1}^{2}\left(\frac{\partial\varphi}{\partial r_i}\right)_m^2\sigma_{r_i}^2=\frac{9}{8}\ \Omega^2,\ \sigma_r\approx 1.06\ \Omega.$$

In this case the maximum error is 3.2 Ω, and that is 0.7 per cent (rather than 1 per cent) of the rated value.

7.87. The resonance frequency f_r of an oscillatory circuit is found from the expression $f_r = 1/(2\pi\sqrt{LC})$, where L is the inductance of the circuit and C is the capacitance of the circuit. Find an approximation for the mean value of the resonance frequency of the circuit and its mean square deviation if $m_l = 50$ μH, $m_C = 200$ pF, $\sigma_l = 0.5$ μH and $\sigma_C = 1.5$ pF.

Solution. $m_{f_r}=1/(2\pi\sqrt{m_lm_C})=1.59$ μH.

$$\left(\frac{\partial f_r}{\partial l}\right)_m=\frac{-1}{2\pi m_C^{1/2}m_l^{3/2}2}=m_{f_r}\frac{-1}{2m_l};$$

$$\left(\frac{\partial f_r}{\partial C}\right)_m=\frac{-1}{2\pi m_l^{1/2}m_C^{3/2}2}=m_{f_r}\frac{-1}{2m_C};$$

$$\mathrm{Var}_{f_r}=\left(\frac{\partial f_r}{\partial l}\right)_m^2\sigma_l^2+\left(\frac{\partial f_r}{\partial G}\right)_m^2\sigma_C^2=\frac{m_{f_r}^2}{4}\left(\frac{\sigma_l^2}{m_l^2}+\frac{\sigma_C^2}{m_C^2}\right)$$

$$=\overline{m}_{f_r}^2\left(\frac{5}{8}\right)^2\cdot 10^{-4}$$

$$\sigma_{f_r}=m_{f_r}\frac{5}{8}\cdot 10^{-2}=1.0\cdot 10^{-2}\ \mu\text{H},$$

and this is 0.62 per cent of the rated frequency.

CHAPTER 8

Distributions of Functions of Random Variables. The Limit Theorems of Probability Theory

8.0. If X is a continuous random variable with a probability density $f(x)$, and a random variable Y is in a functional relationship with it

$$Y = \varphi(X),$$

where φ is a differentiable function, monotonic on the whole interval of the possible values of the argument X, then the probability density of Y is expressed by the formula

$$g(y) = f(\psi(y)) \mid \psi'(y) \mid, \tag{8.0.1}$$

where ψ is the inverse of φ.

If φ is a nonmonotonic function, then its inverse is nonsingle-valued and the probability density of Y is defined as the sum of the terms of the form (8.0.1), the number of summands being equal to the number of the values (for a given y) of the inverse function:

$$g(y) = \sum_{i=1}^{k} f(\psi_i(y)) \mid \psi_i'(y) \mid, \tag{8.0.2}$$

where $\psi_1(y), \psi_2(y), \ldots, \psi_k(y)$ are the values of the inverse function for a given y.

For a function of several random variables, it is more convenient to seek the distribution function rather than the probability density. In particular, for a function of two arguments $Z = \varphi(X, Y)$ the distribution function

$$G(z) = \iint\limits_{D(z)} f(x, y)\, dx\, dy, \tag{8.0.3}$$

where $f(x, y)$ is the joint probability density of the variables X and Y, and $D(z)$ is the domain on the x, y-plane for which $\varphi(x, y) < z$.

The probability density function $g(z)$ can be determined by means of differentiation of $G(z)$: $g(z) = G'(z)$.

The probability density of the sum of two random variables $Z = X + Y$ is expressed by either of the formulas

$$g(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx, \quad g(z) = \int_{-\infty}^{\infty} f(z - y, y)\, dy, \tag{8.0.4}$$

where $f(x, y)$ is the joint probability density of the variables X and Y.

In particular, when the random variables X and Y are mutually independent, then $f(x, y) = f_1(x) f_2(y)$ and

$$g(z) = \int_{-\infty}^{\infty} f_1(x) f_2(z - x)\, dx, \tag{8.0.5}$$

or

$$g(z) = \int_{-\infty}^{\infty} f_1(z - y) f_2(y)\, dy. \tag{8.0.6}$$

In that case the distribution of the sum $g(z)$ is called the *convolution* (composition) of the distributions of the terms $f_1(x)$ and $f_2(y)$.

If a random variable has a normal distribution and is subjected to a linear transformation, then the resulting random variable will again have a normal distribution. In particular, if a random variable X has a normal distribution with parameters m_x, σ_x, then the random variable $Y = aX + b$ (where a and b are nonrandom) has a normal distribution with parameters $m_y = am_x + b$, $\sigma_y = |a| \sigma_x$.

A convolution of two normal distribution functions $f_1(x)$ and $f_2(y)$ with characteristics m_x, σ_x, m_y, σ_y respectively results in a normal distribution with characteristics

$$m_z = m_x + m_y, \quad \sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}. \tag{8.0.7}$$

The addition of two normally distributed random variables X, Y with parameters m_x, σ_x, m_y, σ_y and the correlation coefficient r_{xy} results in a random variable Z, *which also has a normal distribution with characteristics*

$$m_z = m_x + m_y, \quad \sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2 + 2r_{xy}\sigma_x\sigma_y}. \tag{8.0.8}$$

A linear function of several independent normally distributed random variables $X_1, X_2, \ldots, X_n$

$$Z = \sum_{i=1}^{n} a_i X_i + b,$$

where a_i, b are nonrandom coefficients, also has a normal distribution with parameters

$$m_z = \sum_{i=1}^{n} a_i m_{x_i} + b, \quad \sigma_z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_{x_i}^2}, \tag{8.0.9}$$

where m_{x_i}, σ_{x_i} are the parameters of the random variable X_i $(i = 1, \ldots, n)$.

If the arguments $X_1, X_2, \ldots, X_n$ are correllated, then the distribution of the linear function remains normal, but has parameters

$$m_z = \sum_{i=1}^{n} a_i m_{x_i} + b, \quad \sigma_z = \sqrt{\sum_{i=1}^{n} a_i^2 \sigma_{x_i}^2 + 2\sum_{i<j} a_i a_j r_{x_i x_j} \sigma_{x_i} \sigma_{x_j}}, \tag{8.0.10}$$

where $r_{x_i x_j}$ is the correlation coefficient of the variables X_i, X_j $(i = 1, \ldots, n;$ $j \neq i)$.

The *convolution of two normal distributions on a plane* is the distribution of a random vector (X, Y) with components $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where (X_1, Y_1), (X_2, Y_2) are uncorrelated random vectors $(r_{x_1x_2} = r_{x_1y_2} = r_{y_1x_2} = r_{y_1y_2} = 0)$.

A convolution of two normal distributions on a plane results in a normal distribution with parameters

$$\begin{aligned} m_x &= m_{x_1} + m_{x_2}, \quad m_y = m_{y_1} + m_{y_2}, \\ \sigma_x &= \sqrt{\sigma_{x_1}^2 + \sigma_{x_2}^2}, \quad \sigma_y = \sqrt{\sigma_{y_1}^2 + \sigma_{y_2}^2}, \end{aligned} \tag{8.0.11}$$

whence

$$\begin{aligned} \mathrm{Cov}_{xy} &= \mathrm{Cov}_{x_1y_1} + \mathrm{Cov}_{x_2y_2}, \\ r_{xy} &= (r_{x_1y_1}\sigma_{x_1}\sigma_{y_1} + r_{x_2y_2}\sigma_{x_2}\sigma_{y_2})/(\sigma_x\sigma_y). \end{aligned} \tag{8.0.12}$$

When a random point (X, Y), which is normally distributed on a plane, is projected onto the z-axis, which passes through the centre of scattering and at an angle α to the x-axis, a random point Z results which has a normal distribution

with parameters

$$m_z = m_x \cos\alpha + m_y \sin\alpha,$$

$$\sigma_z = \sqrt{\sigma_x^2 \cos^2\alpha + \sigma_y^2 \sin^2\alpha + r_{xy}\sigma_x\sigma_y \sin 2\alpha}. \tag{8.0.13}$$

Probability theory limit theorems form two groups: (1) a law of large numbers, and (2) a central limit theorem. The law of large numbers has several forms, each of which establishes the *stability of the average* for a large number of observations.

1. *Bernoulli's theorem* (the simplest form of the law of large numbers). As the number n of independent trials, in each of which an event A occurs with probability p, tends to infinity, the frequency P_n^* of the event A converges in probability to the probability p of the event:

$$\lim_{n\to\infty} \mathrm{P}\{|P_n^* - p| < \varepsilon\} = 1, \tag{8.0.14}$$

where ε is an arbitrarily small positive number.

2. *Poisson's theorem*. As the number n of independent trials, in each of which an event A occurs with probabilities $p_1, p_2, \ldots, p_n$, tends to infinity, the frequency P_n^* of the event A converges in probability to the average probability of the event:

$$\lim_{n\to\infty} \mathrm{P}\left\{\left| P_n^* - \frac{1}{n}\sum_{i=1}^{n} p_i \right| < \varepsilon\right\} = 1. \tag{8.0.15}$$

3. *Chebyshev's theorem* (*the law of large numbers*). As the number n of independent trials, in each of which a random variable X, with expectation m_x, assumes a value X_i, tends to infinity, the arithmetic mean of these values converges in probability to the expectation of the random variable X:

$$\lim_{n\to\infty} \mathrm{P}\left\{\left| \frac{1}{n}\sum_{i=1}^{n} X_i - m_x \right| < \varepsilon\right\} = 1. \tag{8.0.16}$$

4. *Markov's theorem* (*the law of large numbers for varying experimental conditions*). If $X_1, X_2, \ldots, X_n$ are independent random variables with expectations $m_{x_1}, m_{x_2}, \ldots, m_{x_n}$ and variances $\mathrm{Var}_{x_1}, \mathrm{Var}_{x_2}, \ldots, \mathrm{Var}_{x_n}$, all the variances being bounded from above by the same number L, i.e. $\mathrm{Var}_{x_i} < L$ $(i = 1, \ldots, n)$, then, as n tends to infinity, the arithmetic mean of the observed values of the random variables converges in probability to the arithmetic mean of their expectations:

$$\lim_{n\to\infty} \mathrm{P}\left\{\left| \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} m_{x_i} \right| < \varepsilon\right\} = 1. \tag{8.0.17}$$

When the speed of convergence in probability of various averages to constant values is to be estimated, *Chebyshev's inequality* is used:

$$\mathrm{P}\{|X - m_x| \geqslant \alpha\} \leqslant \mathrm{Var}_x/\alpha^2, \tag{8.0.18}$$

where $\alpha > 0$, and m_x and Var_x are the mean value and variance of the random variable X.

The **central limit theorem** has different forms of which we give three.

1. *Laplace's theorem*. If n independent trials are made, in each of which an event A has a probability p, then, as $n \to \infty$, the distribution of a random variable X, the number of times the event occurs, *tends to the normal distribution* with parameters $m = np$ and $\sigma = \sqrt{npq}$ $(q = 1 - p)$. We can go on from this to calculate the probability that the random variable X falls on any interval (α, β): for a suf-

ficiently large n

$$P\{X \in (\alpha, \beta)\} \approx \Phi\left(\frac{\beta - np}{\sqrt{npq}}\right) - \Phi\left(\frac{\alpha - np}{\sqrt{npq}}\right). \qquad (8.0.19)$$

Instead of formula (8.0.19), we often utilize the expression for the probability that the normalized variable

$$Z = (X - m_x)/\sigma_x = (X - np)/\sqrt{npq}, \quad m_z = 0, \quad \sigma_z = 1$$

(rather than the random variable X itself) falls on that interval. For a sufficiently large n

$$P\{Z \in (\alpha, \beta)\} \approx \Phi(\beta) - \Phi(\alpha). \qquad (8.0.20)$$

2. *The central limit theorem for similarly distributed terms.* If $X_1, X_2, \ldots, X_n$ are similarly distributed independent random variables with mean value m_x and mean square deviation σ_x, then, for a sufficiently large n, *their sum* $Y = \sum_{i=0}^{n} X_i$ *has an approximately normal distribution* with parameters

$$m_y = nm_x, \quad \sigma_y = \sqrt{n}\,\sigma_x. \qquad (8.0.21)$$

3. *Lyapunov's theorem.* If $X_1, X_2, \ldots, X_n$ are independent random variables with mean values $m_{x_1}, m_{x_2}, \ldots, m_{x_n}$ and variances $\mathrm{Var}_{x_1}, \mathrm{Var}_{x_2}, \ldots, \mathrm{Var}_{x_n}$, and the restriction

$$\lim_{n\to\infty} \left[\sum_{i=1}^{n} b_i \Big/ \left(\sum_{i=1}^{n} \mathrm{Var}_{x_i}\right)^{3/2}\right] = 0, \qquad (8.0.22)$$

where $b_i = M[|X_i|^3]$, is satisfied, then, for a sufficiently large n, the random variable $Y = \sum_{i=0}^{n} X_i$ has an *approximately normal distribution* with parameters

$$m_y = \sum_{i=1}^{n} m_{x_i}, \quad \sigma_y = \sqrt{\sum_{i=1}^{n} \mathrm{Var}_{x_i}}. \qquad (8.0.23)$$

The sense of the restriction (8.0.22) is that the random variables should be comparable as concerns the order of their effect on the scattering of the sum.

Problems and Exercises

8.1. A continuous random variable X has a probability density $f(x)$. Find the probability density function $g(y)$ of the random variable $Y = aX + b$, where a and b are not random.

Solution. Since the function $\varphi(x) = ax + b$ is monotonic, we can use formula (8.0.1). The inverse function $\psi(y)$ can be found by solving the equation $y = ax + b$ with respect to x; we get $\psi(y) = (y - b)/a$. We now find that $\psi'(y) = 1/a$ and $|\psi'(y)| = 1/|a|$, hence

$$g(y) = \frac{1}{|a|} f\left(\frac{y-b}{a}\right). \qquad (8.1)$$

8.2. A random variable X is uniformly distributed in the interval $(-\pi/2, \pi/2)$. Find the distribution of the random variable $Y = \sin X$.

Solution. In the interval $(-\pi/2, \pi/2)$ the function $y = \sin x$ is monotonic and, therefore, the probability density of the variable Y can be found from formula (8.0.1): $g(y) = f(\psi(y)) \mid \psi'(y) \mid$. It is convenient to arrange the solution of the problem as two columns, writing the designations of the functions in a general case on the left and those of the specific functions corresponding to the example on the right, e.g.

$f(x)$	$1/\pi$ for $x \in (-\pi/2, \pi/2)$
$y = \varphi(x)$	$y = \sin x$
$x = \psi(y)$	$x = \arcsin y$
$\psi'(y)$	$1/\sqrt{1-y^2}$
$g(y) = f(\psi(y))\psi'(y)$	$g(y) = 1/(\pi\sqrt{1-y^2})$ for $y \in (-1, 1)$.

The interval $(-1, 1)$, which includes the values of the random variable Y, is defined by the range of the function $y = \sin x$ for $x \in (-\pi/2, \pi/2)$*).

8.3. A random variable X is uniformly distributed in the interval $(-\pi/2, \pi/2)$. Find the probability density function $g(y)$ of the random variable $Y = \cos X$.

Solution. The function $y = \cos x$ is nonmonotonic in the interval $(-\pi/2, \pi/2)$. We find the solution by analogy with the preceding problem, only in this case the inverse function will have two values for each y [see (8.0.2)]. We again arrange the solution in two columns:

$f(x)$	$1/\pi$ for $x \in (-\pi/2, \pi/2)$
$y = \varphi(x)$	$y = \cos x$
$x = \begin{cases} \psi_1(y) \\ \psi_2(y) \end{cases}$	$x_1 = -\arccos y$ $x_2 = \arccos y$
$\lvert\psi_1'(y)\rvert, \lvert\psi_2'(y)\rvert$	$1/\sqrt{1-y^2}$
$g(y) = \sum_{i=1}^{k} f(\psi_i(y)) \lvert\psi_i'(y)\rvert$	$g(y) = 2/(\pi\sqrt{1-y^2})$ for $y \in (0, 1)$

8.4. A random variable X is uniformly distributed in the interval $(-\pi/2, \pi/2)$. Find the probability density function of the random variable $Y = \mid \sin X \mid$.

Answer. $g(y) = 2/(\pi\sqrt{1-y^2})$ for $y \in (0, 1)$.

8.5. A random variable X has a probability density $f(x)$. Find the probability density function of the random variable $Y = \mid 1 - X \mid$.

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*) In what follows, when solving similar problems, we shall write everywhere the expressions for the probability density only on the interval where it is nonzero, assuming that it is zero outside the interval.

Solution. The function $y = |1 - x|$ is nonmonotonic. We find the solution by analogy with Problem 8.3:

$f(x)$	$f(x)$
$y=\varphi(x)$	$y=\|1-x\|$
$x=\begin{cases}\psi_1(y)\\ \psi_2(y)\end{cases}$	$x_1=1-y$ $x_2=1+y$
$\|\psi_1'(y)\|,\ \|\psi_2'(y)\|$	1
$g(y)=\sum\limits_{i=1}^{k} f(\psi_i(y))\,\|\psi_i'(y)\|$	$g(y)=f(1-y)+f(1+y)$ for $y>0$.

8.6. A continuous random variable X has a probability density $f_x(x)$. We consider the variable $Y = -X$. Find its probability density function $f_y(y)$.

Answer. $f_y(y) = f_x(-y)$.

8.7. A continuous random variable X is uniformly distributed on the interval (α, β) $(\beta > \alpha)$. Find the distribution of the variable $Y = -X$.

Answer. It is uniform on the interval $(-\beta, -\alpha)$.

8.8. A continuous random variable X has a probability density $f_x(x)$. Find the probability density function $f_y(y)$ of its modulus $Y = |X|$.

Answer. $f_y(y) = f_x(-y) + f_x(y)$ for $y > 0$.

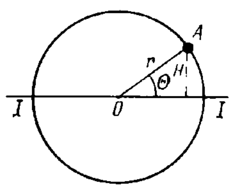

Fig. 8.10

8.9. A random variable X has a normal distribution with parameters m_x, σ_x. Write the probability density function of its modulus $Y = |X|$. In particular, what will be the distribution for $m_x = 0$?

Answer.

$$f_y(y)=\frac{1}{\sigma_x\sqrt{2\pi}}\left\{\exp\left[-\frac{(y+m_x)^2}{2\sigma_x^2}\right]+\exp\left[-\frac{(y-m_x)^2}{2\sigma_x^2}\right]\right\} \text{ for } y>0.$$

If $m_x=0$, then

$$f_y(y)=\frac{2}{\sigma_x\sqrt{2\pi}}\exp\left(-\frac{y^2}{2\sigma_x^2}\right) \text{ for } y>0.$$

8.10. A circular wheel, fixed at its centre O (Fig. 8.10), is rotated, the rotation being damped by friction. As a result, a fixed point A on the rim of the wheel stops at a certain height H (positive or negative) from the horizontal line I-I which passes through the centre of the wheel. The height H depends on the random angle Θ at which the rotation terminates. Find (1) the distribution of the height H, and (2) the distribution of the absolute value of H.

Solution. $H = r\sin\Theta$, where the angle Θ is a random variable uniformly distributed in the interval $(0, 2\pi)$. The solution of the problem will evidently not change if the random variable Θ is assumed to be uniformly distributed in the interval $(-\pi/2, \pi/2)$; then H is a monotonic function of Θ.

The distribution density of the variable $|H|$ is

$$g(h) = \left(\pi r \sqrt{1 - \left(\frac{h}{r}\right)^2}\right)^{-1} \quad \text{for} \quad -r < h < r.$$

The distribution density of $|H|$ is

$$g_1(d) = 2\left(\pi r \sqrt{1 - \left(\frac{d}{r}\right)^2}\right)^{-1} \quad \text{for} \quad 0 < d < r.$$

8.11. A random variable X has a Rayleigh distribution with probability density $f(x) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)$ for $x > 0$. Find the probability density function $g(y)$ of the variable $Y = e^{-X^2}$.

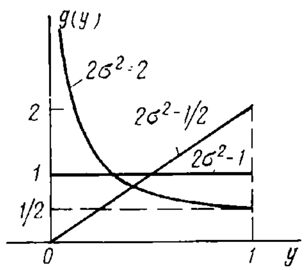

Fig. 8.11

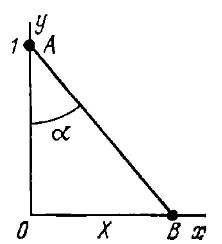

Fig. 8.13

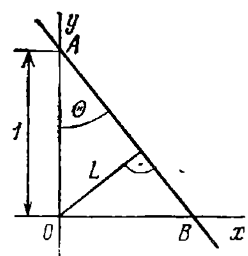

Fig. 8.15

Solution. On the interval of the possible values of the argument the function $y = e^{-x^2}$ is monotonic. Applying the general rule, we get

$f(x)$	$\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\ (x > 0)$
$y = \varphi(x)$	$y = e^{-x^2}$
$x = \psi(y)$	$x = \sqrt{-\ln y}$
$\lvert\psi'(y)\rvert$	$(2y\sqrt{-\ln y})^{-1}$
$g(y) = f(\psi(y))\,\lvert\psi'(y)\rvert$	$g(y) = \frac{1}{2\sigma^2 y}\exp\left(\frac{\ln y}{2\sigma^2}\right)$ $= \frac{1}{2\sigma^2}\,y^{\frac{1-2\sigma^2}{2\sigma}}$ for $0 < y < 1$.

The graphs of $g(y)$ for different σ are given in Fig. 8.11.

8.12. A random variable X has a Cauchy distribution with probability density $f(x) = [\pi(1 + x^2)]^{-1}$ $(-\infty < x < \infty)$. Find the probability density function $g(y)$ of the inverse variable $Y = 1/X$.

Solution. Bearing in mind that despite the discontinuity of the function $y = 1/x$, the inverse function $x = 1/y$ is single-valued, and solving the problem according to the rules for a monotonic function,

we get

$$g(y)=\frac{1}{\pi\left[1+\frac{1}{y}\right]^2}\frac{1}{y^2} \quad \text{or} \quad g(y)=\frac{1}{\pi(1+y^2)} \quad (-\infty<y<\infty),$$

i.e. *the inverse of a variable which has a Cauchy distribution also has a Cauchy distribution.*

8.13. Through a point A on the y-axis at a unit distance from the origin a straight line AB is drawn at an angle α to the y-axis (Fig. 8.13). All the magnitudes of the angle α from $-\pi/2$ to $\pi/2$ are equiprobable. Find the probability density function $g(x)$ of the abscissa X of the point B where the straight line cuts the abscissa axis.

Solution. $X = \tan\alpha$; this function is monotonic on the interval $-\pi/2 < \alpha < \pi/2$. We have $g(x) = [\pi(1 + x^2)]^{-1}$ $(-\infty < x < \infty)$, i.e. the random variable X has a Cauchy distribution.

8.14. A discrete random variable X has an ordered series

X:

−2	−1	0	1	2
0.1	0.2	0.3	0.3	0.1

Construct the ordered series of the random variables $Y = X^2 + 1$; $Z = |X|$.

Solution. For each X we find the respective value of Y and Z and place them in increasing order. We get an ordered series

Y:

1	2	5
0.3	0.5	0.2

Z:

0	1	2
0.3	0.5	0.2

8.15. A straight line AB is drawn through a point A with coordinates (0, 1) at a random angle Θ to the axis of ordinates (Fig. 8.15). The distribution of the angle Θ has the form $f(\Theta) = 0.5\cos\vartheta$ for $-\pi/2 < \vartheta < \pi/2$. Find the distribution of the distance L from the line AB to the origin.

Solution. We have $L = |\sin\Theta|$. The function $l = |\sin\vartheta|$ is nonmonotonic on the interval $(-\pi/2, \pi/2)$. Applying the usual scheme for a nonmonotonic function, we obtain

$$g(l)=\frac{1}{\sqrt{1-l^2}}\cos(\arcsin l)$$

or, taking into account that $\cos(\arcsin l) = \sqrt{1-l^2}$, we have $g(l) = 1$ for $l \in (0, 1)$, i.e. the distance L has a uniform distribution in the interval (0, 1).

8.16. The radius R of a circle is a random variable which has a Rayleigh distribution:

$$f(r)=\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right) \quad \text{for } r>0.$$

Find the distribution of the area S of the circle.

Solution. The function $S = \pi R^2$ is monotonic on the interval of possible values R $(0, \infty)$ and, consequently,

$$g(s) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{s}{2\pi\sigma^2}\right) \text{ for } s > 0,$$

i.e. the distribution of the area of the circle is exponential with parameter $1/(2\pi\sigma^2)$.

8.17. A random variable X has a distribution function $F(x)$, and a random variable Y is in a functional relationship with it: $Y = 2 - 3X$. Find the distribution function $\tilde{F}(y)$ of the random variable Y.

Solution.

$$\tilde{F}(y) = \mathrm{P}\{Y < y\} = \mathrm{P}\{2 - 3X < y\} = \mathrm{P}\left\{X > \frac{2-y}{3}\right\}$$
$$= 1 - \mathrm{P}\left\{X \leqslant \frac{2-y}{3}\right\} = 1 - F\left(\frac{2-y}{3}\right) - \mathrm{P}\left\{X = \frac{2-y}{3}\right\}.$$

If the random variable X is continuous, then the probability that it assumes any particular value is zero, and

$$\tilde{F}(y) = 1 - F\left(\frac{2-y}{3}\right).$$

If X is a mixed or discrete variable, then $\mathrm{P}\left\{X = \frac{2-y}{3}\right\}$ may be nonzero and equal to a jump of the function $F(x)$ at a point $(2-y)/3$. Thus, in the general case,

$$\tilde{F}(y) = 1 - F\left(\frac{2-y}{3}\right) + \lim_{\Delta x \to 0}\left[F\left(\frac{2-y}{3} + \Delta x\right) - F\left(\frac{2-y}{3}\right)\right].$$

8.18. Given a continuous random variable X with a distribution function $F(x)$ (Fig. 8.18), find and construct the distribution function $G(y)$ of the random variable $Y = |X|$.

Solution. For $X > 0$ we get $Y = X$; for $X < 0$ we get $Y = -X$.

$$G(y) = \mathrm{P}\{Y < y\} = \mathrm{P}\{|X| < y\}.$$

For $y < 0$ we get $G(y) = 0$; for $y > 0$ we get $G(y) = \mathrm{P}\{-y < X < y\} = F(y) - F(-y)$. The function $G(y)$ is shown by a dash line in Fig. 8.18 [it then merges with $F(x)$].

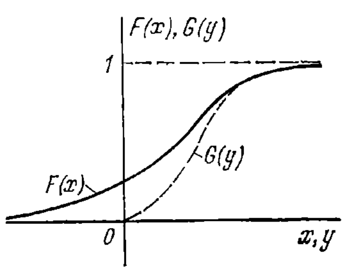

Fig. 8.18

8.19. A mixed random variable X assumes two values, each with a nonzero probability: a negative value $(-x_1)$ with probability p_1 and a positive value (x_2) with probability p_2. In the interval between $-x_1$ and x_2 the distribution function $F(x)$ is continuous (Fig. 8.19*a*); $x_1 < x_2$. Find and construct the distribution function $G(y)$ of the random variable $Y = |X|$.

Solution. For $y \leqslant 0$ we get $G(y) = 0$ (Fig. 8.19*b*); for $y > 0$ we get $G(y) = \mathrm{P}\{-y < X < y\} = F(y) - F(-y)$. To construct $G(y)$, we

must subtract from each value of $F(y)$ the value of the function at the point $-y$, which is the mirror reflection of y about the axis of ordinates (see Fig. 8.19*b*, where $F(x)$ is shown by a dash line and $G(y)$ by a solid line). The function $G(y)$ will have two jumps at points x_1 and x_2, which are equal to p_1 and p_2 respectively.

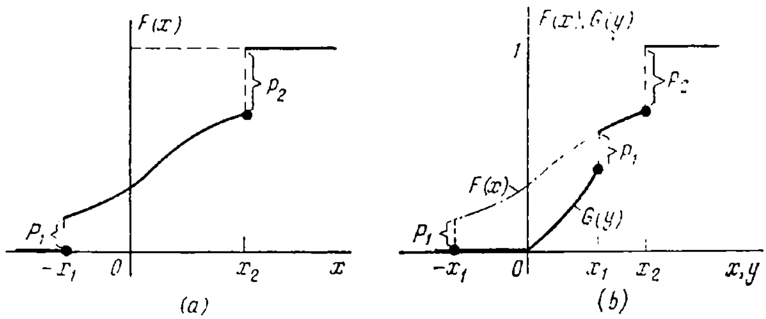

Fig. 8.19

8.20. A random variable X has a normal distribution with parameters m and σ (Fig. 8.20*a*). Find and construct the probability density function $g(y)$ of its modulus $Y = |X|$.

Solution. To construct the probability density of the random variable Y, we must sum up each ordinate of the distribution curve of $f(x)$ with

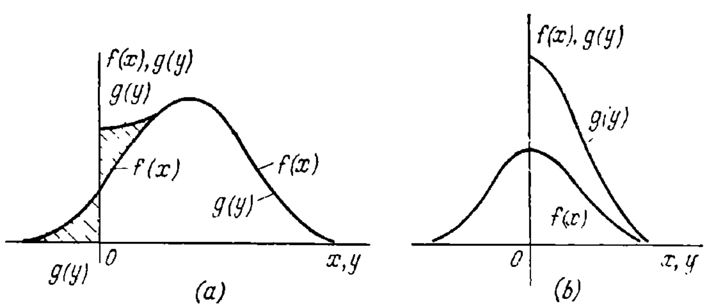

Fig. 8.20

the ordinate corresponding to the value of the probability density at the point x (see Fig. 8.20*a*). For $m = 0$ new density is double the probability density $f(x)$ (Fig. 8.20*b*).

8.21. A random variable X has a normal distribution with parameters $m = 0$, σ. Find and construct the probability density function $g(y)$ of the random variable $Y = X^2$.

Solution. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$, $Y = \varphi(X) = X^2$. The inverse function of $y = x^2$ has two values: $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$. By

formula (8.0.2)

$$g(y)=f(\sqrt{-y})\frac{1}{2\sqrt{y}}+f(\sqrt{y})\frac{1}{2\sqrt{y}}=\frac{1}{2\sqrt{y}}\left\{\exp\left(-\frac{y}{2\sigma^2}\right)\right.$$
$$\left.+\exp\left(-\frac{y}{2\sigma^2}\right)\right\}=\frac{1}{\sqrt{y}}\exp\left(-\frac{y}{2\sigma^2}\right)\quad \text{for}\quad y>0.$$

For $y = 0$ the probability density function $g(y)$ has a point of discontinuity of the 2nd kind (it tends to infinity, see Fig. 8.21).

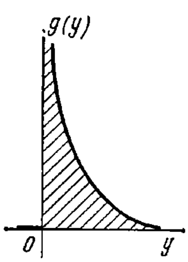

Fig. 8.21

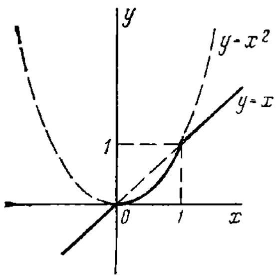

Fig. 8.23

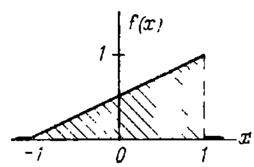

Fig. 8.24

8.22. Given a continuous random variable X with a probability density $f(x)$, find the distribution of the random variable

$$Y=\operatorname{sign} X=\begin{cases}+1 & \text{for } X>0,\\ 0 & \text{for } X=0,\\ -1 & \text{for } X<0,\end{cases}$$

and its numerical characteristics m_y and Var_y.

Solution. A discrete random variable Y has only two values: -1 and $+1$ (the probability that $Y = 0$ is zero).

$$\mathrm{P}\{Y=-1\}=\mathrm{P}\{X<0\}=\int_{-\infty}^{0} f(x)\,dx=F(0),$$
$$\mathrm{P}\{Y=+1\}=\mathrm{P}\{X>0\}=1-F(0),$$
$$m_y=-1\cdot F(0)+1\cdot[1-F(0)]=1-2F(0),$$
$$\alpha_2[Y]=1\cdot F(0)+1\cdot[1-F(0)]=1,$$
$$\mathrm{Var}_y=\alpha_2[Y]-m_y^2=4F(0)[1-F(0)],$$

where $F(x)$ is a distribution function.

8.23. There is a continuous random variable X with probability density $f(x)$. Find the distribution of the random variable $Y = \min\{X, X^2\}$, i.e. of the variable that is equal to X if $X < X^2$ and to X^2 if $X^2 < X$.

Solution. The function $y = \varphi(x)$ is monotonic (shown by a solid line in Fig. 8.23):

$$\varphi(x) = \begin{cases} x^2 & \text{for} \quad x \in (0, 1), \\ x & \text{for} \quad x \notin (0, 1). \end{cases}$$

Since the interval (0, 1) of the x-axis is mapped onto the interval (0, 1) of the y-axis, it follows, by the general rule, that

$$g(y) = \begin{cases} f(\sqrt{y})/(2\sqrt{y}) & \text{for} \quad y \in (0, 1), \\ f(y) & \text{for} \quad y \notin (0, 1) \end{cases}$$

8.24. A random variable X has a probability density $f(x)$ defined by the graph (Fig. 8.24). A random variable Y is related to X as $Y = 1 - X^2$. Find the probability density function $g(y)$ of the variable Y.

Solution. The probability density $f(x)$ is defined by the function $f(x) = 0.5(x + 1)$ for $x \in (-1, +1)$. The function $y = 1 - x^2$ is nonmonotonic on that interval; the inverse of the function has two values: $x_1 = -\sqrt{1 - y}$, $x_2 = +\sqrt{1 - y}$. Hence

$$g(y) = (4\sqrt{1-y})^{-1}[(1 - \sqrt{1-y}) + (1 + \sqrt{1-y})]$$

or

$$g(y) = (2\sqrt{1-y})^{-1} \quad \text{for} \quad 0 < y < 1.$$

8.25. A random variable X has a probability density $f(x)$. Find the probability density $g(y)$ of its inverse $Y = 1/X$.

Solution. Although the function $y = 1/x$ is nonmonotonic in the ordinary sense of the word (for $x = 0$ it increases jumpwise from $-\infty$ to $+\infty$), its inverse is single-valued, and this means that the problem can be solved in the same way as for monotonic functions,

$$g(y) = f\left(\frac{1}{y}\right)\frac{1}{y^2},$$

for the values of y which can be inverse to the given set of possible values of x.

8.26. *The log-normal distribution.* The natural logarithm of a random variable X has a normal distribution with a centre of scattering m and a mean square deviation σ. Find the probability density of the variable X.

Solution. We designate the normally distributed variable as U and have

$$U = \ln X, \quad X = e^U, \quad f(u) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(u-m)^2}{2\sigma^2}\right].$$

The function e^u is monotonic:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - m)^2}{2\sigma^2}\right]\frac{1}{x}$$
$$= \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - m)^2}{2\sigma^2}\right] \quad \text{for} \quad x > 0.$$

This distribution of the variable X is known as *log-normal*.

8.27. A spot Sp representing a target on a circular radar screen may occupy any position on it (Fig. 8.27), the probability density of the coordinates (X, Y) of the spot being constant within the screen. The radius of the screen is r_0. Find the probability density of the distance R from the spot to the centre of the screen.

Solution. We find the distribution function $G(r) = \text{P}\{R < r\} = \text{P}\{(X, Y) \in K_r\}$, where K_r is a circle of radius r with centre at a point O. Since the probability density is constant within the screen, the probability that the spot will fall in the circle is equal to its relative area:

$$G(r) = (\pi r^2)/(\pi r_0^2) = (r/r_0)^2, \quad \text{whence}$$
$$g(r) = G'(r) = 2r/r_0^2 \quad \text{for} \quad 0 < r < r_0.$$

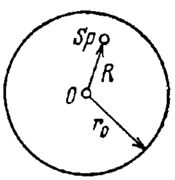

Fig. 8.27

8.28. A random variable X is uniformly distributed in the interval (0, 1). A random variable Y is in a monotonically increasing functional relationship with X: $Y = \varphi(X)$. Find the distribution function $G(y)$ and the probability density $g(y)$ of the random variable Y.

Solution. We have $f(x) = 1$ for $x \in (0, 1)$. We designate the inverse function of $y = \varphi(x)$ as $\psi(y)$. Since $\varphi(x)$ increases monotonically, it follows that $g(y) = f(\psi(y))\psi'(y) = \psi'(y)$, whence $G(y) = \psi(y)$, i.e. the required distribution function is the inverse of φ (in the range of possible values of Y).

8.29. To what transformation must the random variable X, which is uniformly distributed in the interval (0, 1), be subjected in order to obtain a random variable Y which would have an exponential distribution $g(y) = \lambda e^{-\lambda y}$ $(y > 0)$?

Solution. Proceeding from the solution of the preceding problem, we set $Y = G^{-1}(X)$, where G^{-1} is the inverse of the required distribution function $G(y)$ of the random variable Y. We have

$$G(y) = \int_0^y \lambda e^{-\lambda y}\, dy = 1 - e^{-\lambda y} \quad (y > 0).$$

Setting $1 - e^{-\lambda y} = x$ and solving this expression for y, we get the inverse function $y = -(\lambda)^{-1}\ln(1 - x)$. Hence the required relationship is

$$Y = -(\lambda)^{-1}\ln(1 - X) \quad (0 < X < 1).$$

8.30. A random variable X has an exponential distribution $f_1(x) = \lambda e^{-\lambda x}$ $(x > 0)$. To what functional transformations must it be sub-

jected to reduce it to a random variable Y which would have a Cauchy distribution $f_2(y) = [\pi(1+y^2)]^{-1}$.

Solution.

$$F_1(x) = 1 - e^{-\lambda x}\ (x > 0);\quad F_2(y) = \frac{1}{\pi}\left[\arctan y + \frac{\pi}{2}\right].$$

Setting $\frac{1}{\pi}\left[\arctan y + \frac{\pi}{2}\right] = u$ and solving this equation for y, we find the inverse function $F_2^{-1}(u)$:

$$y = F_2^{-1}(u) = \tan(\pi u - \pi/2) = -\cot \pi u.$$

From the preceding problem we get

$$Y = F_2^{-}(F_1(X)) = -\cot \pi(1 - e^{-\lambda X}) = \cot \pi e^{-\lambda X} \qquad (X > 0).$$

8.31. Two people agreed to meet at a certain place between 12.00 and 13.00 hours. Each arrives at the meeting place independently and,

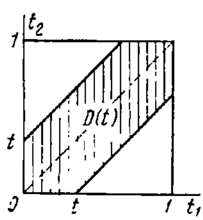

Fig. 8.31

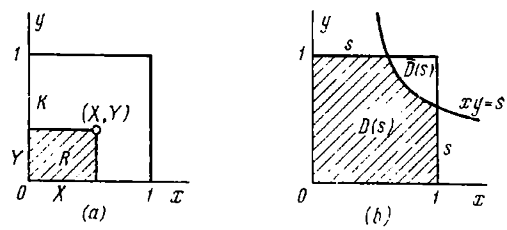

Fig. 8.32

with a constant probability density, at any moment of the assigned time interval. The first to arrive waits for the other person. Find the probability distribution function of the waiting time and the probability that the first will wait no less than half an hour.

Solution. We designate the arrival times of the two people as T_1 and T_2 and take 12.00 as our reference time. Then each of the independent random variables T_1 and T_2 is distributed with a constant density in the interval (0, 1). A random variable T is the waiting time: $T = |T_1 - T_2|$.

Let us find the distribution function $G(t)$ of this variable. We isolate, on the plane t_1Ot_2, a domain $D(t)$ in which $|t_1 - t_2| < t$ (the hatched domain in Fig. 8.31). In this case the distribution function $G(t)$ is equal to the area of this domain: $G(t) = 1 - (1-t)^2 = t(2-t)$, whence we have $g(t) = 2(1-t)$ for $0 < t < 1$.

$$\mathrm{P}\{T > 1/2\} = 1 - G(t/2) = 0.25.$$

8.32. A random point (X, Y) is uniformly distributed in a square K with unit sides (Fig. 8.32a). Find the distribution of the area S of the rectangle R with sides X and Y.

Solution. We isolate, on the x, y-plane, a domain $D(s)$ within which $xy < s$ (Fig. 8.32b). In this case the distribution function is equal to the

area of the domain D (s):

$$G(s)=1-\iint\limits_{D(s)} dx\,dy=1-\int\limits_s^1 dx\int\limits_{s/x}^1 dy=s(1-\ln s).$$

Hence $g(s) = G'(s) = -\ln s$ for $0<s<1$.

8.33. A random variable X has a normal distribution with parameters $m = 0$, σ. Find the distribution of the inverse variable $Y = 1/X$.

Solution. $y = \varphi(x) = 1/x$; the inverse function is single-valued: $x = 1/y$. By the general rule

$$g(y)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2y^2}\right)\frac{1}{y^2}\quad\text{or}\quad g(y)=\frac{1}{\sigma y^2\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2y^2}\right).$$

For $y = 0$ the density function $g(y)$ has a discontinuity of the 2nd kind (see Fig. 8.33).

It is interesting to note that the random variable Y does not have a mean value since the corresponding integral diverges.

8.34. A system of random variables (X, Y) has a joint density function $f(x, y)$. Find the density function $g(z)$ of their ratio $Z = Y/X$.

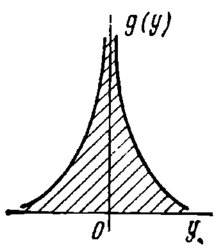

Fig. 8.33

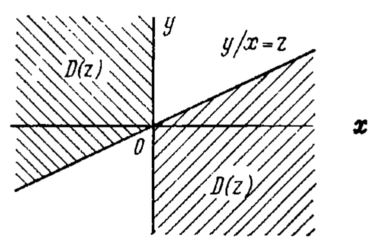

Fig. 8.34

Solution. We specify a certain value of z and construct, on the x, y-plane, a domain D (z), where $y/x < z$ (the hatched area in Fig. 8.34). The distribution function

$$G(z)=\iint\limits_{D\ (z)} f(x,y)\,dx\,dy=\int\limits_{-\infty}^{0}dx\int\limits_{zx}^{\infty}f(x,y)\,dy+\int\limits_{0}^{\infty}dx\int\limits_{-\infty}^{zx}f(x,y)\,dy.$$

Differentiating with respect to z, we have

$$g(z)=-\int\limits_{-\infty}^{0}xf(x,zx)\,dx+\int\limits_{0}^{\infty}xf(x,zx)\,dx.$$

If the random variables X and Y are mutually independent, then

$$g(z) = -\int_{-\infty}^{0} x f_1(x) f_2(zx)\,dx + \int_{0}^{\infty} x f_1(x) f_2(zx)\,dx.$$

8.35. Find the distribution of the ratio $Z = Y/X$ of two independent normally distributed random variables X and Y with characteristics $m_x = m_y = 0$, σ_x, σ_y.

Solution. We first consider a special case $\sigma_x = \sigma_y = 1$. On the basis of the preceding problem

$$y(z) = -\int_{-\infty}^{0} x \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+z^2x^2)}\,dx + \int_{0}^{\infty} x \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+z^2x^2)}\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\frac{x^2}{2}(1+z^2)} x\,dx = \frac{1}{\pi(1+z^2)} \quad \text{(Cauchy's distribution).}$$

In the general case the ratio $Z = X/Y$ can be represented as $Z = (Y_1\sigma_y)/(X_1\sigma_x)$, where the variables $X_1 = X/\sigma_x$ and $Y_1 = Y/\sigma_y$ have a normal distribution with variance unity; therefore,

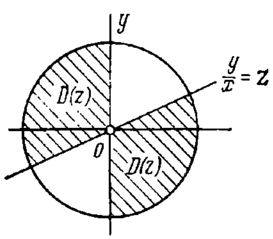

Fig. 8.36

$$g(z) = \frac{1}{\pi\,[1+(\sigma_x z)^2 \sigma_y^{-2}]} \frac{\sigma_x}{\sigma_y}.$$

In particular, if $\sigma_x = \sigma_y$, then

$$g(z) = [\pi(1+z^2)]^{-1}$$

8.36. A random point (X, Y) is uniformly distributed in a circle K of radius 1. Find the distribution of the random variable $Z = Y/X$.

Solution. In this case $G(z)$ is the relative area of the domain $D(z)$ (Fig. 8.36):

$$G(z) = \frac{1}{\pi}\left(\arctan z + \frac{\pi}{2}\right),$$

whence $g(z) = G'(z) = [\pi(1 + z^2)]^{-1}$ (Cauchy's distribution).

8.37. *Erlang's distribution of order 2.* Form a convolution of two exponential distributions with parameter λ, i.e. find the distribution of the sum of two independent random variables X_1 and X_2 which have probability densities

$$f_1(x_1) = \lambda e^{-\lambda x_1} \quad (x_1 > 0), \quad f_2(x_2) = \lambda e^{-\lambda x_2} \quad (x_2 > 0)$$

Solution. By the general formula (8.0.5) for the convolution of distributions we have

$$g(z) = \int_{-\infty}^{\infty} f_1(x_1) f_2(z - x_1)\,dx_1.$$

Since the functions f_1 and f_2 are zero for negative values of the arguments, the integral assumes the form

$$g(z)=\int_0^z f_1(x_1)f_2(z-x_1)\,dx_1=\lambda^2\int_0^z e^{-\lambda x_1}e^{-\lambda(z-x_1)}\,dx_1=\lambda^2 z e^{-\lambda z}\quad(z>0).\tag{8.37}$$

The distribution with a probability density (8.37) is known as *Erlang's distribution of order 2.* It originates as follows. Assume that there is an elementary flow of events with intensity λ on the t-axis, and only every other point (event) is retained in this flow, the intermediate points being deleted. Then the interval between adjacent events in the so rarefied flow has an Erlang order 2 distribution.

8.38. *Erlang's distribution of order n.* Form a composition of n exponential distributions with parameter λ, i.e. the distribution of the sum of n

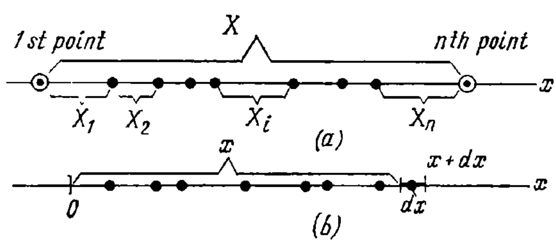

Fig. 8.38

independent random variables $X_1, X_2, \ldots, X_n$ which have an exponential distribution with parameter λ.

Solution. We could solve the problem by consecutively finding the convolutions of two (see Problem 8.37), three, etc. distributions, but it is simpler to solve the problem proceeding from the elementary flow, retaining every nth point in it and deleting the intermediate points (Fig. 8.38*a*). $X=\sum_{i=1}^{n} X_i$, where X_i is a random variable which has an exponential distribution. We find the probability density $f_n(x)$ of the random variable X, first finding the element of probability $f_n(x)\,dx$. This is the probability that the random variable X falls in the elementary interval $(x, x+dx)$.

For X to fall in that interval, it is necessary that exactly $n-1$ events fall in the interval x and one event in the interval dx (Fig. 8.38*b*). The probability of this occurrence is $f_n(x)\,dx = \mathrm{P}_{n-1}\lambda\,dx$, where P_{n-1} is the probability that $n-1$ events fall in the interval x. But the number of events of the elementary flow falling on the interval x has a Pois-

son distribution with parameter $a = \lambda x$, and hence

$$f_n(x)\,dx = \frac{(\lambda x)^{n-1}}{(n-1)!}\lambda e^{-\lambda x}\,dx,$$

$$f_n(x) = \frac{\lambda(\lambda x)^{n-1}}{(n-1)!}e^{-\lambda x} \quad (x>0). \tag{8.38.1}$$

The distribution with probability density (8.38.1) is known as *Erlang's distribution of order n*.

We could seek the distribution function $G_n(x)$ for Erlang's distribution of order n by integrating (8.38.1) from 0 to x, but it is simpler to derive it using the elementary flow again:

$$G_n(x) = \mathrm{P}\{X < x\} = 1 - \mathrm{P}\{X > x\}.$$

For the event $\{X > x\}$ to occur, it is necessary that no more than $n-1$ events fall on the interval x, i.e. $0, 1, 2, \ldots, n-1$ events; hence

$$G_n(x) = 1 - \sum_{m=0}^{n-1}\frac{(\lambda x)^m}{m!}e^{-\lambda x} \quad (x>0). \tag{8.38.2}$$

This expression can be reduced to the tabulated function $R(m, a)$ (see Appendix 2):

$$G_n(x) = 1 - R(n-1, \lambda x).$$

8.39. *Erlang's generalized distribution.* Form the convolution of two exponential distributions with different parameters:

$$f_1(x_1) = \lambda_1 e^{-\lambda_1 x_1} \quad (x_1 > 0), \qquad f_2(x_2) = \lambda_2 e^{-\lambda_2 x_2} \quad (x_2 > 0).$$

Solution. We designate $X = X_1 + X_2$, where X_1 and X_2 have the distributions $f_1(x_1)$, $f_2(x_2)$.

In accordance with the general formula for a convolution of distributions

$$g(x) = \int_{-\infty}^{\infty} f_1(x_1) f_2(x - x_1)\,dx_1.$$

But in this case both distributions are nonzero only for a positive value of the argument, and this means that $f_1(x_1) = 0$ for $x_1 < 0$ and $f_2(x - x_1) = 0$ for $x_1 > x$. For $x > 0$

$$g(x) = \int_0^x \lambda_1 e^{-\lambda_1 x_1}\lambda_2 e^{-\lambda_2(x-x_1)}\,dx_1 = \frac{\lambda_1\lambda_2 e^{-\lambda_2 x}}{\lambda_2-\lambda_1}[e^{(\lambda_2-\lambda_1)x} - 1]$$

$$= \frac{\lambda_1\lambda_2(e^{-\lambda_1 x} - e^{-\lambda_2 x})}{\lambda_2 - \lambda_1} \quad (x > 0).$$

(Erlang's generalized distribution of order 2). For $\lambda_1 = \lambda_2 = \lambda$, we can evaluate the indeterminate form and obtain Erlang's order 2 distri-

bution:

$$g(x) = \lambda^2 x e^{-\lambda x} \quad (x > 0).$$

Remark. We can prove by induction that the distribution of the sum of n independent random variables $X_1, \ldots, X_n$, which have exponential distributions with different parameters $\lambda_1, \ldots, \lambda_n$, i.e. Erlang's generalized distribution of order n has a probability density

$$g_n(x) = (-1)^{n-1} \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{e^{-\lambda_j x}}{\prod\limits_{k \neq j}^{n} (\lambda_j - \lambda_k)} \quad (x > 0).$$

(The notation $\prod\limits_{k \neq j}^{n}$ signifies that we take the product of all binomials of the form $\lambda_j - \lambda_k$ for $k = 1, 2, \ldots, j-1, j+1, \ldots, n$, i.e. all except for $\lambda_j - \lambda_k$). In a special case, when $\lambda_i = i\lambda$,

$$g_n(x) = \sum_{i=1}^{n} (-1)^{j-1} C_n^j \lambda_j e^{-\lambda_j x}.$$

The distribution function of Erlang's generalized distribution of order n has the form

$$G_n(x) = (-1)^{n-1} \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{1 - e^{-\lambda_j x}}{\lambda_j \prod\limits_{k \neq j}^{n} (\lambda_j - \lambda_k)} \quad (x > 0).$$

If $\lambda_i = i\lambda$, then

$$G_n(x) = \sum_{j=1}^{n} (-1)^{j-1} C_n^j [1 - e^{-j\lambda x}] \quad (x > 0).$$

If $\lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$, we get Erlang's distribution of order n:

$$g_n(x) = \frac{\lambda (\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} = \lambda P(n-1, \lambda x) \quad (x > 0),$$

$$G_n(x) = \int_0^x \lambda P(n-1, \lambda x)\, dx = 1 - \int_x^\infty \lambda P(n-1, \lambda x)\, dx = 1 - R(n-1, \lambda x) \quad (x > 0),$$

where

$$P(m, a) = \frac{a^m}{m!} e^{-a}, \quad R(m, a) = \sum_{k=0}^{m} \frac{a^k}{k!} e^{-a} \quad \text{(Appendices 1, 2)}.$$

8.40. *The distribution of the maximal of two random variables.* Given two random variables X and Y with a joint probability density $f(x, y)$, find the distribution function $G(z)$ and the probability density function $g(z)$ of the maximal of these two variables $Z = \max\{X, Y\}$.

Solution. We shall seek the distribution function of the random variable Z: $G(z) = P\{Z < z\}$. For the maximal of the variables X

and Y to be smaller than z, each of the variables must be smaller than z:

$$G(z) = \mathrm{P}\{X < z,\ Y < z\} = F(z,\ z),$$

where

$$F(x,\ y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,\ y)\,dx\,dy.$$

Thus,

$$G(z) = \int_{-\infty}^{z} \int_{-\infty}^{z} f(x,\ y)\,dx\,dy.$$

To find the probability density function $g(z)$, we differentiate $G(z)$ with respect to z, which appears in the limits of integration of the double integral. We differentiate $G(z)$ as a composite function of the two variables z_1 and z_2, each of which depends on z ($z_1 = z$, $z_2 = z$).

$$g(z) = \frac{dG(z)}{dz} = \frac{d}{dz}\left\{\int_{-\infty}^{z_1}\left[\int_{-\infty}^{z_2} f(x,\ y)\,dy\right]dx\right\} = \frac{\partial G(z)}{\partial z_1}\frac{dz_1}{dz} + \frac{\partial G(z)}{\partial z_2}\frac{dz_2}{dz}$$

$$= \int_{-\infty}^{z} f(z,\ y)\,dy + \int_{-\infty}^{z} f(x,\ z)\,dz.$$

In the special case when X and Y are independent, $f(x,\ y) = f_1(x)\cdot f_2(y)$, we have

$$g(z) = f_1(z)\int_{-\infty}^{z} f_2(y)\,dy + f_2(z)\int_{-\infty}^{z} f_1(x)\,dx,$$

or, in a more concise form,

$$g(z) = f_1(z)\,F_2(z) + f_2(z)\,F_1(z).$$

If the random variables X and Y are independent and have the same distribution, then $f_1(x) = f_2(x) = f(x)$ and $g(z) = 2f(z)\,F(z)$.

8.41. *The distribution of the minimal of two random variables.* A system of two random variables (X, Y) has a joint probability density $f(x, y)$. Find the distribution function $G(u)$ and the probability density function $g(u)$ of the minimal of these variables: $U = \min\{X, y\}$.

Solution. We seek the complement of the distribution function with respect to unity:

$$1 - G(u) = \mathrm{P}(U > u) = \mathrm{P}\{X > u,\ Y > u\}.$$

This is the probability that the random point (X, Y) falls in the domain $D(u)$ hatched in Fig. 8.41. It is evident that $1 - G(u) = 1 - F(u,\ \infty) - F(\infty,\ u) + F(u,\ u)$, whence $G(u) = F(u,\ \infty) + F(\infty,\ u) - F(u,\ u) = F_1(u) + F_2(u) - F(u,\ u)$. Differentiating with respect to

u, we have (see Problem 8.40)

$$g(u)=f_1(u)+f_2(u)-\int_{-\infty}^{u} f(u, y)\,dy-\int_{-\infty}^{u} f(x, u)\,dx.$$

When the variables X and Y are mutually independent, we have

$$g(u) = f_1(u)[1 - F_2(u)] + f_2(u)[1 - F_1(u)].$$

If the random variables X and Y are mutually independent and have the same distribution, then $f_1(x) = f_2(x) = f(x)$ and $g(u) = 2f(u)[1 - F(u)]$.

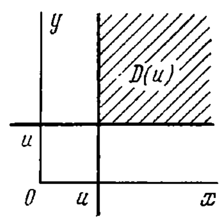

Fig. 8.41

8.42. *The distribution of the maximal and the minimal of several random variables.* Given n independent random variables $X_1, X_2, \ldots, X_n$ distributed with probability densities $f_1(x_1)$, $f_2(x_2), \ldots, f_n(x_n)$, find the probability density function of the maximal of them $Z = \max\{X_1, X_2, \ldots, X_n\}$ and the minimal of them $U = \min\{X_1, X_2, \ldots, X_n\}$, i.e. of the random variable which assumes a maximum (minimum) value as a result of an experiment.

Solution. We designate the distribution function of the variable Z as $G_z(z)$. We have

$$G_z(z)=\mathrm{P}\{Z<z\}=\prod_{i=1}^{n} F_i(z),$$

where $F_i(z)=\int_{-\infty}^{z} f_i(x_i)\,dx_i \quad (i=1, 2, \ldots, n).$

Differentiating, we get a sum whose every term results from the multiplication of the derivative of one of the distribution functions $F_1(x), F_2(x), \ldots, F_n(x_n)$ by the product of the other distribution functions. The result can be written in the form

$$g_z(z)=\sum_{j=1}^{n} \frac{f_j(z)}{F_j(z)} \prod_{i=1}^{n} F_i(z).$$

By analogy, designating the distribution function of the variable U as $G_u(u)$, we obtain

$$G_u(u)=1-\prod_{i=1}^{n}[1-F_i(u)].$$

Differentiating, we get

$$g_u(u)=\sum_{i=1}^{n} \frac{f_j(u)}{1-F_j(u)} \prod_{i=1}^{n}[1-F_i(u)].$$

8.43. There are n independent random variables $X_1, X_2, \ldots, X_n$ which have the same distribution with probability density $f(x)$. Find the distribution of the maximal of them $Z = \max\{X_1, X_2, \ldots, X_n\}$ and the minimal of them $U = \min\{X_1, X_2, \ldots, X_n\}$.

Solution. On the basis of the solution of the preceding problem

$$G_z(z) = (F(z))^n, \quad g_z(z) = n(F(z))^{n-1} f(z),$$
$$G_u(u) = 1 - [1 - F(u)]^n, \quad g_u(u) = n[1 - F(u)]^{n-1} f(u).$$

8.44. Three independent shells are fired at an x, y-plane. The centre of scattering coincides with the origin, the scattering is normal and circular, $\sigma_x = \sigma_y = \sigma$. The hit nearest to the centre of scattering is chosen. Find the probability density function $g(r)$ of the distance R from the nearest hit to the centre.

Solution. We have $R = \min\{R_1, R_2, R_3\}$. From the solution of the preceding problem it follows that $g(r) = 3[1 - F(r)]^2 f(r)$, where $F(r)$ and $f(r)$ are the distribution function and the probability density function, respectively, of the distance from any hit to the centre of scattering;

$$F(r) = 1 - \exp\left[-\frac{r^2}{2\sigma^2}\right],$$
$$f(r) = \frac{r}{\sigma^2}\exp\left[-\frac{r^2}{2\sigma^2}\right] \quad (r > 0).$$

Hence

$$g(r) = \frac{3r}{\sigma^2}\exp\left[-\frac{3r^2}{2\sigma^2}\right] \quad (r > 0)$$

i.e. the probability density of the distance from the nearest hit to the centre of scattering has the same form as that for each of them, the only condition being that the parameter σ is decreased $\sqrt{3}$ times, i.e. replaced by $\sigma' = \sigma/3$.

8.45. Find the distribution $g_u(u)$ of the minimal of two independent random variables T_1 and T_2 which have an exponential distributions:

$$f_1(t_1) = \lambda_1 e^{-\lambda_1 t_1} \quad (t_1 > 0), \qquad f_2(t_2) = \lambda_2 e^{-\lambda_2 t_2} \quad (t_2 > 0)$$

Solution. On the basis of the solution of Problem 8.42,

$$g_u(u) = f_1(u)[1 - F_2(u)] + f_2(u)[1 - F_1(u)]$$
$$= \lambda_1 e^{-\lambda_1 u} e^{-\lambda_2 u} + \lambda_2 e^{-\lambda_2 u} e^{-\lambda_1 u} = (\lambda_1 + \lambda_2)\, e^{-(\lambda_1+\lambda_2)u},$$

i.e. the distribution of the minimal of two independent random variables which have an exponential distributions is also an exponential distribution whose parameter is equal to the sum of the parameters of the initial distributions.

We can arrive at this conclusion much easier if we use the concept of a flow of events. Assume that there is an elementary flow with intensity λ_1 on the t_1-axis and an elementary flow with intensity λ_2 on the t_2-axis. We bring these two flows together on the t-axis, i.e. we super-

impose them. It is easy to verify that the result of the superposition of two elementary flows is also elementary (the properties of stationariness, ordinariness and the absence of aftereffects are retained). The intensity of the combined flow is $\lambda_1 + \lambda_2$. The distribution of the distance from a given point to the nearest event of the flow is exponential with parameter $\lambda_1 + \lambda_2$.

The same is evidently true for the distribution of the minimal of any number of n independent random variables which have exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. It is exponential with parameter $\sum\limits_{i=1}^{n} \lambda_i$.

8.46. From the conditions of the preceding problem find the distribution $g_z(z)$ of the maximal of the variables T_1, T_2.

Solution.

$$\begin{aligned} g_z(z) &= f_1(z) F_2(z) + f_2(z) F_1(z) = \lambda_1 e^{-\lambda_1 z}[1 - e^{-\lambda_2 z}] + \lambda_2 e^{-\lambda_2 z}[1 - e^{-\lambda_1 z}] \\ &= \lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z} - (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)z} \quad (z > 0). \end{aligned}$$

This is not an exponential distribution. For $\lambda_1 = \lambda_2 = \lambda$

$$g_z(z) = 2\lambda e^{-\lambda z}(1 - e^{-z\lambda}) \quad (z > 0).$$

8.47*. A random variable X which has a probability density $f(x)$ is subjected to n independent trials. The results of the trials are arranged in increasing order. A series of random variables $Z_1, Z_2, \ldots, Z_k, \ldots, Z_n$ results. Consider the kth of them, Z_k. Find its distribution function $G_k(z)$ and probability density function $g_k(z)$.

Solution. $G_k(z) = P\{Z_k < z\}$. For the kth (in an increasing order) of the random variables $Z_1, Z_2, \ldots, Z_k, \ldots, Z_n$ to be smaller than z, it is necessary that at least k of them be smaller than z:

$$G_k(z) = \sum_{m=k}^{n} P_m,$$

where P_m is the probability that exactly m of the values of the random variable X in n trials are smaller than z. By the theorem on the repetition of trials

$$P_m = C_n^m [F(z)]^m [1 - F(z)]^{n-m},$$

whence

$$G_k(z) = \sum_{m=k}^{n} C_n^m [F(z)]^m [1 - F(z)]^{n-m},$$

where $F(z) = \int\limits_{-\infty}^{z} f(x)\, dx$.

The probability density $g_k(z)$ can be found by differentiating this expression and taking into account that

$$C_n^m m = \frac{n!}{(m-1)!\,(n-m)!} = nC_{n-1}^{m-1},$$

$$C_n^m (n-m) = nC_{n-1}^m \quad (m < n).$$

After some simple transformations we obtain

$$g_k(z) = nC_{n-1}^{k-1} f(z)\,[F(z)]^{k-1}\,[1-F(z)]^{n-k}.$$

It is much easier, however, to obtain $g_k(z)$ directly, using the following simple reasoning. The element of probability $g_k(z)\,dz$ is approximately the probability that the random variable Z_k (the kth largest value of the variable X) falls on the interval $(z, z+dz)$. For this to occur, it is necessary that the following events are superimposed:

(1) one of the values of the random variable X falls on the interval $(z, z+dz)$;

(2) $(k-1)$ other values prove to be smaller than z;

(3) $(n-k)$ remaining values prove to be larger than z [we neglect the probability that more than one value fall on the elementary interval $(z, z+dz)$].

The probability that every such combination of events occurs is $f(z)\,dz\,[F(z)]^{k-1}\,[1-F(z)]^{n-k}$. The number of combinations is equal to the product of the number n of ways in which we can choose one value out of the n values to place it on the interval $(z, z+dz)$ by the number C_{n-1}^{k-1} of ways in which we can choose $k-1$ values out of the remaining $n-1$ values to place them to the left of z. Consequently,

$$g_k(z)\,dz = nC_{n-1}^{k-1} f(z)\,[F(z)]^{k-1}\,[1-F(z)]^{n-k}\,dz,$$

whence

$$g_k(z) = nC_{n-1}^{k-1} f(z)\,[F(z)]^{k-1}\,[1-F(z)]^{n-k}.$$

8.48. There are four regulators (thermocouples) in an electric furnace, each of which shows the temperature with an error normally distributed with zero mean value and the mean square deviation σ_t. The furnace is heated. At the moment when two of the four thermocouples show the temperature not lower than the critical temperature τ_0, the furnace is automatically switched off. Find the probability density of the temperature Z at which the furnace will be switched off.

Solution. The temperature Z at which the furnace is switched off is the second smallest (i.e. the third largest) of the four values of the random variable T which has a normal distribution with the centre of scattering τ_0 and the mean square deviation σ_t:

$$f(t) = \frac{1}{\sigma_t\sqrt{2\pi}} \exp -\frac{(t-\tau_0)^2}{2\sigma_t^2}.$$

The corresponding distribution function

$$F(t) = \Phi\left(\frac{t-\tau_0}{\sigma_t}\right) + 0.5.$$

Using the results of the preceding problem for $n=4$, $k=3$, we obtain

$$g_3(t)=\frac{12}{\sigma_t\sqrt{2\pi}}\exp\left[-\frac{(t-\tau_0)^2}{2\sigma_t^2}\right]\left[\Phi\left(\frac{t-\tau_0}{\sigma_t}\right)\right.$$
$$\left.+0.5\right]^2\left[0.5-\Phi\left(\frac{t-\tau_0}{\sigma_t}\right)\right].$$

8.49. There are n independent random variables $X_1, X_2, \ldots, X_n$, whose distribution functions are power functions

$$F_i(x_i)=\begin{cases}0 & \text{for } x_i\leqslant 0,\\ x_i^{k_i} & \text{for } 0<x_i\leqslant 1 \quad (i=1, 2, \ldots, n)\\ 1 & \text{for } x_i>1,\end{cases}$$

where k_i is a positive integer. The value of each random variable is considered and the maximum value Z is chosen. Find the distribution $G(z)$ of that random variable.

Solution. On the basis of the solution of Problem 8.42

$$G(z)=\prod_{i=1}^{i}F_i(z)=\prod_{i=1}^{n}z^{k_i} \quad \text{for} \quad 0<z\leqslant 1,$$

or, if we designate $k=\sum_{i=1}^{n}k_i$,

$$G(z)=\begin{cases}0 & \text{for } z\leqslant 0,\\ z^k & \text{for } 0<z\leqslant 1,\\ 1 & \text{for } z>1,\end{cases}$$

i.e. the maximal of several random variables, which are distributed according to a power law in the interval (0, 1), is also distributed according to a power law with an exponent equal to the sum of the exponents of separate distributions.

8.50. The discrete random variables $X_1, X_2, \ldots, X_n$ are mutually independent and have a Poisson distribution with parameters a_1, $a_2, \ldots, a_n$. Show that their sum $Y=\sum_{i=1}^{n}X_i$ also has a Poisson distribution with parameter $a=\sum_{i=1}^{n}a_i$.

Solution. We first prove that the sum of two random variables X_1 and X_2 has a Poisson distribution, for which purpose we find the probability that $X_1+X_2=m$ $(m=0, 1, 2, \ldots)$.

$$\mathrm{P}\{X_1+X_2=m\}=\sum_{k=0}^{m}\mathrm{P}\{X_1=k\}\,\mathrm{P}\{X_2=m-k\}$$
$$=\sum_{k=0}^{m}\frac{a_1^k}{k!}e^{-a_1}\frac{a_2^{m-k}}{(m-k)!}e^{-a_2}.$$

Taking into account that $C_m^k = \frac{m!}{(k!)(m-k)!}$, we represent the expression in the form

$$\frac{e^{-(a_1+a_2)}}{m!}\sum_{k=0}^{m} C_m^k a_1^k a_2^{m-k} = \frac{(a_1+a_2)^m}{m!} e^{-(a_1+a_2)},$$

and this is a Poisson distribution with parameter $a_1 + a_2$.

We have thus proved that the sum of two independent random variables, which have a Poisson distribution, also has a Poisson distribution. This result can be extended to any number of terms by induction.

8.51. A system of random variables (X, Y) is normally distributed with characteristics m_x, m_y, σ_x, σ_y and z_{xy}. Random variables (U, V) are related to (X, Y) as $U = aX + bY + c$ and $V = kX + lY + m$. Find the distribution of the system of random variables (U, V).

Answer. The system (U, V) is normally distributed with characteristics

$$m_u = am_x + bm_y + c, \quad m_v = km_x + lm_y + m,$$

$$\sigma_u = \sqrt{a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_x\sigma_y r_{xy}},$$

$$\sigma_v = \sqrt{k^2\sigma_x^2 + l^2\sigma_y^2 + 2kl\sigma_x\sigma_y r_{xy}},$$

$$z_{uv} = \frac{ak\sigma_x^2 + bl\sigma_y^2 + (bk+al)\,\sigma_x\sigma_y r_{xy}}{\sigma_u\sigma_v}.$$

8.52. Form the convolution of two binomial distributions with parameters (n, p) and (k, p).

Solution. Assume that X is the number of occurrences of an event A in n independent trials, in each of which it occurs with probability p; X has a binomial distribution with parameters (n, p). Similarly, Y is the number of occurrences of the event A in k independent trials under the same conditions; it has a binomial distribution with parameters (k, p). $Z = X + Y$ is the number of occurrences of the event A in a series of $n + k$ trials with probability p of the event A on each trial; the variable z also has a binomial distribution with parameters $(n + k, p)$.

Note that whereas the probabilities p are different in different series, the convolution of two binomial distributions results in a nonbinomial distribution.

8.53. Using Chebyshev's inequality, estimate from above the probability that a random variable X, having a mean value m and a mean square deviation σ, deviates from m by less than 3σ.

Solution. Chebyshev's inequality (8.0.18) yields

$$\mathrm{P}\{|X-m| \geqslant 3\sigma\} \leqslant \frac{D\,|X|}{(3\sigma)^2} = \frac{\sigma^2}{(3\sigma)^2} = \frac{1}{9}, \quad \mathrm{P}\{|X-m| < 3\sigma\} \geqslant \frac{8}{9}.$$

Thus *any random variable deviates from its mean value by less than* 3σ *with probability not less than* 8/9*).

8.54. A large number n of independent trials are made, in each of which a random variable X is uniformly distributed on the interval (1, 2). We consider the arithmetic mean of the observed values of the random variable X: $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$. Proceeding from the law of large numbers, find the number a to which the variable Y will tend (converge in probability) as $n \to \infty$. Estimate the maximum practically possible error of the approximate equality $Y \approx a$.

Solution.

$$a = M[Y] = \frac{1}{n}\sum_{i=1}^{n} M[X_i] = \frac{1}{n}\cdot n\cdot 1.5 = 1.5,$$

$$\operatorname{Var}[Y] = \frac{1}{n^2}\sum_{i=1}^{n}\operatorname{Var}[X_i] = \frac{n}{n^2}\,\frac{1}{12} = \frac{1}{12n}, \quad \sigma_y = \sqrt{\operatorname{Var}[Y]} = \frac{1}{2\sqrt{3n}}.$$

The maximum possible value of the error is $3\sigma_y$.

8.55. We consider a sequence of n random variables $X_1, X_2, \ldots, X_n$ uniformly distributed on the intervals (0, 1), (0, 2), ..., (0, n) respectively. What happens to their arithmetic mean $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$ when n increases?

Solution. For a given n

$$M[Y] = \frac{1}{n}\sum_{i=1}^{n} M[X_i] = \frac{1}{n}\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\ldots+\frac{n}{2}\right)$$

$$= \frac{1}{2n}\,\frac{n(n+1)}{2} = \frac{n+1}{4}.$$

As $n\to\infty$, $M[Y]$ increases indefinitely and there is no stability of the arithmetic mean.

8.56. Random variables $X_1, X_2, \ldots, X_n$ are uniformly distributed on the intervals $(-1, 1), (-2, 2), \ldots, (-n, n)$. Will the arithmetic mean of the random variables X_i, i.e. $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$, converge to zero in probability with an increase of n?

*) This is the extreme, most unfavourable case. For random variables usually encountered in practical applications this probability is much closer to unity. For instance, for a normal distribution it is 0.997; for a uniform distribution it is unity, and for an exponential distribution it is 0.982.

Solution. $M[Y] = \frac{1}{n}\sum_{i=1}^{n} M[X_i] = 0$, but the random variable Y will not converge to zero in probability, since the conditions of Markov's theorem are violated: the variances of the random variables X_i are not limited by the same number L but increase indefinitely with an increase in n.

8.57. A computer produces random binary digits such that the symbols 0 and 1 may appear with equal probability at each position and independently of other positions. The sequence of symbols is divided into groups consisting of the same symbols, say, 000 111 0 1 00 1111 00 11 0000 1 0 . . . . The number of symbols in each group is calculated and is divided by the number of groups. How will the quotient behave when the number of groups n increases indefinitely?

Solution. A random variable X_i, the number of symbols in the ith group, has a geometric distribution beginning with unity (see Chapter 4), $M[X_i] = 1/p = 2$; $\mathrm{Var}[X_i] = q/p^2 = 0.5/0.5^2 = 2$.

If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, then

$$M[Y_n] = \frac{1}{n} n2 = 2, \quad \mathrm{Var}[Y_n] = \frac{1}{n^2} n2 = \frac{2}{n}, \quad \sigma_{y_n} = \sqrt{\frac{2}{n}}.$$

On the basis of the law of large numbers, the variable Y_n converges in probability to $M[Y_n] = 2$ as $n \to \infty$; $\lim\limits_{n \to \infty} P\{|Y_n - 2| < \varepsilon\} = 1$. By the three-sigma rule the error of the approximate equality $Y_n \approx 2$ does not exceed $3\sqrt{2/n}$.

8.58. A factory shop produces balls for ball-bearings. The shop manufactures $n = 10\,000$ balls per shift. The probability that a ball will be defective is 0.05. The causes of defects are independent for separate balls. The ready balls are inspected immediately after their manufacture. The defective balls are rejected and dumped into a bin, while the sound balls are sent to the assembling shop. Find the number of balls for which the bin must be designed so that it is not overfilled during a shift with probability 0.99.

Solution. The number of rejected balls X has a binomial distribution; since n is large, we can assume, on the basis of the Laplace limit theorem, that the distribution is approximately normal with characteristics $m_x = np = 10\,000 \cdot 0.05 = 500$, $\mathrm{Var}_x = npq = 500 \times 0.95 = 475$, $\sigma_x \approx 21.8$.

We find the value of l such that $P\{X < l\} = 0.99$, or

$$\Phi\left(\frac{l - m_x}{\sigma_x}\right) + 0.5 = \Phi\left(\frac{l - 500}{21.8}\right) + 0.5 = 0.99.$$

From the tables for the function $\Phi(x)$ we find $(l - 500)/21.8 \approx 2.33$, whence $l \approx 551$, i.e. the bin, designed for approximately 550 balls, will not be overfilled during a shift with probability 0.99.

8.59. There is a queue of $n = 60$ people to the cashier, the payment to each person being a random variable. The mean payment $m_x = 50$ roubles, the mean square deviation of the payment $\sigma_x = 20$ roubles. The payments are independent. (1) How much money must the cashier have so that all 60 people get their money with probability 0.95. (2) How much money b will remain with the same probability 0.95, after the cashier pays all 60 people, if he starts with 3500 roubles?

Solution. (1) The total payment $Y = \sum_{i=1}^{60} X_i$, where X_i is the payment to the ith person. On the basis of the central limit theorem for uniformly distributed terms, Y has an approximately normal distribution with parameters $m_y = 3000$ roubles, $\sigma_y = \sqrt{60}\, 20 \approx 154.8$ roubles. We find the necessary initial reserve of money l from the condition $\Phi[(l - m_y)/\sigma_y] + 0.5 = 0.95$. By the tables for the error function (Appendix 5), $(l - m_y)/\sigma_y \approx 1.65$; $l = m_y + 1.65\,\sigma_y \approx 3256$ roubles.

(2) The money b that remains with probability 0.95, can be obtained by subtracting the sum l found in (1), i.e. $b = 3500 - 3256 = 244$ roubles, from 3500.

8.60*. A lottery is organized as follows. The participants buy tickets with tables of numbers 1, 2, . . ., 90. A participant must choose five numbers at random, mark them on his ticket and send the ticket to the organizers who store all the tickets till the day of the drawing. On an assigned day five different numbers are drawn at random from 90 numbers and the numbers drawn are reported to the participants. If a participant guessed less than two numbers (0 or 1), he gets no prize. If two of his numbers coincide with two of those drawn, he wins a rouble; if he guessed three numbers, he gets 100 roubles, if he guessed four numbers, he gets 10 000 roubles, and if he guessed all five numbers, he gets 1 000 000 roubles. Find: (1) the lowest ticket price at which on the average the lottery organizers will not lose; (2) the average income M of the lottery organizers when 1 000 000 people participate in the lottery, each choosing numbers independently and each buying one 30-kopeck ticket; (3) using the three-sigma rule, find the boundaries of the practically possible payments in the lottery; can we consider the total payment in the lottery to have an approximately normal distribution?

Solution. (1) We designate as p_i the probability that exactly i of the five numbers chosen by a participant coincide with the drawn numbers. We find that

$$p_2 = \frac{C_5^2 C_{85}^3}{C_{90}^5} \approx 2.25 \times 10^{-2}, \quad p_3 = \frac{C_5^3 C_{85}^2}{C_{90}^5} \approx 8.12 \times 10^{-4},$$

$$p_4 = \frac{C_5^4 C_{85}^1}{C_{90}^5} \approx 9.67 \times 10^{-6}, \quad p_5 = \frac{1}{C_{90}^5} \approx 2.28 \times 10^{-8}.$$

The minimum price of a ticket must be equal to the mean value of the prize received by the participant who bought the ticket: $m = 2.25 \times 10^{-2} + 8.12 \times 10^{-4} \times 10^2 + 9.67 \times 10^{-6} \times 10^4 + 2.28 \times$

$10^{-8} \times 10^6 = 22.3 \cdot 10^{-2}$ roubles, i.e. the minimum price of a ticket is about 23 kopecks.

(2) $M = (0.30 - 0.223) \times 10^6 = 77 \times 10^3$ roubles.

(3) The total winnings X which the organizers of the lottery must pay is the sum of the winnings of the participants: $X = \sum_{i=1}^{1\,000\,000} X_i$, where X_i is the amount won by the ith participant.

We assume that the participants mark their numbers independently so that the variables X_i ($i = 1, 2, \ldots, 1\,000\,000$) are independent. We know from the central limit theorem that the sum of a sufficiently large number of independent random variables which have the same distribution has approximately a normal distribution. We must find out whether the number of terms $n = 1\,000\,000$ is sufficient in this case for the variable X to be considered normally distributed.

We find the mean value m_x and the mean square deviation σ_x of the random variable X. For any $i = 1, 2, \ldots, 1\,000\,000$ we get $m_{x_i} = 22.3 \times 10^{-2} = 0.223$, $\alpha_2[X_i] = 2.25 \times 10^{-2} + 8.12 + 9.67 \times 10^2 + 2.28 \times 10^4 = 2.38 \times 10^4$; $\mathrm{Var}_{x_i} = 2.38 \times 10^4 - 0.22^2 \approx 2.38 \times 10^4$. Hence

$$m_x = 10^6 \times m_{x_i} = 2.23 \times 10^5, \quad \mathrm{Var}_x = 10^6 \times \mathrm{Var}_{x_i} = 2.38 \times 10^{10},$$

$$\sigma_x = 10^5 \sqrt{2.38} \approx 1.54 \times 10^5.$$

We know that for the random variable X, which has a normal distribution, the range of practically possible values is $m_x \pm 3\sigma_x$. In our case the lower bound of the possible values of the random variable X, provided that it was normally distributed, would be $m_x - 3\sigma_x = -2.39 \times 10^5$. The negative value of this bound signifies that we cannot consider the random variable X to be normally distributed since there cannot be negative winnings.

8.61*. Find the limit $\lim\limits_{a\to\infty} \sum\limits_{m=0}^{a} \frac{a^m}{m!} e^{-a}$, where a is a positive integer.

Solution. $\sum\limits_{m=0}^{a} \frac{a^m}{m!} e^{-a}$ is the probability that the random variable X, which has a Poisson distribution, will not exceed its mean value a. But as the parameter a tends to infinity, the Poisson distribution approaches the normal distribution. For the normal distribution, the probability that the random variable will not exceed its mean value is $1/2$, and this means that $\lim\limits_{a\to\infty} \sum\limits_{m=0}^{a} \frac{a^m}{m!} e^{-a} = \frac{1}{2}$

8.62. Independent random variables $X_1, X_2, \ldots$ have the same exponential distribution with parameter λ: $f(x) = \lambda e^{-\lambda x}$.

We consider the sum of a random number of these variables $Z = \sum_{i=1}^{Y} X_i$, where the random variable Y has a geometric distribution beginning with unity:

$$P_n = \mathrm{P}\{Y = n\} = pq^{n-1} \quad (0 < p < 1;\ n = 1, 2, \ldots).$$

Find the distribution and the numerical characteristics of the random variable Z.

Solution. The sum of the fixed number n of random variables $\sum_{i=1}^{n} X_i$ has Erlang's distribution of order n (see Problem 8.38) with probability density

$$f_n(x) = \frac{\lambda(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \quad (x > 0).$$

The probability density $\varphi(z)$ of the random variable Z can be found from the total probability formula on the hypotheses $\{Y = n\}$:

$$\varphi(z) = \sum_{n=1}^{\infty} f_n(z) P_n = \sum_{n=1}^{\infty} \frac{\lambda(\lambda z)^{n-1}}{(n-1)!} e^{-\lambda z} pq^{n-1}$$

$$= p\lambda e^{-\lambda z} \sum_{k=0}^{\infty} \frac{(\lambda q z)^k}{k!} = p\lambda e^{-\lambda z} e^{\lambda q z} = p\lambda e^{-\lambda p z} \quad (z > 0),$$

i.e. the random variable Z also has an exponential distribution, but with parameter λp. Consequently

$$m_z = 1/(\lambda p), \quad \mathrm{Var}_z = 1/(\lambda^2 p^2).$$

8.63. *The distribution of the sum of a random number of random terms.* We consider the sum of a random number of random terms $Z = \sum_{i=1}^{y} X_i$, where $X_1, X_2, \ldots$ is a sequence of independent random variables which have the same distribution with density $f(x)$; Y is a positive integer-valued variable, independent of them, which has a distribution $\mathrm{P}\{Y = n\} = P_n$ $(n = 1, 2, \ldots, N)$. Find the distribution and the numerical characteristics of the random variable Z.

Solution. We suppose that the random variable Y assumes the value n $(n = 1, 2, \ldots, N)$ with probability P_n. On this hypothesis $Z = \sum_{i=1}^{n} X_i$. We designate the probability density of the sum of n independent random variables $X_1, X_2, \ldots, X_n$, which have the same distribution, as $f^{(n)}(x)$. We can find the densities $f^{(n)}(x)$ successively: we first find $f^{(2)}(x)$, i.e. the convolution of two similar distributions $f(x)$ and $f(x)$, then] $f^{(3)}(x)$, the convolution of $f^{(2)}(x)$ and $f(x)$, and so on.

By the total probability formula, the probability density $\varphi(z)$ of the random variable Z is

$$\varphi(z) = \sum_{n=1}^{N} f^{(n)}(z) P_n. \tag{8.63}$$

We found the numerical characteristics of the random variable Z in Problem 7.64:

$$\text{M}[Z] = m_x m_y, \quad \text{Var}[Z] = \text{Var}_x m_y + m_x^2 \text{Var}_y,$$

where m_x, m_y; Var_x, Var_y are mean values and variances of the random variables X and Y.

When certain conditions are fulfilled, we can assume, with an accuracy sufficient for applications, that the random variable Z is normally distributed with the indicated parameters $\text{M}[Z]$, $\text{Var}[Z]$. Let us show this. Without violating the generality of our reasoning, we set $m_x = 0$ for the sake of simplicity. On the basis of the central limit theorem, we can assume for terms having the same distribution (see Sec. 8.0) that for $n > m_y - 3\sigma_y > 20$ the distribution density $f^{(n)}(z)$ in formula (8.63) is normal with parameters $m^{(n)} = 0$, $\text{Var}^{(n)} = n \text{Var}_x$ [see (5.0.33)]:

$$f^{(n)}(z) = \frac{1}{\sqrt{2\pi n \text{Var}_x}} \exp\left(-\frac{z^2}{2n \text{Var}_x}\right).$$

Formula (8.63) expresses the mean value of the function of the random argument Y:

$$\varphi(z) = \sum_{n=1}^{N} f^{(n)}(z) p_n = \text{M}[f^{(Y)}(z)].$$

For functions sufficiently close to linear functions in the range of practically possible values of a random argument, we can assume that the mean value of the function is equal to the same function of the mean value of the random argument, i.e. that

$$\text{M}[f^{(Y)}(z)] \approx f^{(m_y)}(z) \quad \text{or} \quad \varphi(z) \approx \frac{1}{\sqrt{2\pi m_y D_x}} \exp\left(-\frac{z^2}{2m_y \text{Var}_x}\right),$$

which is a normal distribution. The approximation is the more precise the closer to a linear function the function $f^{(Y)}(z)$ of the random argument Y is in the range of its practically possible values: $m_y \pm 3\sigma_y$.

Calculations have shown that we can consider this function to be approximately linear provided that $m_y - 3\sigma_y > 20$. For example, if the random variable Y has a Poisson distribution with parameter a [see (4.0.26)], then the condition will be fulfilled for $a > 40$; if it has a binomial distribution with parameters n, p [see (4.0.24)] (and a small parameter $p < 0.1$), then $np > 40$. Note that in both cases the random variable Y (the number of random terms) is approximately normally distributed with parameters $m_y = \text{Var}_y = a$ (for a Poisson distribution) and $m_y = np$; $\text{Var}_y = npq$ (for a binomial distribution).

8.64*. Independent random variables $X_1, X_2, \ldots, X_n, \ldots$ have the same exponential distribution with parameter λ. A random variable $Y = U + 1$, where the random variable U has a Poisson distribution with parameter a. Find the distribution and the numerical characteristics of the random variable $Z = \sum_{i=1}^{Y} X_i$.

Solution. The sum $\sum_{i=1}^{n} X_i$ has Erlang's distribution of order n (see Problem 8.38) with parameter λ:

$$f_n(x) = \frac{\lambda (\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \quad (x > 0).$$

By the total probability formula the probability density of the random variable Z is

$$\varphi(z) = \sum_{n=1}^{\infty} f_n(z) P_n = \sum_{n=1}^{\infty} \frac{\lambda (\lambda z)^{n-1}}{(n-1)!} e^{-\lambda z} \frac{a^{n-1}}{(n-1)!} e^{-a}$$

$$= \lambda e^{-\lambda z - a} \sum_{k=0}^{\infty} \frac{(\lambda z a)^k}{(k!)^2} \quad \text{for} \quad z > 0.$$

We can express this density as follows:

$$\varphi(z) = \lambda e^{-\lambda z - a} I_0 (2\sqrt{\lambda z a}) \quad \text{for} \quad z > 0,$$

where

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2}$$

is a modified cylindrical Bessel function. Then, from Problem 8.66, we have

$$M[Z] = m_x m_y = (a+1)/\lambda, \quad \mathrm{Var}[Z] = \mathrm{Var}_x m_y + m_x^2 \mathrm{Var}_y$$
$$= (a+1)/\lambda^2 + a/\lambda^2 = (2a+1)/\lambda^2.$$

8.65. Consider a system of random variables X_i $(i = 1, 2, \ldots, n)$ related to a discrete random variable Y thus:

$$X_i = \begin{cases} 1, & \text{if} \quad i \leqslant Y, \\ 0, & \text{if} \quad i > Y. \end{cases}$$

The distribution function $F(y)$ of the random variable Y is known. Find the distribution of each random variable X_i and the numerical characteristics of the system of random variables $(X_1, X_2, \ldots, X_n)$.

Solution. The ordered series of the random variable X_i has the form

$$X_i: \left| \begin{array}{c|c} 0 & 1 \\ \hline P\{Y < i\} & P\{Y \geqslant i\} \end{array} \right|, \quad (i = 1, 2, \ldots, n).$$

Since $\mathrm{P}\{Y < i\} = F(i)$, the ordered series has the form

$$X_i: \left| \begin{array}{c|c} 0 & 1 \\ \hline F(i) & 1-F(i) \end{array} \right|,$$

whence $m_{x_i} = 1 - F(i)$, $\mathrm{Var}_{x_i} = F(i)[1 - F(i)]$.

Let us find the covariances of the random variables X_i and X_j, for which purpose we determine $\mathrm{M}[X_iX_j]$. For $i < j$ the product X_iX_j can assume only two values. viz. 1 if $X_j = 1$, and 0 if $X_j = 0$. Consequently $\mathrm{M}[X_iX_j] = m_{x_j} = 1 - F(j)$ $(i < j)$, whence $\mathrm{Cov}_{ij} = \mathrm{M}[X_iX_j] - m_{x_i}m_{x_j} = 1 - F(j) - [1 - F(i)][1 - F(j)] = F(i)[1 - F(j)]$ $(i < j)$, and the correlation coefficient

$$r_{ij} = \frac{\mathrm{Cov}_{ij}}{\sigma_{x_i}\sigma_{x_j}} = \frac{F(i)[1-F(j)]}{\sqrt{F(i)F(j)[1-F(i)][1-F(j)]}} = \sqrt{\frac{F(i)[1-F(j)]}{F(j)[1-F(i)]}}.$$

8.66. A number of independent trials are made in each of which an event A may occur with probability p. The trials are terminated as soon as the event A occurs n times $(n > 1)$. Find the distribution and the numerical characteristics of the number X of "failures" in which the event A does not occur.

Solution. We find the probability that the random variable X assumes a value k. For that event to occur, it is necessary that the total number of trials should be equal to $n + k$ (k outcomes are failures and n outcomes are successes). By the hypothesis, the last trial must be successful and in the preceding $n + k - 1$ trials $n - 1$ successes and k failures must be distributed at random. The probability of this occurrence is

$$\mathrm{P}\{X = k\} = C_{n+k-1}^{k} p^n q^k, \quad \text{where } q = 1 - p \ (k = 0, 1, \ldots).$$

The distribution obtained is a natural generalization of a geometric distribution; we shall call it a *generalized geometric distribution* of order n. It is a convolution of n geometric distributions with the same parameter p: $X = \sum_{s=1}^{n} X_s$, where each random variable X_s has a geometric distribution

$$\mathrm{P}\{X_s = k\} = pq^k \quad (k = 0, 1, \ldots).$$

Indeed, the total number of failures is the sum of: (1) the number of failures till the first occurrence of the event A; (2) the number of failures from the first to the second occurrence of the event A, and so on. Hence we obtain the numerical characteristics of the variable X: $m_x = nq/p$, $\mathrm{Var}_x = nq/p^2$.

8.67. The hypothesis is the same as in Problem 8.66, except that the random variable Y is the **total number of trials** (both successful and unsuccessful), made till the nth occurrence of the event A. Find the distribution and the numerical characteristics of the random variable Y.

Solution. $Y = X + n$, where X is the random variable in the preceding problem. Hence

$$P\{Y=k\}=P\{X=k-n\}=C_{k-1}^{k-n}p^nq^{k-n}=C_{k-1}^{n-1}p^nq^{k-n}$$
$$(k=n,\ n+1,\ \ldots).$$

The numerical characteristics of the variable Y are

$$m_y = m_x + n = nq/p + n = n/p, \quad \mathrm{Var}_y = \mathrm{Var}_x = nq/p^2.$$

8.68. There is a random variable Y which has an exponential distribution with parameter λ: $f(y) = \lambda e^{-\lambda y}$ $(y > 0)$. For a given value of the variable $Y = y$ a random variable X has a Poisson distribution with parameter y:

$$P\{X=k|Y=y\}=\frac{y^k}{k!}e^{-y} \quad (k=0,\ 1,\ 2,\ \ldots).$$

Find the marginal distribution of the random variable X.

Solution. The total probability of the event $X=k$ is

$$P\{X=k\}=\int_0^\infty \frac{y^k}{k!}e^{-y}\lambda e^{-\lambda y}\,dy=\frac{\lambda}{k!}\int_0^\infty y^k e^{-(1+\lambda)y}\,dy$$
$$=\frac{\lambda}{k!}k!\,(1+\lambda)^{-(k+1)}=\frac{\lambda}{(1+\lambda)^{k+1}} \quad (k=0,\ 1,\ 2,\ \ldots).$$

If we introduce the designations $\lambda/(1 + \lambda) = p$, $1/(1 + \lambda) = q = 1 - p$, we obtain $P\{X = k\} = pq^k$ $(k = 0, 1, 2, \ldots)$, i.e. the random variable X has a geometric distribution with parameter $p = \lambda/(1 + \lambda)$.

8.69. A Geiger-Müller counter is mounted in a spaceship to find the number of particles falling in it during a random time interval T which has an exponential distribution with parameter μ. Particles arrive in a Poisson flow with intensity λ; each particle is recorded by the counter with probability p. A random variable X is the number of recorded particles. Find its distribution and the characteristics m_x and Var_x.

Solution. We assume that $T = t$ and find the conditional probability that $X = m$ $(m = 0, 1, 2, \ldots)$.

$$P\{X=m|t\}=\frac{(\lambda pt)^m}{m!}e^{-\lambda pt}.$$

Then the total probability of the event $\{X=m\}$

$$P\{X=m\}=\int_0^\infty \frac{(\lambda pt)^m}{m!}e^{-\lambda pt}\mu e^{-\mu t}\,dt=\frac{\mu}{m!}(\lambda p)^m\int_0^\infty t^m e^{-(\lambda p+\mu)t}\,dt$$
$$=\mu\frac{(\lambda p)^m}{(\lambda p+\mu)^{m+1}}=\frac{\mu}{\lambda p+\mu}\left(\frac{\lambda p}{\lambda p+\mu}\right)^m \quad (m=0,\ 1,\ 2,\ \ldots).$$

This is a geometric distribution with parameter $\mu/(\lambda p+\mu)$ (see the preceding problem), and, therefore, [see formulas (4.0.30) and (4.0.31)]:

$$m_x=\left(\frac{\lambda p}{\lambda p+\mu}\right)\Big/\left(\frac{\mu}{\lambda p+\mu}\right)=\frac{\lambda p}{\mu},$$

$$\mathrm{Var}_x=\left(\frac{\lambda p}{\lambda p+\mu}\right)\Big/\left(\frac{\mu}{\lambda p+\mu}\right)^2=\frac{\lambda p(\lambda p+\mu)}{\mu^2}$$

$$=\left(\frac{\lambda p}{\mu}\right)^2+\frac{\lambda p}{\mu}=m_x(m_x+1).$$

8.70. Solve Problem 8.69 on the hypothesis that the counter is switched on for a random time T with density $f(t)$ $(t>0)$.

Solution. As in the preceding problem, for $T=t$ the conditional distribution of the variable X is

$$\mathrm{P}\{X=m\mid t\}=\frac{(\lambda pt)^m}{m!}\mathrm{e}^{-\lambda pt}\quad(m=0,\ 1,\ 2,\ \ldots).$$

The marginal distribution

$$\mathrm{P}\{X=m\}=\int_0^\infty\frac{(\lambda pt)^m}{m!}\mathrm{e}^{-\lambda pt}f(t)\,dt\quad(m=0,\ 1,\ 2,\ \ldots).$$

We find the numerical characteristics of the random variable X. The conditional expectation $m_{x|t}=\lambda pt$; the absolute expectation $m_x=\int_0^\infty\lambda ptf(t)\,dt=\lambda p\int_0^\infty tf(t)\,dt=\lambda pm_t$, where $m_t=\mathrm{M}[T]$. By analogy we find the second moment about the origin of the random variable X (note that in this way we can only find absolute moments about the origin and not central moments):

$$\alpha_2[X\mid t]=\lambda pt+(\lambda pt)^2,$$

$$\alpha_2[X]=\lambda p\int_0^\infty tf(t)\,dt+(\lambda p)^2\int_0^\infty t^2f(t)\,dt$$

$$=\lambda pm_t+(\lambda p)^2\alpha_2[T]=\lambda pm_t+(\lambda p)^2(\mathrm{Var}_t+m_t^2),$$

where Var_t is the variance of the random variable T. Hence

$$\mathrm{Var}_x=\alpha_2[X]-m_x^2=\lambda pm_t+(\lambda p)^2\,\mathrm{Var}_t.$$

8.71. Solve the preceding problem for the special case when $f(t)$ is Erlang's distribution of order $(k+1)$ with parameter μ:

$$f(t)=f_{k+1}(t)=\frac{\mu(\mu t)^k}{k!}\mathrm{e}^{-\mu t}\quad(t>0).$$

Solution.

$$P(X=m)=\int_0^\infty \frac{(\lambda pt)^m}{m!} e^{-\lambda pt} \frac{\mu(\mu t)^k}{k!} e^{-\mu t}\,dt$$

$$=\frac{\mu(\lambda p)^m\mu^k}{m!\,k!}\int_0^\infty t^{m+k} e^{-(\lambda p+\mu)t}\,dt=\frac{\mu(\lambda p)^m\mu^k(m+k)!}{m!\,k!\,(\lambda p+\mu)^{m+k+1}}$$

$$=C_{m+k}^m\left(\frac{\mu}{\lambda p+\mu}\right)^{k+1}\left(\frac{\lambda p}{\lambda p+\mu}\right)^m.$$

Thus the random variable X has a generalized geometric distribution of order $k+1$ (see Problem 8.66) with parameter

$$p_1=\frac{\mu}{\lambda p+\mu}.$$

The mean value of the random variable T, which has Erlang's distribution of order $k+1$, is $m_t=(k+1)/\mu$, and the variance $\mathrm{Var}_t=(k+1)/\mu^2$. Consequently, from the formulas obtained in the preceding problem we have

$$m_x=\frac{\lambda p}{\mu}(k+1),\quad \mathrm{Var}_x=\frac{\lambda p}{\mu}(k+1)\left(1+\frac{\lambda p}{\mu}\right),$$

which can be represented, as in the previous problem, in the form

$$m_x=\frac{(k+1)}{p_1}q_1,\quad \mathrm{Var}_x=\frac{(k+1)}{p_1^2}q_1,\quad \text{where}\quad q_1=1-p_1.$$

8.72. When physical quantities are measured, the results are usually rounded off to the least scale division of the instrument. Then a continuous random variable turns into a discrete variable whose possible values are partitioned by intervals equal to the scale division. The following problem arises correspondingly. A continuous random variable X, which has a distribution with probability density $f(x)$, is rounded off to the nearest integer; a discrete random variable $Y = V(X)$ results, where $V(X)$ is an integer nearest to X. Find the ordered series of the random variable Y and its numerical characteristics.

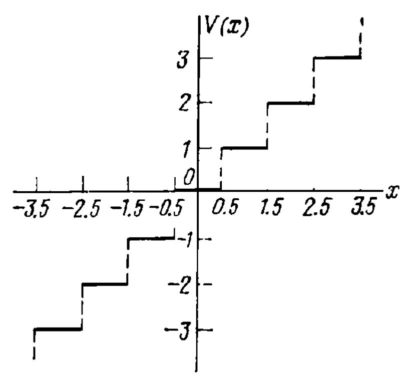

Fig. 8.72

Solution. The graph of the function $V(X)$ is given in Fig. 8.72. When the distances from the value of x to two nearest integral values are equal, the rule of rounding off is inessential since the probability that a continuous random variable falls in any point is zero.

The probability that the random variable Y assumes an integral value k is

$$P\{Y=k\}=\int_{k-0.5}^{k+0.5} f(x)\,dx \quad (k=0, \pm 1, \pm 2, \ldots),$$

whence

$$m_y=\sum_{k=-\infty}^{\infty} k \int_{k-0.5}^{k+0.5} f(x)\,dx, \quad \mathrm{Var}_y=\sum_{k=-\infty}^{\infty} k^2 \int_{k-0.5}^{k+0.5} f(x)\,dx-m_y^2\ {}^{*)}.$$

8.73. The random variables X and Y are mutually independent and have Poisson's distributions with parameters a and b. Find the distribution of their difference $Z = X - Y$ and of the modulus of their difference $U = |X - Y| = |Z|$.

Solution. The random variable Z may be either positive or negative. The probability that Z assumes the value $k > 0$ is equal to the sum of probabilities that X and Y will assume two values differing by k (X being greater than or equal to Y):

$$P\{Z=k\}=\sum_{m=0}^{\infty} \frac{b^m}{m!}\frac{a^{m+k}}{(m+k)!} e^{-(a+b)} \quad (k \geqslant 0).$$

The probability that Z will assume a negative value $-k$ is

$$P\{Z=-k\}=\sum_{m=0}^{\infty} \frac{a^m}{m!}\frac{b^{(m+k)}}{(m+k)!} e^{-(a+b)} \quad (k>0).$$

For the random variable U

$$P\{U=0\}=\sum_{m=0}^{\infty} \frac{(ab)^m}{(m!)^2} e^{-(a+b)},$$

$$P\{U=k\}=\sum_{m=0}^{\infty} \frac{(ab)^m\,(a^k+b^k)}{m!\,(m+k)!} e^{-(a+b)} \quad (k>0).$$

These probabilities can be written by means of a modified cylindrical Bessel functions of the 1st kind:

$$I_k(x)=I_{-k}(x)=\sum_{m=0}^{\infty} \frac{1}{m!\,(k+m)!}\left(\frac{x}{2}\right)^{k+2m} \quad (k=0, 1, 2, \ldots)\ {}^{**)}.$$

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*) If the unit of measurements (the value of the division of the instrument) is small as compared to the range of the possible values of the random variable X, then $m_y \approx m_x$, $\mathrm{Var}_y \approx \mathrm{Var}_x$.

**) The tables of cylindrical Bessel functions of the 1st kind can be found in reference books.

In this case

$$\mathrm{P}\{Z=k\}=I_k\left(2\sqrt{ab}\right)\left(\frac{q}{b}\right)^{k/2}\mathrm{e}^{-(a+b)} \quad (k=0, \pm 1, \pm 2, \ldots),$$

$$\mathrm{P}\{U=0\}=I_0\left(2\sqrt{ab}\right)\mathrm{e}^{-(a+b)},$$

$$\mathrm{P}\{U=k\}=I_k\left(2\sqrt{ab}\right)\left[\left(\frac{a}{b}\right)^{k/2}+\left(\frac{a}{b}\right)^{-k/2}\right]\mathrm{e}^{-(a+b)} \quad (k>0).$$

8.74. Given the joint probability density $f(x, y)$ of the random variables (X, Y), find the probability density $g(z)$ of the difference $Z = X - Y$.

Solution. The probability density function of the system $(X, -Y)$ is $f(x, -y)$ and, therefore, we find from the equation $X - Y = X + (-Y)$ that

$$g(z)=G'(z)=\int_{-\infty}^{\infty} f(x, x-z)\,dx.$$

If the random variables X and Y are mutually independent, then

$$g(z)=\int_{-\infty}^{\infty} f_1(x) f_2(x-z)\,dx=\int_{-\infty}^{\infty} f_1(y-z) f_2(y)\,dy.$$

8.75. Find the probability density of the difference of two independent random variables which have an exponential distribution with parameters λ and μ: $Z = X - Y$, $f_1(x) = \lambda e^{-\lambda x}$ $(x > 0)$, $f_2(y) = \mu e^{-\mu y}$ $(y > 0)$.

Solution. $g(z)=\int_{-\infty}^{\infty} f_1(x) f_2(x-z)\,dx$; $f_1(x)$ is nonzero for $x>0$; $f_2(x-z)$ is nonzero for $x-z>0$.

(a) $z>0$, $\quad g(z)=\int_{z}^{\infty} \lambda e^{-\lambda x}\mu e^{-\mu(x-z)}\,dx=\frac{\lambda\mu e^{-\lambda z}}{\lambda+\mu}$,

(b) $z<0$, $\quad g(z)=\int_{0}^{\infty} \lambda e^{-\lambda x}\mu e^{-\mu(x-z)}\,dx=\frac{\lambda\mu e^{\mu z}}{\lambda+\mu}$.

Consequently

$$g(z)=\begin{cases} \lambda\mu e^{-\lambda z}(\lambda+\mu)^{-1} & \text{for} \quad z>0, \\ \lambda\mu e^{\mu z}(\lambda+\mu)^{-1} & \text{for} \quad z<0. \end{cases}$$

The parameters of this distribution are

$$m_z=\frac{1}{\lambda}-\frac{1}{\mu}=\frac{\mu-\lambda}{\lambda\mu}, \quad \mathrm{Var}_z=\frac{1}{\lambda^2}+\frac{1}{\mu^2}=\frac{\lambda^2+\mu^2}{(\lambda\mu)^2}.$$

The distribution curve has the form shown in Fig. 8.75*a*. For $\lambda = \mu$ we get $g(z) = \frac{\lambda}{2} e^{-\lambda|z|}$ (Fig. 8.75*b*). This distribution is known as a Laplace distribution.

8.76. *A convolution of two distributions of two nonnegative random variables.* Given two nonnegative random variables X and Y with probability densities $f_1(x)$ $(x > 0)$ and $f_2(y)$ $(y > 0)$, form the convolution of their distributions.

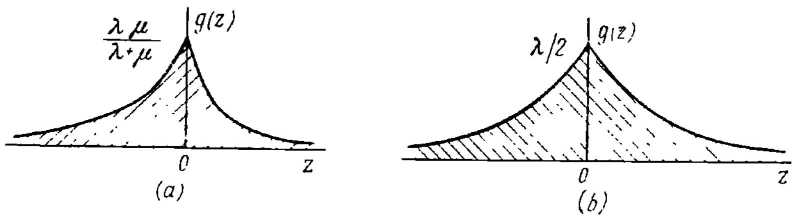

Fig. 8.75

Solution. Assume $Z = X + Y$; we find the probability density $g(z)$ of the random variable Z. By the general formula (8.0.5)

$$g(z) = \int_{-\infty}^{\infty} f_1(x) f_2(z-x)\, dx.$$

Taking into account that $f_1(x) = 0$ for $x < 0$ and $f_2(z - x) = 0$ for $x > z$, we obtain

$$g(z) = \int_{0}^{z} f_1(x) f_2(z-x)\, dx. \tag{8.76}$$

8.77. There are two mutually independent random variables X and Y which have the same normal distribution with parameters $m_x = m_y = 0$, σ_x; σ_y. Find the probability density of the sum of their moduli $Z = |X| + |Y|$.

Solution. We designate $U = |X|$; $V = |Y|$. Their probability density functions $f_1(u)$ and $f_2(v)$ are

$$f_1(u) = \begin{cases} \frac{2}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{u^2}{2\sigma_x^2}\right) & \text{for} \quad u > 0, \\ 0 & \text{for} \quad u \leqslant 0, \end{cases}$$

$$f_2(v) = \begin{cases} \frac{2}{\sqrt{2\pi}\,\sigma_y} \exp\left(-\frac{v^2}{2\sigma_y^2}\right) & \text{for} \quad v > 0, \\ 0 & \text{for} \quad v \leqslant 0, \end{cases}$$

respectively.

Taking formula (8.76) into account, we get for the convolution of the

distributions of nonnegative random variables

$$g(z) = \int_0^z f_1(u) f_2(z-u)\,du \quad (z > 0)$$

or

$$g(z) = \left(\frac{2}{\sqrt{2\pi}}\right)^2 \cdot \frac{1}{\sigma_x \sigma_y} \int_0^z \exp\left[-\left(\frac{u^2}{2\sigma_x^2} + \frac{z^2}{2\sigma_y^2} - \frac{zu}{\sigma_y^2} + \frac{u^2}{2\sigma_y^2}\right)\right] du. \quad (8.77)$$

This integral can be expressed in terms of the error function $\Phi(x)$, for which purpose we isolate a perfect square in the exponent of expression (8.77). After the requisite transformations we obtain

$$g(z) = \frac{2\sqrt{2}}{\sqrt{\pi(\sigma_x^2 + \sigma_y^2)}} \exp\left(\frac{-z^2}{2(\sigma_x^2 + \sigma_y^2)}\right)$$
$$\times \left[\Phi\left(\frac{z\sigma_y}{\sigma_x \sqrt{\sigma_x^2 + \sigma_y^2}}\right) + \Phi\left(\frac{z\sigma_x}{\sigma_y \sqrt{\sigma_x^2 + \sigma_y^2}}\right)\right].$$

8.78. A number of messages are sent over a radio channel. The length of a message X is a random variable which has an exponential distribution with parameter λ (Fig. 8.78). The interval L between the messages

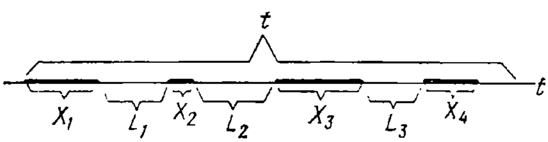

Fig. 8.78

is a random variable which has an exponential distribution with parameter μ. The lengths of separate messages and of the intervals between them are not correlated. Find the probability that no less than m messages will be sent during the time t $(m > 1)$.

Solution. The total length T of m messages plus the intervals between them is a random variable which has a generalized Erlang's distribution of order $2m - 1$ with parameters

$$\underbrace{\lambda, \lambda, \ldots, \lambda}_{m \text{ times}}; \quad \underbrace{\mu, \mu, \ldots, \mu}_{(m-1) \text{ times}}.$$

The probability that no less than m messages will be sent during the time t is none other than the distribution function of the random variable T:

$$\mathrm{P}\{\text{no less than } m \text{ messages during the time } t\} = \mathrm{P}\{T < t\} = G(t).$$

The random variable T is the sum of two independent random variables: $T = T_1 + T_2$, where T_1 has Erlang's distribution of order m with parameter λ:

$$g_1(t) = \frac{(\lambda t)^{m-1}}{(m-1)!} e^{-\lambda t} \quad (t > 0);$$

T_2 has Erlang's distribution of order $m-1$ with parameter μ:

$$g_2(t) = \frac{(\mu t)^{m-2}}{(m-2)!} e^{-\mu t} \quad (t > 0).$$

Consequently, for the case $\lambda > \mu$ we have

$$g(t) = \int_0^\infty g_1(\tau) g_2(t-\tau)\, d\tau = \int_0^t \frac{(\lambda\tau)^{m-1}}{(m-1)!} e^{-\lambda\tau} \frac{[\mu(t-\tau)]^{m-2}}{(m-2)!} e^{-\mu(t-\tau)}\, d\tau$$

$$= \frac{\lambda^{m-1}\mu^{m-2}}{(m-1)!\,(m-2)!} \sum_{i=0}^{m-2} C_{m-2}^i t^{m-2-i} (-1)^i \frac{(m-1+i)!}{(\lambda-\mu)^{m+i}}$$

$$\cdot \left[1 - \sum_{k=0}^{m-1+i} \frac{[(\lambda-\mu)t]^k}{k!} e^{-(\lambda-\mu)t}\right] e^{-\mu t} \quad (t > 0).$$

For the case $\mu > \lambda$ we have

$$g(t) = \int_0^t g_1(t-\tau) g_2(\tau)\, d\tau = \frac{\lambda^{m-1}\mu^{m-2}}{(m-1)!\,(m-2)!} \sum_{i=1}^{m-1} C_{m-1}^i t^{m-1-i} (-1)^i$$

$$\cdot \frac{(m+i)!}{(\mu-\lambda)^{m+1+i}} \left[1 - \sum_{k=0}^{m-1+i} \frac{[(\mu-\lambda)t]^k}{k!} e^{-(\mu-\lambda)t}\right] e^{-\lambda t} \ (t > 0).$$

For the case $\lambda = \mu$ we have

$$g(t) = \lambda \frac{(\lambda t)^{2m-2}}{(2m-2)!} e^{-\lambda t} \quad (t > 0).$$

8.79. A pendulum makes free continuous oscillations, the angle φ (Fig. 8.79) varying with the time t according to the harmonic law: $\varphi = a \sin(\omega t + \Theta)$, where a is the amplitude, ω is the frequency, Θ is the oscillation phase. At a moment $t = 0$, which is not related to the position of the pendulum, it is photographed. Find the distribution of the angle Φ which the axis of the pendulum makes with the vertical at the moment of photographing. Find the mean value and variance of the angle Φ.

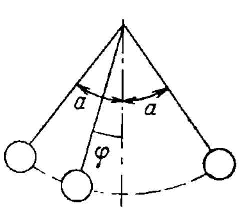

Fig. 8.79

Solution. $\Phi = a \sin \Theta$, where the phase Θ is uniformly distributed in the interval $(0, 2\pi)$; on that interval the function $\varphi = a \sin \vartheta$ is nonmonotonic. The solution of the problem will evidently not change if the variable Θ is considered to be uniformly distributed in the interval $(-\pi/2, \pi/2)$, where the function φ is monotonic. The probability density of the angle Φ is

$$g(\varphi) = \frac{1}{\pi} \frac{1}{\sqrt{1-\left(\frac{\varphi}{a}\right)^2}} \frac{1}{a} = \frac{1}{\pi\sqrt{a^2-\varphi^2}} \quad \text{for} \quad |\varphi| < a.$$

Since the distribution $g(\varphi)$ is symmetric, its mean value $m_\varphi = 0$. The variance of the angle Φ

$$\mathrm{Var}_\varphi = \frac{1}{\pi}\int_{-a}^{a}\frac{\varphi^2\,d\varphi}{\sqrt{a^2-\varphi^2}} = \frac{a^2}{2}.$$

8.80. Given n mutually independent random variables T_i ($i = 1, 2, \ldots, n$), each of which is uniformly distributed on the interval $(0, c)$, find the distribution of the number Y of the random variables (points) T_i falling on the interval $(a, b) \subset (0, c)$. What is the limiting distribution of the random variable Y when $n \to \infty$, $c \to \infty$ and the average number of points on the interval (a, b) remains constant?

Solution. We introduce a random variable Y_i, which is an indicator of the event $T_i \in (a, b)$:

$$Y_i = \begin{cases} 1, & \text{if} \quad T_i \in (a, b); \\ 0, & \text{if} \quad T_i \notin (a, b). \end{cases}$$

It is evident that

$$Y = \sum_{i=1}^{n} Y_i.$$

We designate

$$\mathrm{P}\{Y_i = 1\} = p = \frac{b-a}{c}, \quad \mathrm{P}\{Y_i = 0\} = 1 - p = \frac{c-b+a}{c}.$$

Considering n values $T_1, \ldots, T_n$ to be n independent trials, in each of which the event $T_i \in (a, b)$ may occur, we see that the random variable Y has a binomial distribution with parameters n and p and mean value $\mathrm{M}[Y] = np$:

$$\mathrm{P}\{Y = k\} = C_n^k p^k (1-p)^{n-k} \quad (k = 0, \ldots, n).$$

Let us consider the case when $c \to \infty$, $n \to \infty$, $n/c = \lambda = \text{const}$. In this case $p \to 0$ but the average number of points falling on the interval (a, b) is constant: $\mathrm{M}[Y] = \lambda(b-a) = \text{const}$. We know (see Sec. 4.0) that in this case the limiting distribution of the random variable Y is a Poisson distribution with parameter $d = \lambda(b-a)$:

$$\mathrm{P}\{Y = k\} = \frac{d^k}{k!}e^{-d} \quad (k = 0, 1, 2, \ldots).$$

The number of events in a stationary Poisson flow with intensity λ, falling on the interval (a, b), has the same distribution.

Thus we can make the following conclusion: a stationary flow of events with intensity λ can be considered to be a limit case of a collection of n independent random points on the interval $(0, c)$, each of which has a uniform distribution on that interval, provided that $n \to \infty$, $c \to \infty$, but $n/c = \lambda = \text{const}$. We shall need this model of an elementary (stationary Poisson) flow later on.

CHAPTER 9

Random Functions

9.0. A function $X(t)$ is called a *random function* if its value is a random variable for any argument t. Examples of random functions: $V(t)$, the supply voltage of a computer depending on time t; $T(h)$, the air temperature at a given point at a given moment depending on the altitude h above the ground, $Q(t)$, the number of times a computer fails during the time from 0 to t.

The concept of a random function is a generalization of the concept of a random variable. Since we can consider a random variable X to be a function of an elementary event ω (see Sec. 4.0): $X = \varphi(\omega)$ $(\omega \in \Omega)$, where Ω is a space of elementary events, or sample space, it follows that the random function $X(t)$ can be represented as

$$X(t) = \varphi(t, \omega) \quad (\omega \in \Omega, \ t \in T),$$

where t is a nonrandom argument, and T is the domain of the function $X(t)$.

The *realization* of a random function $X(t)$ is a specific form it assumes as a result of an experiment (when the elementary event ω has occurred). For example, registering by an instrument the supply voltage of a computer against time on the interval $(0, \tau)$, we get a realization $v(t)$ of the random function $V(t)$ (Fig. 9.0.1).

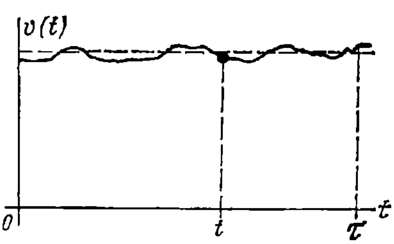

Fig. 9.0.1

A number of trials, the outcome of each of which is a random function $X(t)$, yields a collection of realizations $x_1(t), x_2(t), \ldots, x_n(t)$ of this random function. Realizations inevitably differ from one another due to random causes (Fig. 9.0.2). For a fixed moment t the random function $X(t)$ turns into an ordinary random variable. This random variable is known as a *section* of a random function.

If we consider several sections as a series of points $t_1, t_2, \ldots, t_m$ rather than one section of a random function, we get an m-dimensional random vector which describes it in some approximation (Fig. 9.0.3). In practical applications, if the values of a random function are registered with some interval for the values t_1, $t_2, \ldots, t_n$ of the argument, then we deal with an n-dimensional random vector.

A random function $X(t)$ whose argument is time is usually called a *stochastic*, or *random, process*. If stochastic process takes place in a physical system S, then this means that the state of the system varies at random with time t. If the state of the system S at a moment t can be described by one scalar random variable $X(t)$, then we deal with a *scalar random function* (a scalar stochastic process) $X(t)$. If the state of the system S at a moment t is described by several random variables $X_1(t)$, $X_2(t), \ldots, X_k(t)$, then we deal with a *vector random function* $V(t)$ (a vector stochastic process) with k components, $X_1(t), X_2(t), \ldots, X_k(t)$.

Stochastic processes are classified according to a number of criteria. A stochastic process taking place in a system S is known as a *process with discrete time* if the transformations of the system S from one state to another are only possible at certain predetermined moments $t_1, t_2, \ldots$. Examples of a process with discrete time: a computer that changes its states at moments $t_1, t_2, \ldots$, preset by the machine cycle time; a device inspected at moments $t_1, t_2, \ldots$ and transferred from one category to another as a result.

A stochastic process with discrete time is also called a *random sequence*. If the state of the system S is described by one random variable X, then the stochastic process is a sequence of random variables: $X(t_1), X(t_2), \ldots, X(t_n), \ldots$.

A stochastic process taking place in a system S is known as a *process with continuous time* if the transitions from one state to another can occur at any random time moments which continuously fill the t-axis (or an interval of the t-axis). Examples of a stochastic process with continuous time: a change in the supply

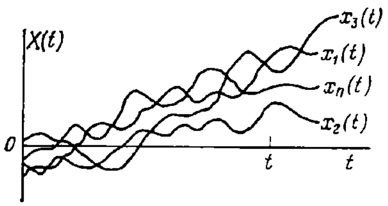

Fig. 9.0.2

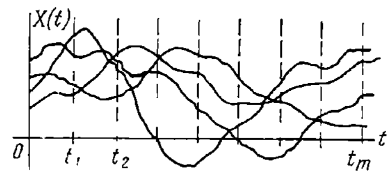

Fig. 9.0.3

voltage of a computer $V(t)$ (see Fig. 9.0.1) or a functioning of a device which fails and is reconditioned at random time moments.

A stochastic process taking place in a system S is known as a *process with discrete states* if the number of possible states of the system S is finite or countable. An example: a device consisting of two units for which the possible states of the system are s_1, both units are sound; s_2, the first unit is faulty and the second unit is sound; s_3, the first unit is sound and the second unit is faulty; s_4, both units are faulty. Another example: transmission of a message over a radio channel. Here the stochastic process $X(t)$ is the number of distorted symbols transmitted up to a moment t. This stochastic process may only have a countable set of states $\{0, 1, \ldots, n, \ldots\}$ and "jumps" up by unity at the moment a distorted symbol is received (Fig. 9.0.4).

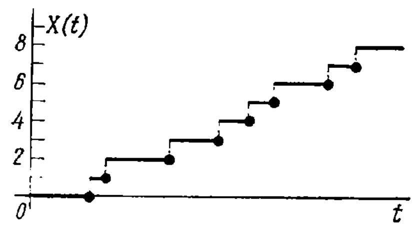

Fig. 9.0.4

A stochastic process taking place in a system S is known as a *process with continuous states* if the set of possible states of the system S is uncountable. Example: the process of placing a spaceship in a predetermined position about the earth. Another example: the supply voltage $V(t)$ of a computer.

By the criteria indicated above stochastic processes are classified as follows.

1a. Processes with discrete states and discrete time.

1b. Processes with discrete states and continuous time.

2a. Processes with continuous states and discrete time.

2b. Processes with continuous states and continuous time.

Examples of a process of type 1a. A certain Petrov bought m tickets of a lottery-loan, which may win or be payed off at certain times (drawings) $t_1, t_2, \ldots$, a stochastic process $X(t)$ is the number of tickets that have won up to and including a moment t. Another example: the state of the on-line storage of a computer; all possible states of the storage can be indicated and changes in state may only occur at discrete moments in accordance with the machine cycle time.

Example of a process of type 1b. An instrument may be in one of the following four states: s_0, sound and not switched on; s_1, sound and switched on, s_2, faulty and not switched on, s_3, faulty and switched on. We shall encounter many examples of a process of type 1b in chapters 10 and 11.

Example of a process of type 2a. The values $X(t_1)$, $X(t_2)$, ... of a continuous random variable X are observed at moments t_1, t_2, ... The sequence of values this variable assumes is a process $X(t)$ with continuous states and discrete time. For example, if the air temperature T is measured twice a day, then the sequence of recorded values of T is a stochastic process with continuous states and discrete time.

Example of a process of type 2b: the process $V(t)$ of variation of the supply voltage of a computer at any moment t or of the level of noise in the transmission of a message.

Let us consider some of the characteristics of a scalar random function $X(t)$.

A *univariate distribution* of a random variable $X(t)$ is the distribution of the section $X(t)$ of that random function for any value of the argument t. If the random variable $X(t)$ is continuous, then this distribution is the probability density function of the section $X(t)$ and is designated as $f(x, t)$. If the random variable $X(t)$ is discrete, then the univariate distribution of the random function $X(t)$ is an ordered series of the probability $P(x_i, t)$ that the random variable $X(t)$ assumes a value x_i at a moment t. For a mixed random variable $X(t)$ the univariate distribution is specified by a distribution function $F(x, t) = P\{X(t) < x\}$. Since a distribution function is the most general form of a distribution and is suitable for any random variable, we can use the general notation $F(x, t)$ for a univariate distribution too.

A *bivariate distribution* of a random function $X(t)$ is a joint distribution of two of its sections $[X(t_1)$ and $X(t_2)$ for any values of t_1 and $t_2]$. This is a function of four arguments: x_1, x_2, t_1, t_2. We can correspondingly consider an *n-variate distribution* which is dependent on $2n$ arguments.

A random function $X(t)$ is said to be *normal* if the joint distribution of any number n of its sections, taken at arbitrary moments $t_1 < t_2 < t_3 < \ldots < t_n$, is an n-variate normal distribution.

The *mean value* of a random function $X(t)$ is a nonrandom function $m_x(t)$ which is the mean value of the corresponding section of the random function for each value of the argument:

$$m_x(t) = \mathrm{M}[X(t)]. \tag{9.0.1}$$

A *correlation function* (or autocorrelation function) of a random function $X(t)$ is a nonrandom function of two arguments $R_x(t, t')$ which is equal to the covariance of the corresponding sections of the random function for each pair of values of the arguments t, t':

$$R_x(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\, \overset{\circ}{X}(t')], \tag{9.0.2}$$

where $\overset{\circ}{X}(t) = X(t) - m_x(t)$ is a *centred random function*.

For $t' = t$ the correlation function turns into a variance of the random function:

$$R_x(t, t') = \mathrm{Var}_x(t) = \mathrm{Var}[X(t)] = [\sigma_x(t)]^2. \tag{9.0.3}$$

The *main properties of a correlation function*:

(1) $R_x(t, t') = R_x(t', t)$, i.e. the function $R(t, t')$ does not vary when t is replaced by t' (symmetry);

(2) $|R_x(t, t')| \leqslant \sigma_x(t)\, \sigma_x(t')$;

(3) the function $R_x(t, t')$ is positive definite, i.e. $\int\limits_{(B)}\int\limits_{(B)} R_x(t, t')\, \varphi(t)\, \varphi(t') \times dt\, dt' \geqslant 0$, where $\varphi(t)$ is any function, and (B) is any domain of integration which is the same for both arguments.

For a normal random function, the characteristics $m_x(t)$, $R_x(t, t')$ are exhaustive and define the distribution of any number of sections.

A *normed correlation function* of a random function $X(t)$ is a function

$$r_x(t, t') = \frac{R_x(t, t')}{\sigma_x(t)\, \sigma_x(t')} = \frac{R_x(t, t')}{\sqrt{\mathrm{Var}_x(t)\, \mathrm{Var}_x(t')}}, \tag{9.0.4.}$$

i.e. for $t = t'$ the correlation coefficient $r_x(t, t')$ of the sections $X(t)$ and $X(t')$ is equal to unity.

A stochastic process $X(t)$ is known as a *process with independent increments* if, for any values of the argument $t_1 < t_2 < t_3 < \ldots < t_k < t_{k+1} < \ldots$ the random values of the increment of the function $X(t)$

$$U_1 = X(t_2) - X(t_1), \quad U_2 = X(t_3) - X(t_2), \ldots, \quad U_k = X(t_{k+1}) - X(t_k) \quad (9.0.5)$$

are independent.

A normal stochastic process with independent increments is known as a *Wiener process* if its expectation is zero and the variance of the increment is proportional to the length of the interval on which it is attained:

$$m_x(t) = 0, \quad \mathrm{Var}[U_k] = a(t_{k+1} - t_k), \quad (9.0.6)$$

where $a > 0$ is a constant.

When a nonrandom term $\varphi(t)$ is added to a random function $X(t)$, the same nonrandom term is added to its mean value, but the correlation function does not change.

When a random function $X(t)$ is multiplied by a nonrandom factor $\varphi(t)$, its mean value is multiplied by the same factor $\varphi(t)$ and the correlation function is multiplied by $\varphi(t)\varphi(t')$.

If a random function $X(t)$ is subjected to a certain transformation A_t, another random function, $Y(t) = A_t\{X(t)\}$, results.

A transformation $L_t^{(0)}$ is known as a *homogeneous linear* transformation if

$$(1) \quad L_t^{(0)}\left\{\sum_{k=1}^{n} X_k(t)\right\} = \sum_{k=1}^{n} L_t^{(0)}\{X_k(t)\}$$

(i.e. the sum can be transformed term-by-term), and if

$$(2) \quad L_t^{(0)}\{cX(t)\} = cL_t^{(0)}\{X(t)\},$$

(i.e. a factor, which is independent of the parameter t with respect to which the transformation is carried out, can be put before the transformation sign).

A transformation L_t is called a *nonhomogeneous linear* transformation if

$$L_t\{X(t)\} = L_t^{(0)}\{X(t)\} + \varphi(t),$$

where $\varphi(t)$ is a nonrandom function.

If a random function $Y(t)$ is related to a random function $X(t)$ by a linear transformation $Y(t) = L_t\{X(t)\}$, then its mean value $m_y(t)$ results from $m_x(t)$ *upon the same linear transformation*:

$$m_y(t) = L_t\{m_x(t)\}, \quad (9.0.7)$$

and to find the correlation function $R_y(t, t')$, the function $R_x(t, t')$ must be *twice* subjected to the corresponding *homogeneous linear transformation*, once with respect to t and once with respect to t':

$$R_y(t, t') = L_{t'}^{(0)}\{L_t^{(0)}\{R_x(t, t')\}\} \quad (9.0.8)$$

The *crosscorrelation function* $R_{xy}(t, t')$ of two random functions $X(t)$ and $Y(t)$ is a function

$$R_{xy}(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\,\overset{\circ}{Y}(t')]. \quad (9.0.9)$$

It follows from the definition of a crosscorrelation function that

$$R_{xy}(t, t') = R_{yx}(t', t).$$

The *normed crosscorrelation function* of two random functions $X(t)$ and $Y(t)$ is a function

$$r_{xy}(t, t') = \frac{R_{xy}(t, t')}{\sigma_x(t)\sigma_y(t')} = \frac{R_{xy}(t, t')}{\sqrt{\mathrm{Var}_x(t)\,\mathrm{Var}_y(t')}}. \tag{9.0.10}$$

The random functions $X(t)$ and $Y(t)$ are *uncorrelated* if $R_{xy}(t, t') \equiv 0$.

If $Z(t) = X(t) + Y(t)$, then $m_z(t) = m_x(t) + m_y(t)$,

$$R_z(t, t') = R_x(t, t') + R_y(t, t') + R_{xy}(t, t') + R_{xy}(t, t').$$

If two random functions $X(t)$ and $Y(t)$ are *uncorrelated*, then

$$R_z(t, t') = R_x(t, t') + R_y(t, t'). \tag{9.0.11}$$

If

$$Z(t) = \sum_{k=1}^{n} X_k(t), \tag{9.0.12}$$

where $X_1(t), X_2(t), \ldots, X_n(t)$ are uncorrelated random functions, then

$$m_z(t) = \sum_{k=1}^{n} m_{x_k}(t), \quad R_z(t, t') = \sum_{k=1}^{n} R_{x_k}(t, t').$$

When transforming random functions, it is often convenient to write them in a complex form. A *complex random variable* is a random function of the form

$$Z(t) = X(t) + iY(t), \tag{9.0.13}$$

where $X(t)$ and $Y(t)$ are real random functions and i is a unit imaginary number.

The mean value, the correlation function and variance of a complex random function are defined thus:

$$m_z(t) = m_x(t) + im_y(t), \quad R_z(t, t') = \mathrm{M}\,[\overset{\circ}{X}(t)\,\overline{\overset{\circ}{X}(t')}], \tag{9.0.14}$$

where the bar over the letters denotes a complex conjugate quantity and

$$\mathrm{Var}_z(t) = R_z(t, t') = \mathrm{M}\,[|\overset{\circ}{X}(t)|^2]. \tag{9.0.15}$$

Before considering complex random variables and functions, it is necessary to define variance as the mean value of the square of the modulus, and covariance as the mean value of the product of a centred random variance by the complex conjugate of another centred variable*).

The *canonical decomposition* of a random variable $X(t)$ is its representation in the form

$$X(t) = m_x(t) + \sum_{k=1}^{m} V_k \varphi_k(t)^{**)}, \tag{9.0.16}$$

where V_k $(k = 1, 2, \ldots, m)$ are centred uncorrelated random variables with variances Var_k $(k = 1, 2, \ldots, m)$ and $\varphi_k(t)$ $(k = 1, 2, \ldots, m)$ are nonrandom functions. Random variables V_k $(k = 1, 2, \ldots, m)$ are known as *coefficients* and functions $\varphi_k(t)$ $(k = 1, 2, \ldots, m)$ as *coordinate functions* of a canonical decomposition.

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*) In what follows we shall indicate the complex nature of a random function every time; if it is not specified, we shall consider a random function to be real.

**) In particular, the sum can be extended to an infinite (countable) number of terms.

If the random function $X(t)$ admits of a canonical decomposition (9.0.16) in a real form, then the correlation function $R_x(t, t')$ is expressed by the sum

$$R_x(t, t') = \sum_{k=1}^{m} \mathrm{Var}_k\, \varphi_k(t)\, \varphi_k(t'), \tag{9.0.17}$$

which is known as a *canonical decomposition of a correlation function.*

If a random function $X(t)$ admits of a canonical decomposition (9.0.16) in a complex form, then the canonical decomposition of the correlation function has the form

$$R_x(t, t') = \sum_{k=1}^{m} \mathrm{Var}_k\, \varphi_k(t)\, \overline{\varphi_k(t')}, \tag{9.0.18}$$

where a bar over the letters denotes a complex conjugate quantity.

The possibility of a canonical decomposition of a correlation function in the form (9.0.17) or (9.0.18) implies a representability of a random function $X(t)$ in the canonical form (9.0.16), where the random variables V_k $(k = 1, 2, \ldots, m)$ have variances Var_k $(k = 1, 2, \ldots, m)$.

A linear transformation of a random function $X(t)$ defined by the canonical decomposition (9.0.16) results in a random function $Y(t) = L_t\{X(t)\}$ also in a canonical form, i.e.

$$Y(t) = m_y(t) + \sum_{k=1}^{m} V_k\, \psi_k(t), \tag{9.0.19}$$

where

$$m_y(t) = L_t\{m_x(t)\}, \quad \psi_k(t) = L_t^{(0)}\{\varphi_k(t)\}. \tag{9.0.20}$$

That is, *in a linear transformation of a random function, defined by a canonical decomposition, its mean value is subjected to the same linear transformation and the coordinate functions are subjected to the corresponding homogeneous linear transformation.*

A *stationary*) random function* $X(t)$ is a random function whose mean value is constant, $m_x = \text{const}$, and whose correlation function only depends on the difference between its arguments: $R_x(t, t') = R_x(\tau)$, where $\tau = t' - t$.

From the symmetry of the correlation function $R_x(t, t')$ it follows that $R_x(\tau) = R_x(-\tau)$, i.e. *the correlation function of a stationary random function is an even function of the argument* τ.

The variance of a stationary random function is constant:

$$\mathrm{Var}_x = R_x(t, t') = R_x(0) = \text{const}. \tag{9.0.21}$$

The correlation function of a stationary random function possesses the property

$$|R_x(\tau)| \leqslant \mathrm{Var}_x. \tag{9.0.22}$$

The normed correlation function of a stationary random function is

$$\rho_x(\tau) = R_x(\tau)/\mathrm{Var}_x = R_x(\tau)/R_x(0). \tag{9.0.23}$$

The canonical decomposition of a stationary random function is

$$X(t) = m_x + \sum_{k=0}^{\infty} (U_k \cos \omega_k t + V_k \sin \omega_k t), \tag{9.0.24}$$

where U_k, V_k $(k = 0, 1, \ldots)$ are centred uncorrelated random variables with pairwise equal variances $\mathrm{Var}[U_k] = \mathrm{Var}[V_k] = \mathrm{Var}_k$.

Representation (9.0.24) is known as the *spectral* decomposition of a function. A spectral decomposition of a stationary random function is associated with a series

*) Stationary in a broad sense.

expansion of its correlation function:

$$R_x(\tau)=\sum_{k=0}^{\infty} \mathrm{Var}_k \cos \omega_k \tau, \tag{9.0.25}$$

whence

$$\mathrm{Var}_x=\sum_{k=0}^{\infty} \mathrm{Var}_k. \tag{9.0.26}$$

Setting $\omega_0 = 0$, we can rewrite the spectral decomposition (9.0.24) of a stationary random function in a complex form, i.e.,

$$X(t)=m_x+\sum_{k=-\infty}^{\infty} W_k e^{i\omega_k t}, \tag{9.0.27}$$

where

$$\omega_{-k}=-\omega_k, \quad W_0=U_0, \quad W_k=\frac{U_k-iV_k}{2}, \quad W_{-k}=\frac{U_k+iV_k}{2} \ (k=1,\ 2,\ \ldots).$$

The *spectral density* of a stationary random function $X(t)$ is the limit of the ratio of the variance per a given interval of frequencies to the length of the interval as the latter tends to zero. The spectral density $S_x(\omega)$ and the correlation function $R_x(\tau)$ are related by Fourier's transformations. In the real form this relation is

$$S_x(\omega)=\frac{2}{\pi}\int_0^{\infty} R_x(\tau)\cos\omega\tau\, d\tau, \quad R_x(\tau)=\int_0^{\infty} S_x(\omega)\cos\omega\tau\, d\omega. \tag{9.0.28}$$

It follows from the last relation that

$$\mathrm{Var}_x=R_x(0)=\int_0^{\infty} S_x(\omega)\, d\omega. \tag{9.0.29}$$

In a complex form Fourier's transformations, which relate the spectral density $S_x^*(\omega)$ and the correlation function $R_x(\tau)$, have the form

$$S_x^*(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\,\mathrm{e}^{-i\omega\tau}\, d\tau, \quad R_x(\tau)=\int_{-\infty}^{\infty} S_x^*(\omega)\,\mathrm{e}^{i\omega\tau}\, d\omega. \tag{9.0.30}$$

Appendix 7 is a table of relations between certain correlation functions and spectral densities.

Both $S_x(\omega)$ and $S_x^*(\omega)$ are nonnegative real functions; $S_x^*(\omega)$ is an even function defined on the interval $(-\infty, +\infty)$; $S_x(\omega)$ is defined on the interval $(0, +\infty)$, and on this interval $S_x^*(\omega) = 0.5S_x(\omega)$.

A *normed spectral density* $s_x(\omega)$ [or $s_x^*(\omega)$ in a complex form] is a spectral density divided by the variance of the random function $X(t)$:

$$s_x(\omega) = S_x(\omega)/\mathrm{Var}_x, \quad s_x^*(\omega) = S_x^*(\omega)/\mathrm{Var}_x. \tag{9.0.31}$$

If a crosscorrelation function $R_{x,y}(t, t')$ of two stationary random functions $X(t)$ and $Y(t')$ is a function of only $\tau = t' - t$, then we say that they are *stationary connected*. The following relations then exist between the crosscorrelated function $R_{xy}(\tau)$ and the *cross-power spectral density* $S_{xy}^*(\omega)$, which are defined by the Fourier transformation in a complex form:

$$S_{xy}^*(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\omega\tau} R_{xy}(\tau)\, d\tau, \quad R_{xy}(\tau)=\int_{-\infty}^{\infty} \mathrm{e}^{i\omega\tau} S_{xy}^*(\omega)\, d\omega \tag{9.0.32}$$

If the normal stationary random functions $X(t)$ and $Y(t)$ are stationary connected, then the random function $Z(t) = X(t)\,Y(t)$ is stationary with characteristics

$$m_z = m_x m_y + R_{xy}(0), \tag{9.0.33}$$

$$S_z^*(\omega) = \int_{-\infty}^{\infty} S_x^*(\omega - v)\, S_y^*(v)\, dv + m_x m_y\, [S_{xy}^*(\omega) S_{yx}^*(\omega)]$$

$$+ \int_{-\infty}^{\infty} S_{xy}^*(\omega - v)\, S_{yx}^*(v)\, dv + m_x^2 S_y^*(\omega) + m_y^2 S_x^*(\omega). \tag{9.0.34}$$

In a special case, when $Z(t) = X^2(t)$, we have

$$m_z = m_x^2 + \mathrm{Var}_x = m_x^2 + R_x(0), \tag{9.0.35}$$

$$S_z^*(\omega) = 2\int_{-\infty}^{\infty} S_x^*(\omega - v)\, S_x^*(v)\, dv + 4m_x^2 S_x^*(\omega). \tag{9.0.36}$$

White noise (or white noise in a wide sense) is a random function $X(t)$, whose two different (arbitrarily close) sections are uncorrelated and whose correlation function is proportional to the delta-function:

$$R_x(t, t') = G(t)\, \delta(t - t'). \tag{9.0.37}$$

The variable $G(t)$ is known as the *intensity of white noise*. The definitions and the properties of the delta-function are given in Appendix 6.

Stationary white noise is white noise with constant intensity $G(t) = G = \text{const}$. The correlation function of stationary white noise has the form

$$R_x(\tau) = G\delta(\tau), \tag{9.0.38}$$

whence it follows that its spectral density is constant and equal to

$$S_x^*(\omega) = G/2\pi. \tag{9.0.39}$$

The variance of stationary white noise $\mathrm{Var}_x = G\delta(0)$, i.e. is infinite.

If a stationary random function $X(t)$ arrives at the input of a stationary linear system L, then, some time later, the time being sufficient for the transition process to be damped out, the random function $Y(t)$ at the output of the linear system will also be stationary. The spectral densities of the input and output functions are related as

$$S_y^*(\omega) = S_x^*(\omega)\, |\, \Phi(i\omega)\, |^2, \tag{9.0.40}$$

where $\Phi(i\omega)$ is the amplitude-frequency characteristic of the linear system.

The stationary function $X(t)$ is said to *possess an ergodic property* if its characteristics $[m_x, R_x(\tau)]$ can be defined as the corresponding t averages for one realization of sufficient length. The sufficient condition for ergodicity of a stationary random function (with respect to expectation) is the tending of its correlation function to zero as $\tau \to \infty$:

$$\lim_{\tau \to \infty} R_x(\tau) = 0. \tag{9.0.41}$$

For the random function $X(t)$ to be ergodic with respect to variance Var_x, it is sufficient that the random function $Y(t) = X^2(t)$ should possess a similar (the same) property, i.e. $\lim\limits_{\tau \to \infty} R_y(\tau) = 0$ as $\tau \to \infty$.*)

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*) For a random function to be ergodic with respect to a correlation function, it is necessary that the function $Z(t, \tau) = X(t)\, X(t + \tau)$ possesses a similar property.

If the realization of a random function $X(t)$ crosses (upwards) a straight line which is parallel to the t-axis and is at the distance a from it, then the function $X(t)$ is said to *cross* the level a (see Fig. 9.0.5, in which the level crossings are marked by crosses). The number of level crossings X during the time T is a discrete random variable; if level crossings are infrequent, then the variable can be considered to have a Poisson distribution.

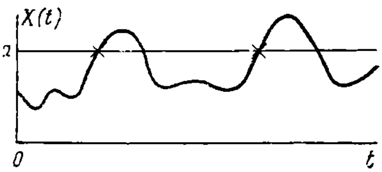


Fig. 9.0.5

For a normal stationary random variable $X(t)$ the average number of crossings of the level a per unit time is

$$\lambda_a = \frac{1}{2\pi} \exp\left(-\frac{(a-m_x)^2}{2\sigma_x^2}\right) \frac{\sigma_y}{\sigma_x}, \tag{9.0.42}$$

where σ_y is the mean square deviation of the derivative of the random function

$$Y(t) = \frac{d}{dt} X(t). \tag{9.0.43}$$

The average number of downward crossings of the given level is the same.

Problems and Exercises

9.1. Considering a nonrandom function of time $\varphi(t)$ to be a special kind of random function $X(t) = \varphi(t)$, find its mean value $m_x(t)$, variance $\mathrm{Var}_x(t)$ and the correlation function $R_x(t, t')$. Is the random function $X(t)$ stationary?

Answer. $m_x(t) = \varphi(t)$, $\mathrm{Var}_x(t) = 0$, $R_x(t, t') = 0$.

In the general case the random function $X(t)$ is nonstationary since for $\varphi(t) \neq \mathrm{const}$ we have $m_x(t) \neq \mathrm{const}$.

9.2. The conditions are the same as in the preceding problem but this time $\varphi(t) = a = \mathrm{const}$, where a is a nonrandom variable: $X(t) = a$. Is the random function $X(t)$ stationary? If it is, then does it possess an ergodic property?

Answer. The random function $X(t)$ is stationary since $m_x(t) = a = \mathrm{const}$, $\mathrm{Var}_x(t) = 0$, $R_x(t, t') = R_x(\tau) = 0$, and possesses an ergodic property.

9.3. In each section a random function $X(t)$ is a continuous random variable with a univariate distribution density $f(x, t)$. Write the expressions for the mean value $m_x(t)$ and variance $\mathrm{Var}_x(t)$ of the random variable $X(t)$.

Answer.

$$m_x(t) = \int_{-\infty}^{\infty} x f(x, t)\, dx, \quad \mathrm{Var}_x(t) = \int_{-\infty}^{\infty} [x - m_x(t)]^2 f(x, t)\, dx.$$

9.4. A random function $X(t)$ is a random variable $X(t) = V$, where V is a continuous random variable with probability density $\varphi(v)$. (1) Write the expression for the univariate distribution (density) $f(x, t)$ of the random function $X(t)$; (2) find the mean value $m_x(t)$ and variance $\mathrm{Var}_x(t)$ of the random function $X(t)$; (3) write the expression

for the bivariate probability distribution function $F(x_1, x_2, t_1, t_2)$ of two sections $X(t_1)$, $X(t_2)$ of the random function $X(t)$; (4) find its correlation function $R_x(t, t')$ and the spectral density $S_x^*(\omega)$.

Solution.

(1) $f(x, t) = \varphi(x)$;

(2) $m_x(t) = \int\limits_{-\infty}^{\infty} x\varphi(x)\,dx = m_v, \quad \mathrm{Var}_x(t) = \int\limits_{-\infty}^{\infty} (x - m_v)^2 \varphi(x)\,dx$:

(3) $F(x_1, x_2, t_1, t_2) = \mathrm{P}\{X(t_1) < x_1, X(t_2) < x_2\} = \mathrm{P}\{V < x_1, Y < x_2\}$.

If $x_1 < x_2$, then $V < x_1$ implies that $V < x_2$ and $\mathrm{P}\{V < x_1, V < x_2\} = \int\limits_{-\infty}^{x_1} \varphi(x)\,dx$. Consequently

$$F(x_1, x_2, t_1, t_2) = \begin{cases} \int\limits_{-\infty}^{x_1} \varphi(x)\,dx = F(x_1) & \text{for} \quad x_1 < x_2, \\ \int\limits_{-\infty}^{x_2} \varphi(x)\,dx = F(x_2) & \text{for} \quad x_2 < x_1, \end{cases}$$

where $F(x) = \int\limits_{-\infty}^{x} \varphi(x)\,dx$ is the distribution function of the variable V;

(4) $R_x(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{X}(t')] = \mathrm{M}[\overset{\circ}{V}\overset{\circ}{V}] = \mathrm{Var}_v$,
$\mathrm{Var}_x(t) = R_x(t, t') = \mathrm{Var}_v$.

Since $m_x(t) = \text{const}$ and $R_x(t, t') = \text{const}$, the random function $X(t)$ is stationary. Since the t average for each realization is equal to the value the random variable V assumes in that realization and differs for different realizations, the random function $X(t)$ is *not ergodic*; its spectral density

$$S_x^*(\omega) = \frac{\mathrm{Var}_v}{\pi} \int\limits_{-\infty}^{\infty} e^{-i\omega\tau}\,d\omega = \mathrm{Var}_v\,\delta(\omega),$$

where $\delta(\omega)$ is the delta-function. We can immediately verify this since

$$R_x(\tau) = \int\limits_{-\infty}^{\infty} S_x^*(\omega)\,e^{i\omega\tau}\,d\omega = \int\limits_{-\infty}^{\infty} \mathrm{Var}_v\,\delta(\omega)\,e^{i\omega\tau}\,d\omega = \mathrm{Var}_v\,e^{i0\tau} = \mathrm{Var}_v.$$

9.5. A random function $X(t)$ is given in the form $X(t) = Vt - b$, where V is a random variable which has a normal distribution with parameters m_v and σ_v; b is a nonrandom variable. Find the univariate distribution density $f(x, t)$ of the section of the random function $X(t)$ and its characterictics: $m_x(t)$, $\mathrm{Var}_x(t)$, $R_x(t, t')$.

Answer. $f(x, t)$ is a normal distribution with parameters $m_v t + b$, $|t|\sigma_v$:

$$f(x, t) = \frac{1}{|t|\sigma_v\sqrt{2\pi}}\exp\left[-\frac{[x-(m_v t+b)]^2}{2t^2\sigma_v^2}\right],$$

$$m_x(t) = m_v t + b, \quad \mathrm{Var}_x(t) = t^2\sigma_v^2, \quad R_x(t, t') = \sigma_v^2 tt'.$$

9.6. Show that any function of two arguments of the form

$$\sum_{i=1}^{n} \mathrm{Var}_i\, \varphi_i(t)\, \varphi_i(t'), \tag{9.6}$$

where Var_i are nonnegative numbers and $\varphi_i(t)$ are any real functions ($i = 1, 2, \ldots, n$), possesses all the properties of a correlation function.

Solution. It is sufficient to show that there is a random function $X(t)$ which has a correlation function (9.6). Let us consider a real random function $X(t)$ given in the form of a canonical decomposition

$$X(t) = m_x(t) + \sum_{i=1}^{n} V_i\varphi_i(t),$$

where $\mathrm{Var}[V_i] = \mathrm{Var}_i$. The correlation function of this random function has the form

$$R_x(t, t') = \sum_{i=1}^{n} \mathrm{Var}_i\, \varphi_i(t)\, \varphi_i(t'),$$

and that is what we wished to prove.

9.7. Given that the characteristics of a normal stochastic process $X(t)$ are $m_x(t)$, $\mathrm{Var}_x(t)$, $R_x(t, t')$ and that the stochastic process is in a state x ($X(t) = x$) at a moment t, find the conditional probability $p_{\alpha\beta}$ that at a moment $t' > t$ the process $X(t')$ will belong to a certain domain (α, β):

$$p_{\alpha\beta} = \mathrm{P}\{X(t') \in (\alpha, \beta) \mid X(t) = x\}.$$

Solution. We designate $X(t)$ X_1, $X(t')$ X_2. A system of random variables X_1, X_2 has a normal distribution $f(x_1, x_2)$ with characteristics

$$m_{x_1} = m_x(t), \quad m_{x_2} = m_x(t'),$$

$$\sigma_{x_1}^2 = \mathrm{Var}_x(t), \quad \sigma_{x_2}^2 = \mathrm{Var}_x(t'), \quad r_{1,2} = \frac{R_x(t, t')}{\sigma_{x_1}\sigma_{x_2}}.$$

The conditional distribution

$$f(x_2 \mid x_1) = f(x_1, x_2)(f_1(x_1))^{-1}, \quad \text{where} \quad f_1(x_1) = \int_{-\infty}^{+\infty} f(x_1, x_2)\, dx_2.$$

This distribution is also normal with characteristics

$$m_{x_2 \mid x_1} = m_2 + r_{1,2}\frac{\sigma_{x_2}}{\sigma_{x_1}}(x_1 - m_1), \quad \sigma_{x_2 \mid x_1} = \sigma_{x_2}\sqrt{1 - r_{1,2}^2}.$$

Hence

$$p_{\alpha\beta}=\int_{\alpha}^{\beta} f(x_2 \mid x_1=x)\,dx_2=\Phi\left(\frac{\beta-\left[m_2+r_{1,2}\frac{\sigma_{x_2}}{\sigma_{x_1}}(x-m_1)\right]}{\sigma_{x_2}\sqrt{1-r_{1,2}^2}}\right)$$

$$-\Phi\left(\frac{\alpha-\left[m_2+r_{1,2}\frac{\sigma_{x_2}}{\sigma_{x_1}}(x-m_1)\right]}{\sigma_{x_2}\sqrt{1-r_{1,2}^2}}\right),$$

where $\Phi(x)$ is the error function.

9.8. Find the univariate and bivariate distributions and the characteristics of a random function $X(t)$ defined by its canonical decomposition

$$X(t)=m_x(t)+\sum_{i=1}^{n} V_i\varphi_i(t),$$

where V_i $(i=1, 2, \ldots, n)$ are crossuncorrelated normally distributed random variables with characteristics $m_i=0$, Var_i.

Solution. The univariate distribution $f(x_1, t)$ is normal with characteristics

$$m_{x_1}(t)=m_x(t), \quad \mathrm{Var}_{x_1}(t)=\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi_i^2(t).$$

The correlation function

$$R_x(t, t')=\mathrm{M}\left[\sum_{i=1}^{n} V_i\varphi_i(t)\sum_{j=1}^{n} V_j\varphi_j(t')\right]=\sum_{i=1}^{n}\mathrm{Var}_i\,\varphi_i(t)\,\varphi_i(t').$$

The bivariate distribution $f(x_1, x_2, t, t')$ is normal with characteristics $m_{x_1}(t)$, $m_{x_2}(t')$, $\mathrm{Var}_{x_1}(t)$, $\mathrm{Var}_{x_2}(t')$, $R_x(t, t')$. The random function $X(t)$ is normal and, therefore, the bivariate distribution is an exhaustive characteristic for any number of sections of the function.

9.9. Given a random function

$$X(t)=V_1e^{-\alpha_1 t}+V_2e^{-\alpha_2 t},$$

where V_1 and V_2 are uncorrelated random variables with characteristics $m_{v_1}=m_{v_2}=0$, Var_{v_1}, Var_{v_2}, find the characteristics of the random function $X(t)$.

Solution. The random function $X(t)$ is given as a canonical decomposition without a free term and, consequently, $m_x(t)=0$,

$$R_x(t, t')=\mathrm{Var}_{v_1}e^{-\alpha_1(t+t')}+\mathrm{Var}_{v_2}e^{-\alpha_2(t+t')},$$

$$\mathrm{Var}_x(t)=\mathrm{Var}_{v_1}e^{-\alpha_1 2t}+\mathrm{Var}_{v_2}e^{-\alpha_2 2t}.$$

9.10. A random function $X(t)$ is defined by its canonical decomposition

$$X(t)=\sum_{i=1}^{n} V_ie^{-\alpha_i t}+a,$$

where V_i are centred random variables with variances Var_{v_i} $(i = 1, 2, \ldots, n)$: $\mathrm{M}[V_i, V_j] = 0$ for $i \neq j$; a is a nonrandom variable. Find the characteristics of the random function $X(t)$.

Answer.

$$m_x(t) = a, \quad R_x(t, t') = \sum_{i=1}^{n} \mathrm{Var}_{v_i} e^{-\alpha_i(t+t')},$$

$$\mathrm{Var}_x(t) = \sum_{i=1}^{n} \mathrm{Var}_{v_i} e^{-2\alpha_i t}.$$

9.11. A random function $X(t)$ is defined by a canonical decomposition

$$X(t) = t + V_1 \cos \omega t + V_2 \sin \omega t,$$

where V_1 and V_2 are uncorrelated random variables with zero mean values and variances $\mathrm{Var}_1 = \mathrm{Var}_2 = 2$. Is the random variable $X(t)$ stationary?

Solution. $m_x(t) = t$, $R_x(t, t') = 2(\cos \omega t \cos \omega t' + \sin \omega t \sin \omega t') = 2 \cos \omega (t - t')$.

The correlation function of the random function $X(t)$ satisfies the condition of stationarity but the mean value $m_x(t)$ depends on time. The random function $X(t)$ is nonstationary but the centred random function $\overset{\circ}{X}(t)$ is stationary.

9.12. Suppose we have two random functions:

$$X(t) = V_1 \cos \omega_1 t + V_2 \sin \omega_1 t \text{ and } Y(t) = U_1 \cos \omega_2 t + U_2 \sin \omega_2 t.$$

The mean values of all the random variables V_1, V_2, U_1 and U_2 are zero, the variances are $\mathrm{Var}_{v_1} = \mathrm{Var}_{v_2} = 1$, $\mathrm{Var}_{u_1} = \mathrm{Var}_{u_2} = 4$. The normed correlation matrix of the system (V_1, V_2, U_1, U_2) has the form

$$\left\| \begin{matrix} 1 & 0 & 0.5 & 0 \\ & 1 & 0 & -0.5 \\ & & 1 & 0 \\ & & & 1 \end{matrix} \right\|.$$

Find the crosscorrelation function $R_{xy}(t, t')$ and the value of the function for $t = 0$, $t' = 1$.

Solution.

$$\begin{aligned} R_{xy}(t, t') &= \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{Y}(t')] \\ &= \mathrm{M}[(V_1 \cos \omega_1 t + V_2 \sin \omega_1 t)(U_1 \cos \omega_2 t' + U_2 \sin \omega_2 t')] \\ &= \cos \omega_1 t \cos \omega_2 t' \mathrm{M}[V_1 U_1] + \cos \omega_1 t \sin \omega_2 t' \mathrm{M}[V_1 U_2] \\ &+ \sin \omega_1 t \cos \omega_2 t' \mathrm{M}[V_2 U_1] + \sin \omega_1 t \sin \omega_2 t' \mathrm{M}[V_2 U_2] \\ &= \cos \omega_1 t \cos \omega_2 t' - \sin \omega_1 t \sin \omega_2 t' = \cos(\omega_1 t + \omega_2 t'), \end{aligned}$$

$$R_{xy}(0, 1) = \cos \omega_2, \quad R_{yx}(0, 1) = \cos \omega_1,$$

$$R_{yx}(t, t') = R_{xy}(t', t) = \cos(\omega_1 t' + \omega_2 t).$$

9.13. *A Poisson process.* We consider an elementary flow of events with intensity λ on the t-axis (points on the t-axis, see Fig. 9.13*a*) and a related function $X(t)$, which is the number of events occurring during the time $(0, t)$. At the moment an event occurs the random function

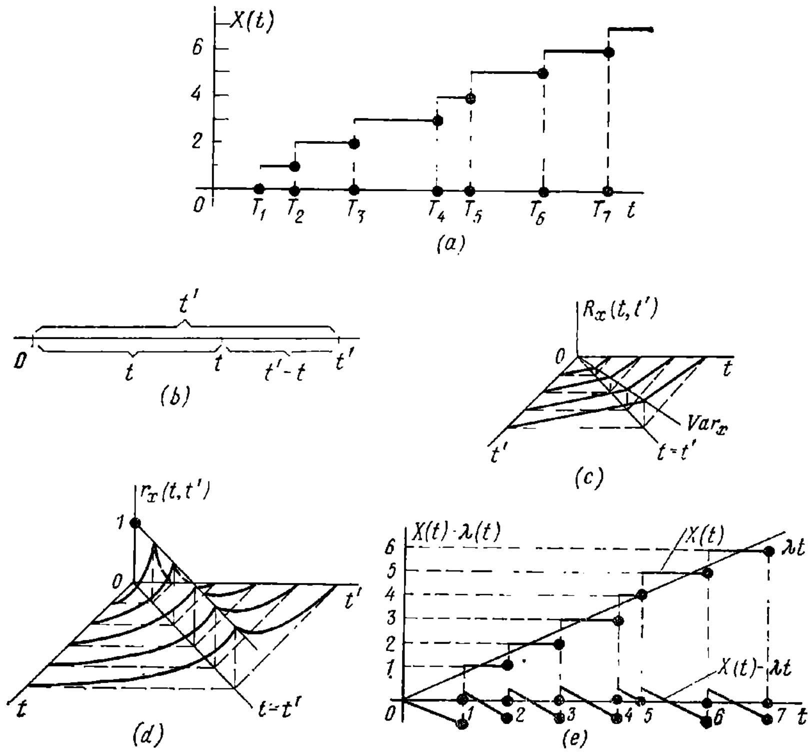


Fig. 9.13

$X(t)$ makes a jump, thereby its value increases by unity (Fig. 9.13*a*)*). The random function $X(t)$ is known as a *Poisson process.*

Find the univariate distribution of the Poisson process and its characteristics $m_x(t)$, $\mathrm{Var}_x(t)$, $R_x(t, t')$, $r_x(t, t')$ as well as the characteristics of the stochastic process $Z(t) = X(t) - \lambda t$.

Solution. The distribution of the section of $X(t)$ is a Poisson distribution with parameter $a = \lambda t$ and, therefore, the probability that the random variable $X(t)$ will assume the value m is expressed by the formula $P_m = (\lambda t)^m (m!)^{-1} e^{-\lambda t}$ $(m = 0, 1, 2, \ldots)$. The mean value and variance of the random function $X(t)$: $m_x(t) = \mathrm{Var}_x(t) = \lambda t$.

Let us find the correlation function $R_x(t, t')$. Assume that $t' > t$ and consider the time interval $(0, t')$ (see Fig. 9.13*b*). We partition the interval into two portions: from 0 to t and from t to t'. The number of

*) We assume that the function $X(t)$ is continuous on the left.

events over the whole interval $(0, t')$ is equal to the sum of the numbers of events on the intervals $(0, t)$ and (t, t')*): $X(t') = X(t) + Y(t' - t)$, where $Y(t' - t)$ is the number of events which occur in the interval (t, t'). Since the flow is stationary, the random function $Y(t' - t)$ has the same distribution as $X(t)$. In addition, according to the properties of a Poisson flow of events, the random variables $X(t)$ and $Y(t' - t)$ are uncorrelated.

We have

$$R_x(t, t') = \mathrm{M}[\mathring{X}(t)\,\mathring{X}(t')] = \mathrm{M}[\mathring{X}(t)(\mathring{X}(t) + \mathring{Y}(t' - t))]$$
$$= \mathrm{M}[(\mathring{X}(t))^2] = \mathrm{Var}_x(t) = \lambda t.$$

Similarly, for $t > t'$ we get $R_x(t, t') = \lambda t'$. Thus $R_x(t, t') = \min\{\lambda t, \lambda t'\} = \lambda \min\{t, t'\}$, where $\min\{t, t'\}$ is the smaller of the values t, t' (if $t = t'$ we can take either t or t' as the minimal value).

Using the notation of a unit function $\mathbf{1}(x)$ (see Appendix 6), we can write the correlation function as

$$R_x(t, t') = \lambda t\,\mathbf{1}(t' - t) + \lambda t'\,\mathbf{1}(t - t').$$

Fig. 9.13*c* illustrates the surface $R_x(t, t')$. In the quadrant $t > 0$ and $t' > 0$ the surface $R_x(t, t')$ consists of two planes which pass through the t- and t'- axes respectively and intersect along the line 0-Var_x, the applicates of whose points are equal to the variance: $\mathrm{Var}_x = \lambda t$.

The normed correlation function

$$r_x(t, t') = \frac{R_x(t, t')}{\sqrt{\mathrm{Var}_x(t)\,\mathrm{Var}_x(t')}} = \sqrt{\frac{t}{t'}}\,\mathbf{1}(t' - t) + \sqrt{\frac{t'}{t}}\,\mathbf{1}(t - t').$$

The surface $r_x(t, t')$ is shown in Fig. 9.13*d*.

A Poisson process is a process with independent increments since its increment at any interval is the number of events occurring on that interval, and for an elementary flow the numbers of events falling on nonoverlapping intervals are independent.

The process $Z(t) = X(t) - \lambda t$ (see Fig. 9.13*e*) results from a nonhomogeneous linear transformation of the stochastic process $X(t)$. Consequently $m_z(t) = m_x(t) - \lambda t = 0$, $\mathrm{Var}_z(t) = \mathrm{Var}_x(t) = \lambda t$, $R_z(t, t') = R_x(t, t') = \lambda \min(t, t')$ since the corresponding homogeneous linear transformation does not change the correlation function of the process $X(t)$.

9.14. A stochastic process $X(t)$ occurs as follows. There is a stationary (elementary) Poisson flow of events with intensity λ on the t-axis. The random function $X(t)$ alternately assumes values $+1$ and -1. Upon the occurrence of an event, it changes its value jumpwise from $+1$ to -1 or vice versa (Fig. 9.14*a*). Initially the random function $X(t)$ may be either $+1$ with probability $1/2$ or -1 with the same prob-

*) We neglect the possibility that an event might occur exactly at the moment t since the probability of this occurrence is zero.

ability. Find the characteristics $m_x(t)$, $\text{Var}_x(t)$ and $R_x(t, t')$ of the random function $X(t)$.

Solution. The section of the random function $X(t)$ has a distribution represented by the ordered series

$$X(t) = \left| \begin{array}{c|c} -1 & +1 \\ \hline 0.5 & 0.5 \end{array} \right| .$$

Indeed, since the moments of sign changes are not connected with the value of the random function, there is no reason for either $+1$ or -1 to be more probable. Hence $m_x(t) = -0.5 + 0.5 = 0$, $\text{Var}_x(t) = (-1)^2/2 + 1^2/2 = 1$.

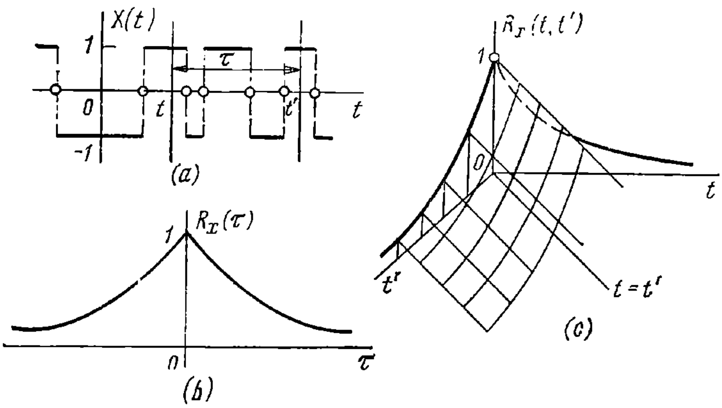


Fig. 9.14

To find the correlation function $R_x(t, t')$, we consider two arbitrary sections of the random function, $X(t)$ and $X(t')$ $(t' > t)$, and find the mean value of their product:

$$R_x(t, t') = \text{M}[\mathring{X}(t)\,\mathring{X}(t')] = \text{M}[X(t)\,X(t')].$$

The product $X(t)\,X(t')$ is equal to -1 if an odd number of events (sign changes) occurred between the points t and t', and to $+1$ if an even number of sign changes (zero inclusive) occurred. The probability that an even number of sign changes will occurr during the time $\tau = t' - t$ is

$$p_{\text{even}} = \sum_{m=1}^{\infty} \frac{(\lambda\tau)^{2m}}{(2m)!} e^{-\lambda\tau} = e^{-\lambda\tau} \frac{e^{\lambda\tau} + e^{-\lambda\tau}}{2},$$

similarly, the probability that an odd number of sign changes will occur during the time τ is

$$p_{\text{odd}} = e^{-\lambda\tau} \frac{e^{\lambda\tau} - e^{-\lambda\tau}}{2}.$$

Hence

$$R_x(t, t') = (+1)\, p_{\text{even}} + (-1)\, p_{\text{odd}} = e^{-2\lambda\tau},$$

where $\tau = t' - t$. Similarly, for $t' < t$

$$R_x(t', t) = e^{-2\lambda(-\tau)}, \quad \text{where} \quad \tau = t' - t.$$

Combining these formulas, we get $R_x(t, t') = R_x(\tau) = e^{-2\lambda|\tau|}$. The graph of this function is shown in Fig. 9.14*b*. The surface $R_x(t, t') = e^{-2\lambda|t'-t|}$ is shown in Fig. 9.14*c*.

The random function $X(t)$ is stationary. Its spectral density

$$S_x^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{\pi} \frac{2\lambda}{(2\lambda)^2 + \omega^2}.$$

9.15. There is an elementary (stationary Poisson) flow of events with intensity λ on the t-axis. A stochastic process $X(t)$ occurs as follows: when the ith event occurs in the flow ($i = 1, 2, \ldots$), it assumes a random value V_i and retains that value till the next event occurs in the flow (Fig. 9.15). At the initial moment $X(0) = V_0$. The random variables $V_0, V_1, V_2, \ldots, V_i, \ldots$ are independent and have the same distribution with density $\varphi(x)$. Find the characteristics of the process, viz. $m_x(t)$, $\text{Var}_x(t)$, $R_x(t, t')$. Is the process stationary?

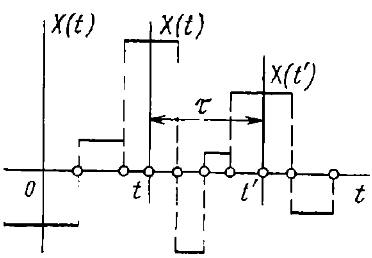


Fig. 9.15

Solution. Any section of the random function $X(t)$ has a distribution $\varphi(x)$; hence

$$m_x(t) = \mathrm{M}[V_i] = \int_{-\infty}^{\infty} x\varphi(x)\, dx = m_v,$$

$$\text{Var}_x(t) = \text{Var}_v = \int_{-\infty}^{\infty} (x - m_v)^2 \varphi(x)\, dx.$$

We find the correlation function $R_x(t, t')$ by the same technique we used in Problem 9.14. Let us consider two sections $X(t)$ and $X(t')$ $(t' > t)$ divided by the interval $\tau = t' - t$. We have

$$R_x(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\, \overset{\circ}{X}(t')].$$

If no events occurred between the points t and t', then $\overset{\circ}{X}(t) = \overset{\circ}{X}(t')$ and $R_x(t, t') = \mathrm{M}[(\overset{\circ}{X}(t))^2] = \text{Var}_x(t) = \text{Var}_v$. If at least one event occurred between the points t and t', then $\mathrm{M}[\overset{\circ}{X}(t)\, \overset{\circ}{X}(t')] = 0$. Hence

$$R_x(t, t') = e^{-\lambda\tau}\, \text{Var}_v + [1 - e^{-\lambda\tau}] \cdot 0 = \text{Var}_v\, e^{-\lambda\tau}.$$

Similarly, for $t' < t$ we have

$$R_x(t, t') = \mathrm{Var}_v \, e^{-\lambda(-\tau)},$$

whence it can be seen that the process is stationary. Its correlation function $R_x(\tau) = \mathrm{Var}_v \, e^{-\lambda|\tau|}$ does not depend on the form of distribution $\varphi(x)$ and only depends on its variance Var_v.

9.16. *A process with independent sections.* We consider the stochastic process $X(t)$, described in the preceding problem, in the limiting case when the intensity λ of the flow of events tends to infinity. Investigate the behaviour of the characteristics of the process as $\lambda \to \infty$.

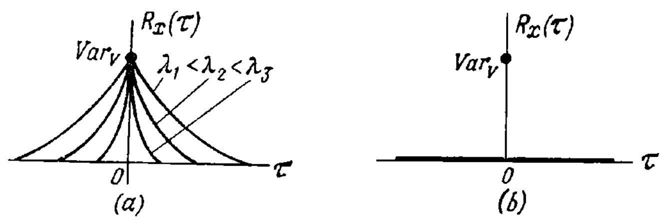


Fig. 9.16

Solution. When λ increases, the correlation function $R_x(\tau) = \mathrm{Var}_v \, e^{-\lambda|\tau|}$ converges to the origin (Fig. 9.16*a*). In the limit we get a correlation function of the form

$$\tilde{R}_x(\tau) = \lim_{\lambda\to\infty} R_x(\tau) = \lim_{\lambda\to\infty} \mathrm{Var}_v \, e^{-\lambda|\tau|} = \mathrm{Var}_v \lim_{\lambda\to\infty} e^{-\lambda|\tau|},$$

i.e. a function which is equal to zero everywhere except for $\tau = 0$, and equal to Var_v for $\tau = 0$ (Fig. 9.16*b*). We can write this as

$$\tilde{R}_x(\tau) = \begin{cases} \mathrm{Var}_v & \text{for } \tau = 0, \\ 0 & \text{for } \tau \neq 0. \end{cases}$$

Since by the hypothesis of Problem 9.15 all the sections of the stochastic process $X(t)$ are independent, we have obtained a model of a stochastic process for which *any two arbitrarily close sections are independent*. We can call this "a process with independent sections". A stochastic process with independent sections $X(t)$ we have in the limiting case when $\lambda \to \infty$ does not possess any points of discontinuity.

9.17. *White noise.* Let us consider a limiting case for the stochastic process $X(t)$, presented in Problem 9.15, under the condition that the intensity of the flow λ tends to infinity and at the same time the variance Var_v of each section tends to infinity with $\mathrm{Var}_v/\lambda = c = \mathrm{const}$. Find the characteristics of this resulting stochastic process $Z(t)$ and show that the random function $Z(t)$ is stationary white noise.

Solution. We investigated the behaviour of the stochastic process $X(t)$ as $\lambda \to \infty$ in Problem 9.16; we shall now take into account the additional conditions $\mathrm{Var}_v \to \infty$, $\mathrm{Var}_v/\lambda = c$. Let us consider the

spectral density $S_x(\omega)$ for the stochastic process $X(t)$ of Problem 9.15:

$$S_x(\omega)=\frac{\mathrm{Var}_v}{\pi}\frac{\lambda}{\lambda^2+\omega^2}=\frac{\mathrm{Var}_v}{\pi\lambda}\frac{\lambda^2}{\lambda^2+\omega^2}$$

From the condition $\mathrm{Var}_v/\lambda=c$ we have $\mathrm{Var}_v/(\pi\lambda)=c/\pi=a=\text{const}$. The correlation function $R_x(\tau)=a\pi\lambda e^{-\lambda|\tau|}$.

For the stochastic process $Z(t)$, which results as $\lambda\to\infty$, $\mathrm{Var}_v\to\infty$. $\mathrm{Var}_v/\lambda=c$, we get

$$R_z(\lambda)=\lim_{\lambda\to\infty}R_x(\lambda)=\lim_{\lambda\to\infty}a\pi\lambda e^{-\lambda|\tau|}=a\pi\lim_{\lambda\to\infty}\lambda e^{-\lambda|\tau|}=a\pi\delta(\tau),$$

where $\delta(\tau)$ is the delta-function (see Appendix 6);

$$S_z(\omega)=\lim_{\lambda\to\infty}\frac{a\lambda^2}{\lambda^2+\omega^2}=a$$

We have thus verified that the stochastic process $Z(t)$ is stationary white noise and constructed a model for its appearance. White noise can be regarded as a limiting case of a sequence of short independent similarly distributed pulses with large variance. In practice we encounter these processes when we consider natural interference such as "thermal" noise in electronic devices, shot effect, and so on.

9.18. We consider the stochastic process $X(t)$, described in Problem 9.15, under the condition that the distribution of each random variable V_i ($i=0, 1, 2, \ldots$) is normal with mean value $m_v=0$ and variance Var_v. Find the characteristics $m_x(t)$, $\mathrm{Var}_x(t)$ and $R_x(t, t')$ of the stochastic process $X(t)$. Is the process stationary? Is it normal?

Solution. Any section of the random function $X(t)$ is normally distributed with parameters m_v, $\sigma_v=\sqrt{\mathrm{Var}_v}$. It follows from the solution of Problem 9.15 that the correlation function $R_x(\tau)=\mathrm{Var}_v\,e^{-\lambda|\tau|}$. The process is stationary. But is it normal? No, it is not, despite the fact that the univariate distribution is normal. The joint distribution of two sections of the stochastic process $X(t)$ is no longer normal since the sections coincide with nonzero probability, which cannot be the case if two random variables have a joint normal distribution.

9.19. *A model of a flux of electrons in a radio valve.* A flux of electrons directed from the cathode to the anode of a radio valve is an elementary flow with intensity λ. When an electron is absorbed by the anode, the voltage of the latter increases jumpwise by unity and then begins to decrease exponentially with parameter α which depends on the characteristics of the electronic circuit (Fig. 9.19*a*). The increment of the voltage, which is due to the arrival of an electron, is summed up with the residual anode voltage. Find the characteristics of the stochastic process $X(t)$, i.e. the anode voltage.

Solution. Electrons arrive at the anode at random moments $T_1, T_2, \ldots, T_i, \ldots$, which form an elementary flow of events. The voltage at moment t, which is due to the action of the ith electron

arriving at the moment T_i, is

$$W_i(t) = \begin{cases} 0 & \text{for} \quad t < T_i, \\ e^{-\alpha(t-T_i)} & \text{for} \quad t \geq T_i \end{cases} = 1\,(t - T_i)\, e^{-\alpha(t-T_i)},$$

where 1 (t) is a unit function; $T_i > 0$, $t > 0$. Let us consider a random variable Y, which is the number of electrons arriving at the anode at time t. This variable has a Poisson distribution with

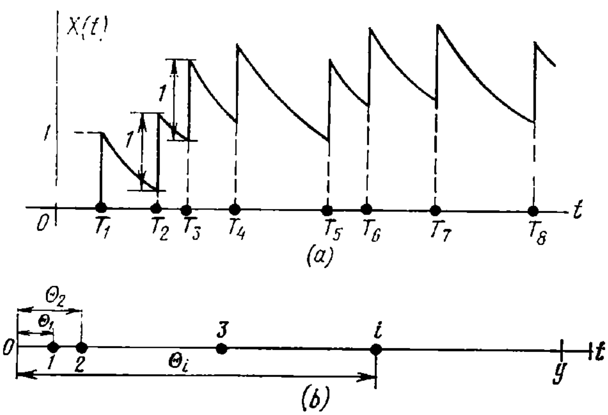


Fig. 9.19

parameter λt. We represent the voltage $X\,(t)$ as the sum of a random number of random terms:

$$X(t) = \sum_{i=1}^{Y} e^{-\alpha(t-T_i)} 1\,(t - T_i). \tag{9.19.1}$$

We have shown somewhat earlier (see Problem 8.80) that a Poisson flow of events on the interval $(0, t)$ can be represented, with sufficient accuracy, as a collection of points on that interval, the coordinate of each of which $\Theta_i \in (0, t)$ is uniformly distributed on that interval (see Fig. 9.19*b*) and does not depend on the coordinates of the other points. Consequently, expression (9.19.1) can be rewritten in the form

$$X(t) = \sum_{i=1}^{Y} e^{-\alpha(t-\Theta_i)}, \tag{9.19.2}$$

where the random variables Θ_i are independent and uniformly distributed in the interval $(0, t)$.

We designate $X_i(t) = e^{-\alpha(t-\Theta_i)} = e^{-\alpha t} e^{\alpha \Theta_i}$, and then

$$X(t) = \sum_{i=1}^{Y} X_i(t) = e^{-\alpha t} \sum_{i=1}^{Y} e^{\alpha \Theta_i}, \tag{9.19.3}$$

where $X_i\,(t)$ are independent similarly distributed random variables, and the random variable Y does not depend on the random variables

$X_i(t)$ either. In accordance with the solution of Problem 7.64, we write the expressions for $m_x(t)$ and $\mathrm{Var}_x(t)$:

$$m_x(t) = m_y(t)\, m_{x_i}(t), \tag{9.19.4}$$

$$\mathrm{Var}_x(t) = m_y(t)\,\mathrm{Var}_{x_i}(t) + \mathrm{Var}_y(t)\, m_{x_i}^2(t). \tag{9.19.5}$$

Since the random variable Y has a Poisson distribution with parameter λt, it follows that $m_y(t) = \mathrm{Var}_y(t) = \lambda t$. Let us find $m_{x_i}(t)$:

$$m_{x_i}(t) = \mathrm{M}[X_i(t)] = \frac{1}{t}\int_0^t e^{-\alpha(t-x)}\,dx = \frac{1-e^{-\alpha t}}{\alpha t}.$$

Next we determine the second moment about the origin of the random variable $X_i(t)$:

$$\mathrm{M}[X_i^2(t)] = \frac{1}{t}\int_0^t [e^{-\alpha(t-x)}]^2\,dx = \frac{1-e^{-2\alpha t}}{2\alpha t}.$$

Consequently,

$$m_x(t) = \lambda\,\frac{1-e^{-\alpha t}}{\alpha}, \tag{9.19.6}$$

$$\mathrm{Var}_x(t) = \lambda t\,[\mathrm{Var}_{x_i}(t) + m_{x_i}^2(t)] = \lambda t\mathrm{M}[X_i^2(t)] = \lambda\frac{1-e^{-2\alpha t}}{2\alpha}. \tag{9.19.7}$$

Note that as $t \to \infty$, the mean value and variance of the process $X(t)$ do not depend on time: $\lim_{t\to\infty} m_x(t) = m_x = \lambda/\alpha$; $\lim_{t\to\infty}\mathrm{Var}_x(t) = \mathrm{Var}_x = \lambda/(2\alpha)$.

To find the distribution of the section of the stochastic process $X_i(t)$ for $m_x = \lambda/\alpha > 20$, we reason as follows. Considering a finite but sufficiently large interval $(0, t)$ and assuming that a sufficiently large number Y of electron emissions take place on that interval, we see that the process $X(t)$ [see formula (9.19.2)] is a sum of independent similarly distributed random variables, which has an approximately normal distribution since in this case the conditions of the central limit theorem are in fact fulfilled (see Chapter 8). Consequently, the section of the stochastic process has a normal distribution with characteristics $m_x = \lambda/\alpha$ and $\mathrm{Var}_x = \lambda/(2\alpha)$. The process will practically become stationary in the time $\tau_{st} = 3/\alpha$.

To find the correlation functions, let us consider two sections of the stochastic process in question at moments t and t' $(t' > t)$. By virtue of the assumptions made, we can assert that the anode voltage $X(t')$ of the valve at the moment t' is equal to the voltage $X(t)$ at the moment t multiplied by the exponent $e^{-\alpha(t'-t)}$ plus the voltage $Y(t'-t)$, which results from the arrival of electrons at the anode in the time interval (t, t'):

$$X(t') = X(t)\,e^{-\lambda(t'-t)} + Y(t'-t). \tag{9.19.8}$$

The stochastic processes $X(t)$ and $Y(t'-t)$ are evidently independent since they are generated by electrons arriving at the anode during different, nonoverlapping time intervals $(0, t)$ and (t, t') respectively. The same can be said of centred stochastic processes $\overset{\circ}{X}(t)$ and $\overset{\circ}{Y}(t'-t)$. Consequently,

$$R_x(t, t') = \mathrm{M}\,[\overset{\circ}{X}(t)\{\overset{\circ}{X}(t)\,\mathrm{e}^{-\alpha(t'-t)} + \overset{\circ}{Y}(t'-t)\}]$$

$$= \mathrm{M}\,[(\overset{\circ}{X}(t))^2]\,\mathrm{e}^{-\alpha(t'-t)} \quad \text{for} \quad t' > t,$$

$$R_x(t, t') = \mathrm{M}\,[(\overset{\circ}{X}(t'))^2]\,\mathrm{e}^{-\alpha(t-t')} \quad \text{for} \quad t > t'.$$

Combining the last two expressions, we obtain

$$R_x(t, t') = \mathrm{Var}_x(\min(t, t'))\,[1 - \mathrm{e}^{2\alpha \min(t, t')}]\,\mathrm{e}^{-\alpha|t'-t|}.$$

Let us consider the limiting behaviour of the stochastic process when $t \to \infty$, $t' \to \infty$, but the value of their difference $\tau = t' - t$ is finite. In this case

$$R_x(\tau) = \mathrm{Var}_x\,\mathrm{e}^{-\alpha|\tau|} = \frac{\lambda}{2\alpha}\,\mathrm{e}^{-\alpha|\tau|}.$$

Thus the stochastic process $X(t)$ we consider in this problem is stationary and practically normal for t tending to infinity and $\lambda/\alpha > 20$.

This problem is a more general case of Problem 9.13. Indeed, as $\alpha \to 0$, the anode voltage of the valve is a Poisson process since with the arrival of each new electron the voltage only increases by unity and does not decrease with time. Consequently, the following equalities must hold for any finite t and t':

$$\lim_{\alpha\to 0} m_x(t) = \lim_{\alpha\to 0} \lambda\,\frac{1-\mathrm{e}^{-\alpha t}}{\alpha} = \lim_{\alpha\to 0} \mathrm{Var}_x(t) = \lim_{\alpha\to 0} \lambda\,\frac{1-\mathrm{e}^{-2\alpha t}}{2\alpha} = \lambda t,$$

$$\lim_{\alpha\to 0} R_x(t, t') = \lim_{\alpha\to 0} \frac{\lambda}{2\alpha}\,[1 - \mathrm{e}^{-2\alpha \min(t, t')}]\,\mathrm{e}^{-\alpha|t'-t|} = \lambda \min(t, t').$$

The reader is invited to verify their validity.

9.20. *The functioning of a linear detector.* Under the conditions of Problem 9.19, we assume that electrons arrive at the anode in "bursts", the moments of arrival of the bursts forming an elementary flow with intensity λ. The number of electrons in the ith burst is a random variable W_i which is independent of the number of electrons in the other bursts and has a distribution $F(w)$ with characteristics m_w, Var_w. This problem is equivalent to that of defining the output voltage of a linear detector, when positive pulses of the random variable (of the highest voltage) W_i arrive at its input at random moments defined by the Poisson flow, and in the intervals between the pulses the voltage drops exponentially (Fig. 9.20). Find the characteristics of the process.

Solution. We can represent this process by a formula analogous to (9.19.2):

$$X(t) = \sum_{i=1}^{Y} W_i \mathrm{e}^{-\alpha(t-\Theta_i)}, \tag{9.20}$$

where the random variables Y, W_i, Θ_i are mutually independent.

We designate $X_i(t) = W_i e^{-\lambda(t-\Theta_i)}$, and then

$$M[X_i(t)] = m_w \frac{1-e^{-\alpha t}}{\alpha t}, \quad M[X_i^2(t)] = (\mathrm{Var}_w + m_w^2)\frac{1-e^{-2\alpha t}}{2\alpha t}.$$

Consequently,

$$m_x(t) = \lambda m_w \frac{1-e^{-\alpha t}}{\alpha}, \quad \mathrm{Var}_x(t) = \lambda(\mathrm{Var}_w + m_w^2)\frac{1-e^{-2\alpha t}}{2\alpha}.$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} m_x(t) = m_x = \frac{\lambda m_w}{\alpha}, \quad \lim_{t\to\infty} \mathrm{Var}_x(t) = \mathrm{Var}_x = \frac{\lambda(\mathrm{Var}_w + m_w^2)}{2\alpha},$$

$$R_x(\tau) = \mathrm{Var}_x e^{-\alpha|\tau|}.$$

Let us consider the limiting behaviour of the process $X(t)$ when the following quantities increase indefinitely: the intensity of the Poisson flow generating the pulses ($\lambda \to \infty$); the variance of the amplitude of each pulse ($\mathrm{Var}_w \to \infty$) and the parameter α ($\alpha \to \infty$). An infinite increase of the quantity α signifies that the pulse voltage drops rapidly, i.e. in the limit, as $\alpha \to \infty$, the area of the pulse tends to zero, the speeds at which the quantities λ and Var_w increase being proportional to the speed of increase of the quantity α: $\lambda = k_1\alpha$, $\mathrm{Var}_w = k_2\alpha$. We obtain (as $t \to \infty$):

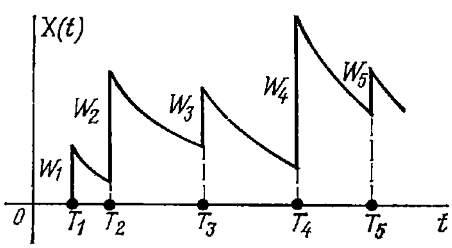


Fig. 9.20

$$\lim_{\lambda,\,\alpha\to\infty} m_x = \lim_{\lambda,\,\alpha\to\infty} \frac{\lambda}{\alpha} m_w = k_1 m_w,$$

$$\lim_{\lambda,\,\alpha,\,\mathrm{Var}_w\to\infty} \mathrm{Var}_x = \lim_{\lambda,\,\alpha,\,\mathrm{Var}_w\to\infty} \frac{\lambda(\mathrm{Var}_w + m_w^2)}{2\alpha} \to \infty,$$

$$\lim_{\lambda,\,\alpha,\,\mathrm{Var}_w\to\infty} R_x(\tau) = \lim_{\lambda,\,\alpha,\,\mathrm{Var}_w\to\infty} \frac{k_1k_2}{2}\alpha e^{-\alpha|\tau|} = k_1k_2\delta(\tau),$$

where $\delta(\tau)$ is the delta-function.

Thus, in the limit we have white noise, which results from an infinitely frequent series of pulses which have a finite expectation of the amplitude of a pulse and an infinite variance of that amplitude, as well as an infinitesimal duration of the pulse itself.

9.21. *The shot effect.* Let us consider a stochastic process $X(t)$ generated by a Poisson process like the case in Problem 9.13. As the ith event of the Poisson flow occurs at a moment T_i, a nonnegative voltage pulse W_i arises in the electric circuit, whose random value (amplitude) then varies according to one and the same law $\varphi(\eta)$, where the variable

η is reckoned from the moment T_i (Fig. 9.21) ($\varphi(0) \geqslant \varphi(\eta)$). The random variables W_i are mutually independent and have the same distribution function $F(w)$. The voltage in the circuit is the total effect of all the pulses with due regard for their variation in time.

The stochastic process being considered is known as the shot effect (shot noise). The stochastic processes investigated in problems 9.19 and 9.20 are special cases of the shot effect. We must find the characteristics of shot noise.

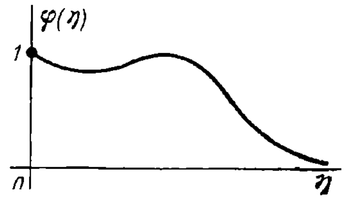


Fig. 9.21

Solution. In accordance with the solution of Problem 9.20 shot noise on the interval $(0, t)$ can be represented, as in (9.20), in the form

$$X(t) = \sum_{i=1}^{Y} W_i \varphi(t - \Theta_i), \quad (9.21)$$

where Y, W_i, Θ_i are mutually independent random variables, and, by analogy with Problem 9.20, the random variable Y has a Poisson distribution with parameter λt and the random variable Θ_i is uniformly distributed in the interval $(0, t)$.

We designate $X_i(t) = W_i \varphi(t - \Theta_i)$ and then $M[X_i(t)] = m_w t^{-1} \Phi(t)$, where $m_w = M[W_i]$, $\Phi(t) = \int_0^t \varphi(t - x)\, dx$ is the area of a pulse (the area bounded by the curve $\varphi(\eta)$ and the coordinate axes):

$$M[X_i^2(t)] = (\mathrm{Var}_w + m_w^2)\, t^{-1} \tilde{\Phi}(t),$$

where $\mathrm{Var}_w = \mathrm{Var}[W_i] = \int_0^t (w - m_w)^2\, dF(w)$, $\tilde{\Phi}(t) = \int_0^t [\varphi(t - x)]^2\, dx$.

Consequently, $m_x(t) = \lambda m_w \Phi(t)$, $\mathrm{Var}_x(t) = \lambda(\mathrm{Var}_w + m_w^2)\tilde{\Phi}(t)$. When $t \to \infty$ and there exist improper integrals,

$$\lim_{t\to\infty} \Phi(t) = \int_0^\infty \varphi(t - x)\, dx = \Phi, \quad \lim_{t\to\infty} \tilde{\Phi}(t) = \int_0^\infty \varphi^2(t - x)\, dx = \tilde{\Phi},$$

$$m_x = \lambda m_w \Phi, \quad \mathrm{Var}_x = \lambda\,(\mathrm{Var}_w + m_w^2)\,\tilde{\Phi}$$

In practice these improper integrals always exist since the area of a pulse with an initial unit amplitude and the area defined by the square of the respective function are finite quantities in practical applications.

Reasoning as we did when deriving formula (9.19.8), we obtain $X(t') = X(t)\,\varphi(t' - t) + Y(t' - t)$ for $t' > t$. Consequently

$$R_x(t, t') = \begin{cases} \mathrm{Var}_x(t)\,\varphi(t' - t) & \text{for} \quad t' > t, \\ \mathrm{Var}_x(t')\,\varphi(t - t') & \text{for} \quad t > t'. \end{cases}$$

Combining these two expressions, we find that

$$R_x(t, t') = \mathrm{Var}_x(\min(t, t'))\,\varphi(|t' - t|).$$

As $t, t' \to \infty$, we get $R_x(\tau) = \mathrm{Var}_x\,\varphi(|\tau|)$ $(\tau = t' - t)$. Thus we see that as $t, t' \to \infty$, the noise effect is a stationary stochastic process.

Let us consider the limiting case when the intensity λ of a flow of pulses tends to infinity, the variance Var_w of the pulse also increases indefinitely, and the area of a pulse with unit amplitude Φ tends to zero, so that the variable $\tilde{\Phi}$ also tends to zero. Then the following conditions are fulfilled:

$$\lim_{\substack{\lambda\to\infty\\ \Phi\to 0}} \lambda m_w \Phi = m, \qquad \lim_{\substack{\lambda,\ \mathrm{Var}_w\to\infty\\ \Phi\to 0}} \lambda(\mathrm{Var}_w + m_w^2)\,\tilde{\Phi} = \mathrm{Var}.$$

In that case shot noise turns into white noise with characteristics $m_x = m$, $R_x(\tau) = \mathrm{Var}\,\delta(\tau)$, where $\delta(\tau)$ is the delta-function.

9.22. *Impulse noise.* Consider shot noise generated by rectangular pulses. The amplitude W_i and the duration $\varkappa_i$ of a pulse arriving at the moment T_i are independent random variables with characteristics m_w, Var_w, $m_\varkappa$ and $\mathrm{Var}_\varkappa$ respectively (Fig. 9.22). Find the mean value and variance of this stochastic process $X(t)$.

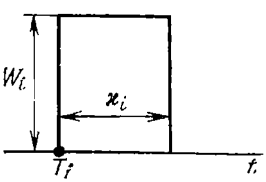


Fig. 9.22

Solution. The impulse noise considered on the interval $(0, t)$ can be represented as

$$X(t) = \sum_{i=1}^{Y} W_i \varphi(t, \Theta_i, \varkappa_i) = \sum_{i=1}^{Y} X_i(t),$$

where $\varphi(t, \Theta_i, \varkappa_i)$ is a pulse which begins at the moment Θ_i and whose height is unity and duration is $\varkappa_i$ (the random variables Θ_i and $\varkappa_i$ are independent): $\varphi(t, \Theta_i, \varkappa_i) = 1(t - \Theta_i)\,1(\Theta_i + \varkappa_i - t)$. The random variable Θ_i is uniformly distributed within the interval $(0, t)$.

Let us find $\mathrm{M}[\varphi(t, \Theta_i, \varkappa_i)]$. We advance a hypothesis that the random variable $\varkappa_i$ assumes the value $\varkappa$. If this hypothesis is satisfied, we find the conditional expectation for sufficiently large values of t:

$$\mathrm{M}_\varkappa[\varphi(t, \Theta_i, \varkappa)] = \frac{1}{t}\int_0^t 1(t-y)\,1(y+\varkappa-t)\,dy.$$

The integrand is a rectangular pulse with unit height and duration $\varkappa$, and the integral of that function is equal to the area of the pulse. Consequently,

$$\mathrm{M}_\varkappa[\varphi(t, \Theta_i, \varkappa)] = \varkappa/t,$$

whence

$$\mathrm{M}[\varphi(t, \Theta_i, \varkappa_i)] = \mathrm{M}[\varkappa_i/t] = m_\varkappa/t.$$

Similarly

$$M_\varkappa [\varphi (t, \Theta_i, \varkappa_i)^2] = \frac{1}{t} \int_0^t [1 (t-y)\, 1 (y+\varkappa-t)]^2\, dy = \frac{\varkappa}{t},$$

$$M [\varphi (t, \Theta_i, \varkappa_i)^2] = \frac{m_\varkappa}{t}.$$

Consequently

$$M [X_i (t)] = M [W_i \varphi (t, \Theta_i, \varkappa_i)] = m_w m_\varkappa / t,$$

$$M [X_i^2 (t)] = M [(W_i \varphi (t, \Theta_i, \varkappa_i))^2] = (\mathrm{Var}_w + m_w^2)\, m_\varkappa / t,$$

$$M [X (t)] = m_x = M [Y]\, M [X_i (t)] = \lambda t m_w m_\varkappa / t = \lambda m_w m_\varkappa, \qquad (9.22.1)$$

$$\mathrm{Var} [X (t)] = \mathrm{Var}_x = M [Y]\, M [X_i^2 (t)] = \lambda (\mathrm{Var}_w + m_w^2)\, m_\varkappa. \qquad (9.22.2)$$

In the limit, when the intensity of an elementary flow and the variance of the amplitude of the pulse Var_w increase indefinitely, and the mean value $m_\varkappa$ and the variance $\mathrm{Var}_\varkappa$ of the duration of the pulse decrease indefinitely, while the quantity $\mathrm{Var}_\varkappa = \lambda (\mathrm{Var}_w + m_w^2)\, m_\varkappa$ remains constant, the impulse noise turns into white noise with characteristics m_x, $R_x (\tau) = \mathrm{Var}_x\, \delta (\tau)$.

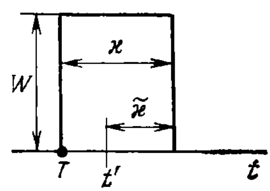


Fig. 9.23

9.23. *Change in the quantity of homogeneous elements.* Consider a process $X (t)$, i.e. the number of homogeneous elements functioning at a moment t. We assume that each element functions for a certain random time $\varkappa$, which has an exponential distribution with parameter μ, similar for all the elements, and then fails ("decays"). The beginning of the functioning (beginning of "life time") of each element is accidental and is defined by an elementary flow with intensity λ. For example, $X (t)$ is the number of computers used, the intensity λ is the number of computers produced per unit time, $\varkappa$ is the random life time of a computer (the average time a computer operates is $1/\mu$). Find the characteristics of the stochastic process $X (t)$.

Solution. The stochastic process $X (t)$ is an impulse noise like the one considered in the preceding problem, the only difference being that $W_i = 1$ for any values of i ($m_w = 1$, $\mathrm{Var}_w = 0$), and the random variable $\varkappa$ has an exponential distribution with parameter μ. Consequently, for sufficiently large t [see formulas (9.22.1), (9.22.2)] $M [X (t)] = \lambda m_w m_\varkappa = \lambda/\mu$; $\mathrm{Var} [X (t)] = \lambda (\mathrm{Var}_w + m_m^2)\, m_w = \lambda\mu$.

We can prove that the univariate distribution of the stochastic process $X (t)$ is a Poisson distribution with the characteristics obtained.

When seeking the correlation function of the stochastic process $X (t)$ being considered, we can use a property of the random variable $\varkappa$, which is the duration of the pulse and has an exponential distribution. The property is that the "remainder" of the pulse duration $\tilde{\varkappa}$ (Fig. 9.23)

reckoned not from its beginning T but from a certain moment t_1 ($T < t_1 < T + \varkappa$) also has an exponential distribution with parameter μ (see problem 5.35).

We advance a hypothesis that the stochastic process $X(t) = n$ ($n = 0, 1, 2, \ldots$). We designate the probability that the hypothesis is true as $P_n(t) = P(X(t) = n)$. Assuming that the hypothesis is true, we can write an expression for the random function $X_n(t')$ ($t' > t$):

$$X_n(t') = \sum_{i=0}^{n} V_i + Y(t'-t),$$

where V_i is a random variable which has an ordered series

$$V_i: \left|\begin{array}{c|c} 0 & 1 \\ \hline 1-e^{-\mu(t'-t)} & e^{-\mu(t'-t)} \end{array}\right| \quad (i \neq 0),$$

$V_0 = 0$, $Y(t'-t)$ is a stochastic process generated by the events occurring in the Poisson flow in the time interval (t, t'). The variable $\sum_{i=0}^{n} V_i$ is the number of elements remaining at the moment t' if they were n in number at the moment t.

Let us find the conditional expectation of the product $X(t)X(t')$ provided that $X(t) = n$:

$$\begin{aligned} M[nX_n(t')] &= nM[X_n(t')] = nM[\sum_{i=1}^{n} V_i + Y(t-t')] \\ &= n\{ne^{-\mu(t'-t)} + M[Y(t'-t)]\} = n^2 e^{-\mu(t'-t)} + n\lambda m_\varkappa, \end{aligned}$$

where $m_\varkappa = 1/\mu$. Consequently, the unconditional expectation of the product $X(t)X(t')$ is

$$\begin{aligned} M[X(t)X(t')] &= \sum_{n=0}^{\infty} M[nX_n(t')]P_n(t) \\ &= \sum_{n=0}^{\infty} n^2 e^{-\mu(t'-t)} P_n(t) + \sum_{n=0}^{\infty} n\lambda m_\varkappa P_n(t) \\ &= M[X^2(t)]e^{-\mu(t'-t)} + m_x^2. \end{aligned}$$

Hence

$$R_x(t, t') - \mathrm{Var}_x e^{-\mu(t'-t)} \quad \text{for} \quad t' > t$$

or

$$R_x(t, t') = \mathrm{Var}_x e^{-\mu(t-t')} \quad \text{for} \quad t > t'.$$

Combining these two formulas, we get ($\tau = t' - t$)

$$R_x(\tau) = \mathrm{Var}_x e^{-\mu|\tau|} = \frac{\lambda}{\mu} e^{-\mu|\tau|}.$$

For $\lambda/\mu > 20$ the process of accumulation of homogeneous elements can be considered to be practically normal with characteristics $m_x = \lambda/\mu$ and $R_x(\tau) = \lambda\mu^{-1}e^{-\mu|\tau|}$.

9.24. *One-dimensional random walk of a particle.* Let us consider, on the x-axis, a particle which changes its position jumpwise (walks) due to random collisions with other particles (Fig. 9.24*a*). At the initial moment the particle is at the origin, at the moment T_1 of the first

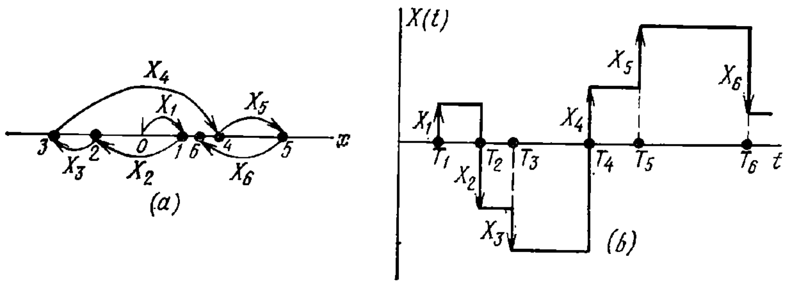


Fig. 9.24

collision it jumps into a point X_1, at the moment T_2 of the second collision it jumps and changes its abscissa by the value X_2, at the moment T_3 it changes its abscissa by the value X_3 and so on. The random variables $X_1, X_2, \ldots, X_i, \ldots$ are independent and have the same distribution with $m_x = 0$ and $\mathrm{Var}_x = \mathrm{Var}$. The moment $T_1, T_2, \ldots, T_i, \ldots$ at which collisions occur form an elementary flow of events with intensity λ.

We consider a stochastic process $X(t)$, the abscissa of a point walking at random as a function of time (Fig. 9.24*b*). (The values of $X(t)$ may be both positive and negative.) The process $X(t)$ is a simplified model of the Brownian movement of a particle. Find the characteristics of the stochastic process $X(t)$.

Solution. The number of events on the interval $(0, t)$ is a random variable Y which has a Poisson distribution with parameter λt. The value of the process $X(t)$ at the moment t is defined by the formula $X(t) = \sum_{i=1}^{Y} X_i$, i.e. is the sum of a random number of random terms; in this case all the random variables X_i and Y are independent. As in Problem 9.20 we have $m_x(t) = \mathrm{M}[Y]\,\mathrm{M}[X_i] = 0$, since $\mathrm{M}[Y] = \lambda t$, $\mathrm{M}[X_i] = m_x = 0$,

$$\mathrm{Var}_x(t) = \mathrm{M}[Y]\,\mathrm{Var}[X_i] + \mathrm{Var}[Y]\,\mathrm{M}[X_i]^2 = \lambda t\,\mathrm{Var} \text{ since } \mathrm{Var}[X_i] = \mathrm{Var}.$$

By analogy with the solution of Problem 9.20, for a sufficiently large t the section of the random function $X(t)$ has a practically normal distribution with the parameters obtained.

We seek the correlation function of the process $X(t)$, for which purpose we consider two sections: $X(t)$ and $X(t')$ $(t' > t)$. It is evident that $X(t') = X(t) + Y(t' - t)$. Here, as in Problem 9.20, the stochastic process $Y(t'-t) = \sum_{i=1}^{Z} X_i$, where Z is the number of events occurring in the flow on the time interval $t'-t$. We have indicated that the stochastic processes $X(t)$, $X(t')$ and $Y(t-t')$ have a zero expectation. Consequently, for $t' > t$ we have

$$\begin{aligned} R_x(t, t') &= \mathrm{M}[X(t)X(t')] = \mathrm{M}[X(t)\{X(t)+Y(t'-t)\}] \\ &= \mathrm{M}[X^2(t)] + \mathrm{M}[X(t)\,Y(t'-t)] \\ &= \mathrm{Var}_x(t) + \mathrm{M}[X(t)]\,\mathrm{M}[Y(t'-t)] = \mathrm{Var}_x(t). \end{aligned}$$

For $t' < t$ we get $R_x(t, t') = \mathrm{Var}_x(t')$. Thus

$$R_x(t, t') = \mathrm{Var}\,\lambda \min(t, t').$$

The stochastic process $X(t)$ we have considered is a process with independent increments.

Remark. A one-dimensional random walk of a particle has the same characteristics as the process $Y(t) = X(t) - \lambda t$, where $X(t)$ is a Poisson process, despite the fact that their realizations differ considerably (compare Fig. 9.13*e* and Fig. 9.24*b*).

9.25. *The Wiener process.* We consider the limiting behaviour of a random process $X(t)$, which is a one-dimensional random walk of a particle (see Problem 9.24) for an infinite increase in the intensity λ of the flow of collisions and simultaneous infinite decrease in the variance Var of the displacement of the particle; in that case the condition $\lambda\,\mathrm{Var} = \alpha = \mathrm{const}$ is satisfied. Show that in this limiting case we have a Wiener process for time intervals sufficiently distant from the origin.

Solution. The condition $m_x(t) = 0$ follows from the hypothesis of the preceding problem. We have also shown there that the increments are independent and the process is normal as $t \to \infty$.

$$\mathrm{Var}[X(t_1) - X(t_2)] = \mathrm{M}[(X(t_1) - X(t_2))^2] \text{ since } m_x = 0,$$

$$\begin{aligned} \mathrm{M}[(X(t_1) - X(t_2))^2] &= \mathrm{M}[X^2(t_1) + X^2(t_2) - 2X(t_1)\,X(t_2)] \\ &= \mathrm{Var}[X(t_1)] + \mathrm{Var}[X(t_2)] - 2R_x(t_1, t_2). \end{aligned}$$

It was shown in the preceding problem that $\mathrm{Var}[X(t)] = \lambda\,\mathrm{Var}\,t$; $R_x(t_1, t_2) = \lambda\,\mathrm{Var}\min\{t_1, t_2\}$. Hence for $t_2 > t_1$

$$\begin{aligned} \mathrm{Var}[X(t_1) - X(t_2)] &= \lambda\,\mathrm{Var}\,t_1 + \lambda\,\mathrm{Var}\,t_2 - 2\lambda\,\mathrm{Var}\min\{t_1, t_2\} \\ &= \lambda\,\mathrm{Var}\,t_1 + \lambda\,\mathrm{Var}\,t_2 - 2\lambda\,\mathrm{Var}\,t_1 = \lambda(t_2 - t_1), \end{aligned}$$

whence it follows that we have a Wiener process.

9.26. A random function $X(t)$ is constructed as follows.

At the point $t = 0$ it assumes at random one of the values $+1$ or -1 with equal probability 1/2 and remains constant until $t = 1$. At the point $t = 1$ it again assumes one of the values $+1$ or -1 with the same probability 1/2 and independently of the value it has assumed on the preceding interval. This value is retained until the next integral-valued point $t = 2$, and so on. In general, the function $X(t)$ is constant on any interval from n to $n + 1$, where n is a natural number, and assumes one of the values $+1$ or -1 with probability 1/2 on the boundary of each new interval independently of the preceding values (one of the

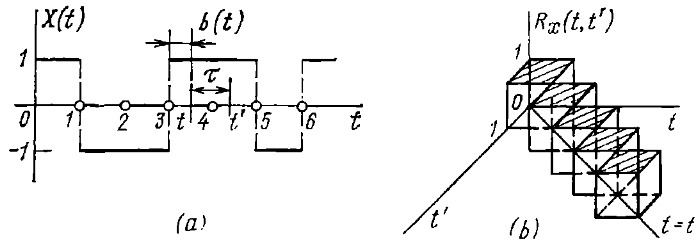


Fig. 9.26

possible realizations of the random function $X(t)$ is shown in Fig. 9.26*a*). Find the characteristics of the random function $X(t)$, viz. the mean value, the variance and the correlation function. Determine whether the random function $X(t)$ is stationary.

Solution.

$$m_x(t) = m_x = (-1)\frac{1}{2} + 1\frac{1}{2} = 0,$$

$$\mathrm{Var}_x(t) = \mathrm{Var}_x = (-1)^2\frac{1}{2} + 1^2\frac{1}{2} = 1.$$

We find the correlation function $R_x(t, t')$. If the points t and t' belong to the same interval $(n, n + 1)$, where n is an integer, then $R_x(t, t') = \mathrm{Var}_x = 1$, otherwise $R_x(t, t') = 0$. We can write this result in a more concise form if we denote the fractional part of the number t by $b(t)$ (see Fig. 9.26*a*). Then we get ($\tau = t' - t$)

$$R_x(t, t') = \begin{cases} 1 & \text{for} \quad |\tau| < 1 - b(\min\{t, t'\}), \\ 0 & \text{for} \quad |\tau| > 1 - b(\min\{t, t'\}), \end{cases} \quad \text{where} \quad \tau = t' - t.$$

This function depends not only on $\tau = t' - t$ but also on where the interval (t, t') is on the t-axis; consequently, the random function $X(t)$ is nonstationary.

The surface $R_x(t, t')$ looks like a number of cubes with the edge equal to unity which are placed on the t, t'-plane along the bisector of the first quadrant $t = t'$ so that the base diagonals coincide with the bisector (Fig. 9.26*b*).

9.27. A random function $X(t)$ is formed in the same way as in Problem 9.26, the only difference being that the points at which the func-

tion assumes new values are not fixed on the t-axis but occupy random positions on it retaining a constant distance equal to unity between them (Fig. 9.27*a*). All the positions of the reference point relative to the sequence of the moments when the function assumes new values are equiprobable. Find the characteristics of the random function $X(t)$, viz. the mean value, variance and correlation function; and determine whether the random function $X(t)$ is stationary.

Solution. As in the preceding problem $m_x(t) = m_x = 0$, $\mathrm{Var}_x(t) = \mathrm{Var}_x = 1$.

Let us find the correlation function. We fix a moment t (Fig. 9.27*a*). This moment is random relative to the points at which the random

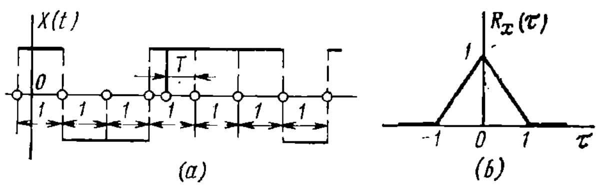


Fig. 9.27

function $X(t)$ assumes new values. We designate as T the time interval which separates the point t from the nearest point at which a new value of $X(t)$ will be assumed. The random variable T is uniformly distributed on the interval from 0 to 1. Assume $t' > t$, $\tau = t' - t > 0$. If $\tau < T$, then $R_x(t, t') = 1$ and if $\tau > T$, then $R_x(t, t') = 0$. Therefore, for $0 < \tau < 1$ we have

$$R_x(t, t') = \mathrm{P}(T > \tau)\cdot 1 + \mathrm{P}(T < \tau)\cdot 0 = \mathrm{P}(T > \tau) = 1 - \tau.$$

Similarly, for $\tau < 0$ we have

$$R_x(t, t') = 1 + \tau \quad \text{for} \quad -1 < \tau < 0.$$

Hence

$$R_x(t, t') = R_x(\tau) = \begin{cases} 1 - |\tau| & \text{for} \quad |\tau| < 1, \\ 0 & \text{for} \quad |\tau| > 1. \end{cases} \tag{9.27}$$

The graph of this function is shown in Fig. 9.27*b*. Since $R_x(t, t') = R_x(\tau)$, the random function $X(t)$ is stationary.

We can write the correlation function (9.27) in a more concise form with the aid of the unit function $\mathbf{1}(x)$:

$$R_x(\tau) = (1 - |\tau|)\,\mathbf{1}(1 - |\tau|).$$

9.28. The conditions of Problem 9.27 are changed so that at each random moment T_i, divided by unit intervals, the random function $X(t)$ assumes (independently of the other functions) the value U_i, which is a random variable with mean value m_u and variance Var_u, and retains that value till the next point. One of the realizations of this random function is shown in Fig. 9.28. Find the characteristics of this random function, viz. the mean value, the variance and the corre-

lation function. Determine whether the random function is stationary, and if it is, then find its spectral density.

Solution. Reasoning as in the preceding problem, we find

$$m_x(t) = \mathrm{M}\,[X(t)] = m_u, \quad \mathrm{Var}_x(t) = \mathrm{Var}\,[X(t)] = \mathrm{Var}_u,$$

$$R_x(\tau) = \begin{cases} \mathrm{Var}_u\,(1-|\tau|) & \text{for } |\tau|<1 \\ 0 & \text{for } |\tau|>1 \end{cases} = \mathrm{Var}_u\,(1-|\tau|)\;1\,(1-|\tau|).$$

The random function $X(t)$ is stationary. Its spectral density $S_x^*(\omega) = \mathrm{Var}_u\,(1-\cos\omega)/(\pi\omega^2)$.

9.29. A random function $X(t)$ is an alternating step function (Fig. 9.29*a*) which assumes values $+1$ and -1 alternately at unit intervals. The position of the step function with respect to the reference point is accidental; the random variable T, characterizing a displacement about the origin of the first point of the sign change, is a random variable uniformly distributed in the interval (0, 1). Find the characteristics of the random function $X(t)$: the mean value, variance and correlation function.

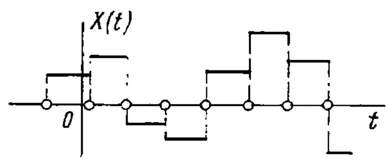


Fig. 9.28

Solution. Let us consider the section of the random function $X(t)$; it may fall, with equal probability, on the interval on which the random

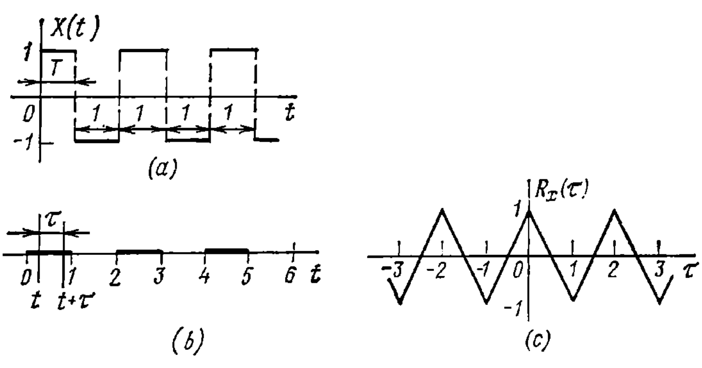


Fig. 9.29

function is equal to $+1$ or on the interval on which it is -1. Consequently, the ordered series of any section has the form

$$X(t):\ \left|\begin{array}{c|c} -1 & +1 \\ \hline \frac{1}{2} & \frac{1}{2} \end{array}\right|,$$

whence $m_x = -1\times\frac{1}{2} + 1\times\frac{1}{2} = 0$ and $\mathrm{Var}_x = (-1)^2\times\frac{1}{2} + (1)^2\times\frac{1}{2} = 1$.

Let us find the correlation function ($\tau = t' - t,\ t' > t$):

$$R_x(t,\ t') = \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{X}(t+\tau)] = \mathrm{M}[X(t)X(t+\tau)].$$

Since the product $X(t)X(t+\tau)$ can assume only two values ($+1$ or -1), it follows that $\mathrm{M}[X(t)X(t+\tau)] = 1p_1 + (-1)(1-p_1) = 2p_1 - 1$, where p_1 is the probability that the points t and $t+\tau$ will fall on the intervals on which $X(t)$ and $X(t+\tau)$ have the same sign.

Since the displacement T on Fig. 9.29*a* is uniformly distributed, we can transfer the reference point to the left end of the interval which includes the point t and consider the point t to be uniformly distributed

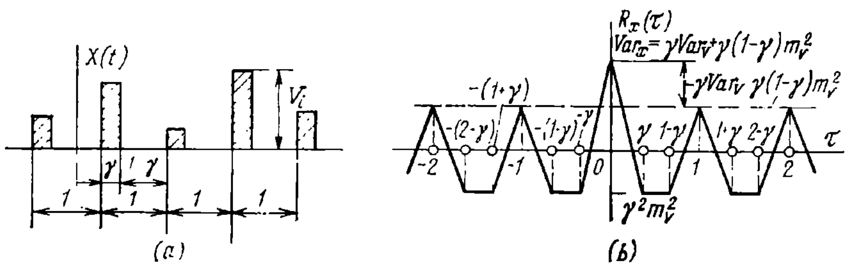


Fig. 9.30

in the interval (0, 1) (Fig. 9.29*b*). In this interpretation p_1 is the probability that the point $(t+\tau)$ will fall in either of the intervals $(2n,\ 2n+1)$, $n = 0,\ \pm 1,\ \pm 2\ \ldots$ (these intervals are marked with thick lines in Fig. 9.29*b*); let us calculate this probability for different values of τ.

For $0 < \tau < 1$ the point $(t+\tau)$ may fall either in the interval (0, 1) or in the interval (1, 2) and, therefore,

$$p_1 = \mathrm{P}\{t+\tau < 1\} = \mathrm{P}\{t < 1-\tau\} = 1-\tau.$$

For $1 < \tau < 2$ the point $t+\tau$ may fall either in the interval (1, 2) or in the interval (2, 3) and, therefore,

$$p_1 = \mathrm{P}\{t+\tau > 2\} = \mathrm{P}\{t > 2-\tau\} = 1-(2-\tau) = \tau - 1.$$

Continuing with our reasoning we get

$$p_1 = \begin{cases} 1-(\tau-2n) & \text{for} \quad 2n < \tau < 2n+1, \\ (\tau-2n)-1 & \text{for} \quad 2n+1 < \tau < 2n+2. \end{cases}$$

It can be seen that p_1 (and, hence $R_x(t,\ t+\tau) = (2p_1 - 1)$ as well, depends only on τ and is an even function of τ. Consequently

$$R_x(t,\ t+\tau) = R_x(\tau) = \begin{cases} 4n+1-2\tau & \text{for } 2n < \tau < 2n+1, \\ 2\tau-(4n+3) & \text{for } 2n+1 < \tau < 2n+2. \end{cases}$$

Fig. 9.29*c* shows a graph of the correlation function.

9.30. A random function $X(t)$ is a sequence of equidistant positive voltage pulses having the same width $\gamma < 1/2$. The beginning of each

pulse is separated by a unit interval from the beginning of a next pulse (Fig. 9.30*a*). The sequence of the pulses occupies a random position on the t-axis relative to the reference point (see the hypothesis of the preceding problem). The potential of the ith pulse V_i is accidental ($i = 1, 2, \ldots$). All the random variables V_i have the same distribution with the mean value m_v and variance Var_v and are independent. Find the characteristics of the random function $X(t)$: the mean value, variance and correlation function.

Solution. By the formula for the complete expectation $m_x = m_v\gamma + 0\cdot(1-\gamma) = m_v\gamma$. We find the variance in terms of the second moment about the origin: $\alpha_2[X(t)] = \alpha_2[V_i]\gamma + 0\cdot(1-\gamma) = (\text{Var}_v + m_v^2)\gamma$. Hence

$$\text{Var}_x = \alpha_2[X(t)] - m_x^2 = (\text{Var}_v + m_v^2)\gamma - m_v^2\gamma$$
$$= \gamma\,\text{Var}_v + \gamma(1-\gamma)m_v^2.$$

In this case the random function $X(t)$ is not centred. We shall seek its correlation function proceeding from the second mixed moment about the origin:

$$R_x(t, t+\tau) = \text{M}[X(t)X(t+\tau)] - m_x^2.$$

Let us find $\text{M}[X(t)X(t+\tau)]$. We use the complete expectation formula to calculate it. As in the preceding problem, we assume the t-axis to be covered with alternating intervals: the intervals marked with a thick line correspond to the pulses and those marked with a thin line correspond to the spaces between them. We denote by T the random value of the left-hand boundary of the interval $(T, T-\tau)$. Three hypotheses are possible:

H_1—both points T and $T+\tau$ fall on the interval of the same pulse;

H_2—one of the points, T or $T+\tau$, falls on the interval of one pulse and the other on the interval of another pulse;

H_3—at least one point, T or $T+\tau$, falls outside of the intervals of pulses.

On the first hypothesis the variables $X(T)$ and $X(T+\tau)$ coincide and $\text{M}[X(T)X(T+\tau)] = \text{M}[V^2] = \text{Var}_v + m_v^2$. On the second hypothesis the variables $X(T)$ and $X(T+\tau)$ are independent random variables with the same mean value m_v; by the theorem on the multiplication of expectations $\text{M}[X(T)X(T+\tau)] = m_v^2$. On the third hypothesis $\text{M}[X(T)X(T+\tau)] = 0$. The complete expectation

$$\text{M}[X(t)X(t+\tau)] = \text{P}\{H_1\}(\text{Var}_v + m_v^2) + \text{P}\{H_2\}m_v^2.$$

The probabilities $\text{P}\{H_1\}$ and $\text{P}\{H_2\}$ and hence the correlation function depend only on τ:

(1) for $0 < \tau < \gamma$

$$\text{P}\{H_1\} = \gamma - \tau,\ \text{P}\{H_2\} = 0,\ \text{M}[X(t)X(t+\tau)] = (\gamma-\tau)(\text{Var}_v + m_v^2)$$
$$R_x(\tau) = (\gamma-\tau)(\text{Var}_v + m_v^2) - \gamma^2 m_v^2,$$

(2) for $\gamma < \tau < 1 - \gamma$

$$\mathrm{P}\{H_1\} = 0, \quad \mathrm{P}\{H_2\} = 0, \quad \mathrm{M}\,[X(t)\,X(t+\tau)] = 0,$$

$$R_x(\tau) = 0 - \gamma^2 m_v^2 = -\gamma^2 m_v^2,$$

(3) for $1 - \gamma < \tau < 1$

$$\mathrm{P}\{H_1\} = 0, \quad \mathrm{P}\{H_2\} = \gamma - (1 - \tau),$$

$$\mathrm{M}\,[X(t)\,X(t+\tau)] = [\gamma - (1 - \tau)]\, m_v^2,$$

$$R_x(\tau) = (\gamma - 1 + \tau - \gamma^2)\, m_v^2.$$

The next intervals of the values of τ can be investigated in the same way.

The graph of the function $R_x(\tau)$ is shown in Fig. 9.30*b*. For $|\tau| > 1/2$ the curve $R_x(\tau)$ is repeated periodically, attaining local maxima equal to $\gamma(1 - \gamma)\, m_v^2$ at integral-valued points.

9.31*. We consider a stationary random function $X(t)$, which is a sawtooth voltage (Fig. 9.31*a*). The reference point occupies a random

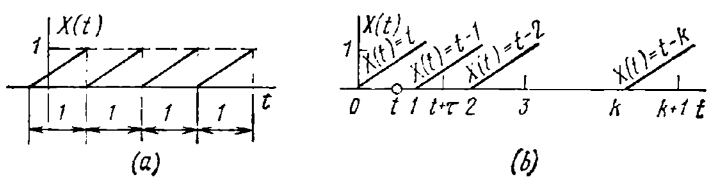


Fig. 9.31

position relative to the tooth, just as in Problem 9.29. Find the mean value, the variance and the correlation function of the random function $X(t)$.

Solution. We can easily find the mean value m_x if we take into account that $X(t)$ has a uniform distribution on the interval (0, 1) for any t. Hence $m_x = 1/2$.

To find the correlation function, we rigidly connect the sequence of teeth with the t-axis and throw the beginning t of the interval $(t, t+\tau)$ at random on the t-axis (Fig. 9.31*b*). Since the teeth are periodic, it is sufficient to throw the point t at random on the first interval (0, 1) distributing it with constant density. As can be seen from Fig. 9.31*b*, in that case $X(t) = t$ and the value $X(t+\tau)$ is equal to the fractional part of the number $t+\tau$, i.e. $X(t+\tau) = t + \tau - E(t+\tau)$, where $E(t+\tau)$ is the integer part of the number $(t+\tau)$.

If the integral part of the number τ is n $(n \leqslant \tau < n+1)$, then

$$E(t+\tau) = \begin{cases} n & \text{for} \quad t+\tau < n+1, \\ n+1 & \text{for} \quad t+\tau \geqslant n+1, \end{cases}$$

and, hence,

$$X(t+\tau)=\begin{cases} t+\tau-n & \text{for} \quad t<n+1-\tau, \\ t+\tau-(n+1) & \text{for} \quad t\geqslant n+1-\tau. \end{cases}$$

From the formula for the mean value of the function of the random variable t we have, for $n \leqslant \tau < n+1$,

$$\begin{aligned} \mathrm{M}\,[X(t)\,X(t+\tau)] &= \int_0^1 X(t)\,X(t+\tau)\cdot 1\cdot dt \\ &= \int_0^{n+1-\tau} t\,(t+\tau-n)\,dt + \int_{n+1-\tau}^{1} t\,(t+\tau-n-1)\,dt \\ &= \frac{1}{2}(n+1-\tau)^2+\frac{1}{2}(\tau-n)-\frac{1}{6} \\ &\qquad (n=0,\ \pm 1,\ \pm 2,\ \ldots). \end{aligned}$$

It follows that the correlation function depends only on τ and for $n \leqslant \tau \leqslant n+1$ $(n=0,\ \pm 1,\ \pm 2,\ \ldots)$ has the form

$$\begin{aligned} R_x(t,\ t') = R_x(\tau) &= \mathrm{M}\,[X(t)\,X(t+\tau)] - m_x^2 \\ &= 0.5\,(n+1-\tau)^2 + (\tau-n)/2 - 5/12. \end{aligned}$$

This is a periodic function with unit period whose graph consists of periodically repeating segments of a parabola which are convex downwards. In the interval $0 \leqslant \tau < 1$ this parabola has the form $R_x(\tau) = (1-\tau^2)/2 + \tau/2 - 5/12$ with vertex at a point $(1/2,\ -1/24)$. Setting $\tau = 0$, we get $\mathrm{Var}_x = R_x(0) = 1/12$.

9.32. We consider a linear transformation of n stochastic processes $X_1(t),\ X_2(t),\ \ldots,\ X_n(t)$ of the form

$$Y(t)=\varphi_0(t)+\sum_{i=1}^{n}\varphi_i(t)\,X_i(t),$$

where $\varphi_i(t)$ $(i=0,\ 1,\ 2,\ \ldots,\ n)$ are nonrandom functions of time.

We know the chatacteristics of the stochastic processes $X_i(t)$: $m_i(t)$, $\mathrm{Var}_i(t)$, $R_i(t,\ t')$ $(i=1,\ 2,\ \ldots,\ n)$ and the crosscorrelation functions $R_{ij}(t,\ t')=\mathrm{M}\,[\mathring{X}_i(t)\,\mathring{X}_j(t')]$ $(i,\ j=1,\ 2,\ \ldots,\ n,\ i\neq j)$. Find the characteristics of the stochastic process $Y(t)$.

Solution. $m_y(t)=\sum_{i=1}^{n}\varphi_i(t)\,m_i(t)$,

$$\begin{aligned} R_y(t,\ t') &= \mathrm{M}\,[\mathring{Y}(t)\,\mathring{Y}(t')] = \mathrm{M}\,\Big[\sum_{i=1}^{n}\varphi_i(t)\,\mathring{X}_i(t)\ \sum_{j=1}^{n}\varphi_j(t')\,\mathring{X}_j(t')\Big] \\ &= \sum_{i=1}^{n}\varphi_i(t)\,\varphi_i(t')\,R_i(t,\ t') \end{aligned}$$

$$+\sum_{i\neq j}\varphi_i(t)\,\varphi_j(t')\,R_{ij}(t,\,t')+\sum_{i\neq j}\varphi_i(t')\,\varphi_j(t)\,R_{ij}(t',\,t),$$

$$\mathrm{Var}_y(t)=R_y(t,\;t)=\sum_{i=1}^{n}\varphi_i^2(t)\,\mathrm{Var}_i(t)+2\sum_{i\neq j}\varphi_i(t)\,\varphi_j(t)\,R_{ij}(t,\,t).$$

If the stochastic processes $X_i(i)$ $(t=1,\,2,\,\ldots.\,n)$ are uncorrelated $(R_{ij}(t,\,t')\equiv 0;\; i,\; j=1,\,2,\,\ldots,\,n)$, then

$$R_y(t,\;t')=\sum_{i=1}^{n}\varphi_i(t)\,\varphi_j(t')\,R_i(t,\;t'),$$

$$\mathrm{Var}_y=\sum_{i=1}^{n}\varphi_i^2(t)\,\mathrm{Var}_i(t).$$

9.33. Given two uncorrelated random function $X(t)$ and $Y(t)$ with characteristics

$$m_x(t)=t^2,\quad R_x(t,\;t')=\mathrm{e}^{\alpha_1(t+t')},$$
$$m_y(t)=1,\quad R_y(t,\;t')=\mathrm{e}^{\alpha_2(t-t')^2},$$

find the characteristics of the random function $Z(t)=X(t)+tY(t)+t^2$. Solve the same problem given that the random functions $X(t)$ and $Y(t)$ are correlated and their crosscorrelated function $R_{xy}(t,\;t')=a\mathrm{e}^{-\alpha|t-t'|}$.

Solution. If $R_{xy}(t,\;t')\equiv 0$, then

$$m_z=m_x(t)+tm_y(t)+t^2=2t^2+t,$$
$$R_z(t,\;t')=R_x(t,\;t')+tt'\,R_y(t,\;t')=\mathrm{e}^{\alpha_1(t+t')}+tt'\,\mathrm{e}^{\alpha_2(t-t')^2}.$$

If $R_{xy}(t,\;t')=a\exp(-\alpha\mid t-t'\mid)$, then $m_z(t)$ does not change;

$$R_z(t,\;t')=R_x(t,\;t')+tt'\,R_y(t,\;t')+t'R_{xy}(t,\;t')+tR_{xy}(t',\;t)$$
$$=\mathrm{e}^{\alpha_1(t+t')}+tt'\,\mathrm{e}^{\alpha_2(t-t')^2}+a\,(t+t')\,\mathrm{e}^{-\alpha|t-t'|}.$$

9.34. Find the mean value and the correlation function of the sum of two uncorrelated random functions $X(t)$ and $Y(t)$ with characteristics

$$m_x(t)=t,\;R_x(t,\;t')=tt',$$
$$m_y(t)=-t,\;R_y(t,\;t')=tt'\,\mathrm{e}^{\alpha(t+t')}.$$

Answer. $m_z(t)=m_x(t)+m_y(t)=0$. $R_z(t,\,t')=R_x(t,\;t')+R_y(t,\,t')=tt'\,[1+\mathrm{e}^{\alpha(t+t')}]$.

9.35. Given a complex random function $Z(t)=X(t)+iY(t)$, where i is the imaginary unit; $X(t)$, $Y(t)$ are uncorrelated random functions with characteristics

$$m_x(t)=t^2,\quad R_x(t,\;t')=\mathrm{e}^{-\alpha_1(t-t')^2},$$
$$m_y(t)=1,\quad R_y(t,\;t')=\mathrm{e}^{2\alpha_2(t+t')},$$

find the characterististics of the random function $Z(t)$: $m_z(t)$, $R_z(t,\;t')$ and $\mathrm{Var}_z(t)$.

Answer. $m_z(t) = t^2 + i$, $R_z(t, t') = e^{-\alpha_1 (t-t')^2} + e^{2\alpha_2 (t+t')}$,

$$\mathrm{Var}_z(t) = R_z(t, t') = 1 + e^{4\alpha_2 t}.$$

9.36. A complex random function $Z(t)$ is given in the form $Z(T) = X(t) + iY(t)$, where

$$X(t) = \sum_{k=1}^{3} (a_k + V_k) e^{-\alpha_k t}, \quad Y(t) = \sum_{k=1}^{3} (b_k + U_k) e^{-\beta_k t}.$$

The mean values of all the random variables V_k and U_k ($k = 1, 2, 3$) are zero and the correlation matrix of the random variables (V_1, V_2, V_3, U_1, U_2, U_3) has the form

$$\left\|\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ & 2 & 0 & 0 & -1 & 0 \\ & & 3 & 0 & 0 & 3 \\ & & & 1 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 3 \end{array}\right\|.$$

Find the characteristics of the random function $Z(t)$.

Answer. $m_z(t) = \sum_{k=1}^{3} a_k e^{-\alpha_k t} + i \sum_{k=1}^{3} b_k e^{-\beta_k t}$,

$$R_z(t, t') = R_x(t, t') + R_y(t, t') + i[R_{xy}(t', t) - R_{xy}(t, t')],$$

$$\text{where} \quad R_x(t, t') = \sum_{k=1}^{3} k e^{-\alpha_k (t+t')},$$

$$R_y(t, t') = \sum_{k=1}^{3} k e^{-\beta_k (t+t')},$$

$$R_{xy}(t', t) = e^{-\alpha_1 t' - \beta_1 t} - e^{-\alpha_2 t' - \beta_2 t} + 3e^{-\alpha_3 t' - \beta_3 t},$$

$$R_{xy}(t, t') = e^{-\alpha_1 t - \beta_1 t'} - e^{-\alpha_2 t - \beta_2 t'} + 3e^{-\alpha_3 t - \beta_3 t'}.$$

9.37. *Correlation function of a product.* Given two uncorrelated centred random functions $X(t)$ and $Y(t)$ and their product $Z(t) = X(t) Y(t)$, prove that the correlation function of the product is equal to the product of the correlation functions of the factors: $R_z(t, t') = R_x(t, t') R_y(t, t')$.

Solution. $R_z(t, t') = \mathrm{M}[\mathring{Z}(t) \mathring{Z}(t')]$; $\mathring{Z}(t) = Z(t) - m_z(t)$. Since the random functions $X(t)$ and $Y(t)$ are uncorrelated and centred, it follows that $m_z(t) = m_x(t) m_y(t) = 0$, hence

$$\mathring{Z}(t) = \mathring{X}(t) \mathring{Y}(t) = X(t) Y(T),$$

$$R_z(t, t') = \mathrm{M}[\mathring{X}(t) \mathring{Y}(t) \mathring{X}(t') \mathring{Y}(t')]$$

$$= \mathrm{M}[\mathring{X}(t) \mathring{X}(t')] \mathrm{M}[\mathring{Y}(t) \mathring{Y}(t')] = R_x(t, t') R_y(t, t').$$

In particular, for $t = t'$

$$\mathrm{Var}_z(t) = \mathrm{Var}_x(t)\,\mathrm{Var}_y(t).$$

9.38. Prove that the correlation function of the product of independent centred random functions $Z(t) = \prod_{i=1}^{n} X_i(t)$ is equal to the product of the correlation functions of the factors:

$$R_z(t, t') = \prod_{i=1}^{n} R_{x_i}(t, t').$$

Solution. The proof is similar to the preceding one, the only difference being that in order to use the theorem on the multiplication of mean values, it is insufficient in this case for the factor to be uncorrelated and it is sufficient for them to be independent.

9.39. A random function $X(t)$ with characteristics $m_x(t) = 0$, $R_x(t, t')$ is subjected to a nonhomogeneous linear transformation:

$$Y(t) = L_t^{(0)}\{X(t)\} + \varphi(t),$$

where $\varphi(t)$ is a nonrandom function. Find the crosscorrelation function $R_{xy}(t, t')$.

Solution. We have $\overset{\circ}{X}(t) = X(t)$; $\overset{\circ}{Y}(t) = L_t^{(0)}\{X(t)\} = L_t^{(0)}\{\overset{\circ}{X}(t)\}$

since when the random function $Y(t)$ is being centred, the nonrandom term $\varphi(t)$ is eliminated; hence

$$R_{xy}(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{Y}(t')] = \mathrm{M}[X(t)\,L_{t'}^{(0)}\{\overset{\circ}{X}(t')\}]$$

$$= L_{t'}^{(0)}\{\mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{X}(t')]\} = L_{t'}^{(0)}\{R_x(t, t')\}.$$

9.40. *Characteristics of the derivative of a stochastic process.* There is a stochastic process $X(t)$ with mean value $m_x(t)$ and correlation function $R_x(t, t')$. Find the characteristics $m_y(t)$, $R_y(t, t')$ and $\mathrm{Var}_y(t)$ of its derivative $Y(t) = dX(t)/dt$. Also find the crosscorrelated function $R_{xy}(t, t')$.

Solution. The random function $Y(t)$ is connected with $X(t)$ by a homogeneous linear transformation. Applying the general rules (9.0.7), (9.0.8), (9.0.9), we obtain

$$m_y(t) = \frac{dm_x(t)}{dt}, \quad R_y(t, t') = \frac{\partial^2 R_x(t, t')}{\partial t\,\partial t'},$$

$$\mathrm{Var}_y(t) = [R_y(t, t')]_{t=t'} = \left[\frac{\partial^2 R_x(t, t')}{\partial t\,\partial t'}\right]_{t=t'},$$

$$R_{xy}(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{Y}(t')] = \mathrm{M}\left[\overset{\circ}{X}(t)\frac{d}{dt'}\overset{\circ}{X}(t')\right] = \frac{\partial}{\partial t'}R_x(t, t').$$

Note that

$$R_{yx}(t, t') = \frac{\partial}{\partial t}R_x(t, t').$$

9.41. A random function $X(t)$ has characteristics $m_x(t) = 1$ and $R_x(t, t') = e^{\alpha(t+t')}$. Find the characteristics of the random function $Y(t) = t\frac{d}{dt}X(t) + 1$. Find out whether the random functions $X(t)$ and $Y(t)$ are stationary.

Solution. Since the transformation $t\frac{dX(t)}{dt} + 1$ is linear,

$$m_y(t) = t\frac{d}{dt}m_x(t) + 1 = 1,$$

$$R_y(t, t') = tt'\frac{\partial^2}{\partial t \partial t'}R_x(t, t') = tt'\alpha^2 e^{\alpha(t+t')}.$$

Neither $X(t)$ nor $Y(t)$ is stationary since their correlation functions depend not only on $\tau = t' - t$ but also on each of the arguments t, t'.

9.42. A random function $X(t)$ has characteristics

$$m_x(t) = t^2 - 1, \quad R_x(t, t') = 2e^{-\alpha(t'-t)^2}.$$

Find the characteristics of the random functions

$$Y(t) = tX(t) + t^2 + 1, \quad Z(t) = 2t\frac{d}{dt}X(t) + (1-t)^2,$$

$$U(t) = \frac{d^2X(t)}{dt^2} + 1.$$

Solution. $m_y(t) = tm_x(t) + t^2 + 1 = t^3 + t^2 - t + 1,$

$$R_y(t, t') = 2tt'e^{-\alpha(t'-t)^2}.$$

$$m_z(t) = 2t\frac{d}{dt}m_x(t) + (1-t)^2 = 1 - 2t + 5t^2,$$

$$R_z(t) = 4tt'\frac{\partial^2}{\partial t \partial t'}R_x(t, t') = 16\alpha tt'e^{-\alpha(t-t')^2}[1 - 2\alpha(t-t')^2],$$

$$m_u(t) = \frac{d^2}{dt^2}m_x(t) + 1 = 3, \quad R_u(t, t') = \frac{\partial^4}{\partial t^2 \partial t'^2}R_x(t, t').$$

When calculating $R_z(t, t')$ we have found $\frac{\partial^2}{\partial t \partial t'}R_x(t, t')$ and, consequently,

$$R_u(t, t') = \frac{\partial^2}{\partial t \partial t'}\{4\alpha e^{-\alpha(t-t')^2}[1 - 2\alpha(t-t')^2]$$
$$= 8\alpha^2 e^{-\alpha(t-t')^2}[3 + 4\alpha^2(t'-t)^4 - 12\alpha(t-t')^2].$$

9.43. A random function $X(t)$ is specified by the expression $X(t) = V\cos\omega t$, where V is a random variable with characteristics $m_v = 2$, $\sigma_v = 3$. Find the characteristics of the random function $X(t)$: $m_x(t)$, $R_x(t, t')$, $\mathrm{Var}_x(t)$. Determine whether the random function $X(t)$ is stationary. Find the characterististics of the random function $Y(t) = X(t) + \alpha\frac{d}{dt}X(t)$, where α is a nonrandom variable. Is the random function $Y(t)$ stationary?

Solution.

$$m_x(t) = m_v \cos \omega t = 2 \cos \omega t,$$
$$R_x(t, t') = \mathrm{Var}_v \cos \omega t \cos \omega t' = 9 \cos \omega t \cos \omega t',$$
$$\mathrm{Var}_x(t) = 9 (\cos \omega t)^2.$$

We can represent the random function $Y(t)$ thus:

$$Y(t) = V \cos \omega t + \alpha \frac{d}{dt} V \cos \omega t = V (\cos \omega t - \alpha\omega \sin \omega t),$$

whence we have $m_y(t) = m_v (\cos \omega t - \alpha\omega \sin \omega t) = 2 (\cos \omega t - \alpha\omega \sin \omega t)$,

$$R_y(t, t') = 9 (\cos \omega t - \alpha\omega \sin \omega t)(\cos \omega t' - \alpha\omega \sin \omega t'),$$
$$\mathrm{Var}_y(t) = 9 (\cos \omega t - \alpha\omega \sin \omega t)^2.$$

The random functions $X(t)$ and $Y(t)$ are nonstationary.

9.44. A telephone exchange receives an elementary flow of requests with intensity λ. A random function $X(t)$ is the number of requests received by the exchange during the time t (see Problem 9.13). Find the characteristics of its derivative $Y(t)$.

Solution. In the ordinary sense, a discontinuous random function, which is a Poisson process, is not differentiable. However, using the generalized delta-function, we can write the characteristics of the derivative. The transformation $Y(t) = dX(t)/dt$, which relates the random functions $Y(t)$ and $X(t)$, is a homogeneous linear transformation. Therefore, from Problem 9.13 we have

$$m_y(t) = \frac{d}{dt} m_x(t) = \frac{d}{dt} t\lambda = \lambda,$$

$$R_y(t, t') = \frac{\partial^2}{\partial t \partial t'} R_x(t, t')$$
$$= \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t'} [\lambda t 1(t' - t) + \lambda t' 1(t - t')] \right)$$
$$= \frac{\partial}{\partial t} [\lambda (t - t') \delta (t - t') + \lambda 1 (t - t')],$$

but

$$(t - t') \delta (t - t') \equiv 0,$$

whence

$$R_y(t, t') = \frac{\partial}{\partial t} (\lambda 1 (t - t')) = \lambda \delta (t - t') = \lambda \delta (\tau).$$

Thus, the correlation function of the random function $Y(t)$ is proportional to the delta-function, i.e. the function $Y(t)$ is stationary white noise with intensity $G = \lambda$ and the average level $m_y = \lambda$. The spectral density of such a white noise is

$$S_y^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda \delta (\tau) e^{-i\omega\tau} dt = \frac{\lambda}{2\pi}.$$

9.45. Find the characteristics of the stochastic process $Y(t)$ which is equal to the derivative of the Wiener process $X(t)$ in Problem (9.25), i.e. $Y(t) = \frac{d}{dt} X(t)$.

Solution. Since $m_x(t) = 0$, it follows that $m_y(t) = 0$. The correlation functions of the Wiener and Poisson processes are equal (with an accuracy to within a constant factor). Therefore, the correlation function of the derivative of the Wiener process (see the solution of the preceding problem) is proportional to the delta-function and the process $Y(t)$ itself is stationary white noise.

9.46. Prove that the derivative of the process of a homogeneous random walk (Brownian movement) of a particle (see Problem 9.24) is stationary white noise.

Hint. Use the solution of Problem 9.44.

9.47. *Characteristics of the integral of a stochastic process.* Given a stochastic process $X(t)$ and its characteristics $m_x(t)$, $R_x(t, t')$, find the characteristics $m_y(t)$, $R_y(t, t')$ of the integral of the stochastic process $Y(t) = \int_0^t X(\tau)\, d\tau$ and the crosscorrelation function $R_{xy}(t, t')$.

Solution.

$$m_y(t) = \int_0^t m_x(t)\, dt, \quad R_y(t, t') = \int_0^t \int_0^{t'} R_x(\tau, \tau')\, d\tau\, d\tau',$$

$$\mathrm{Var}_y(t) = \int_0^{t'} \int_0^t R_x(\tau, \tau')\, d\tau\, d\tau', \quad R_{xy}(t, t') = \int_0^{t'} R_x(t, \tau')\, d\tau'.$$

Note that

$$R_{yx}(t, t') = \int_0^t R_x(\tau, t')\, d\tau.$$

We can prove that there is no random function $X(t)$ different from zero for which $Y(t)$ is stationary.

9.48. A random function $X(t)$ has the following characteristics: $m_x(t) = 0$, $R_x(t, t') = [1-(t'-t)^2]^{-1}$. Find the characteristics of the random function $Y(t) = \int_0^t X(t)\, dt$. Determine whether the random functions $X(t)$ and $Y(t)$ are stationary.

Solution. Since the transformation $\int_0^t X(t)\, dt$ is linear,

$$m_y(t) = \int_0^t m_x(t)\, dt = 0, \quad R_y(t, t') = \int_0^t dt \int_0^{t'} R_x(t, t')\, dt'$$

$$= \int_0^t \left(\int_0^{t'} [1+(t'-t)^2]^{-1} dt' \right) dt$$

$$= t \arctan t + t' \arctan t' - (t-t') \arctan (t-t')$$

$$-\frac{1}{2} \ln \{(1+t^2)(1+t'^2)[1+(t-t')^2]^{-1}\}.$$

The random function $X(t)$ is stationary: $R_x(t, t') = R_x(t-t')$, the random function $Y(t) = \int_0^t X(t)\,dt$ is nonstationary. Indeed, the variance of the random function is $\mathrm{Var}_y(t) = R_y(t, t) = 2t \arctan t - \ln(1+t^2)$, i.e. the variance depends on t.

9.49. A random function $X(t)$ with characteristics $m_x(t) = t^2+3$, $R_x(t, t') = 5tt'$ is subjected to a linear transformation of the form

$$Y(t) = \int_0^t \tau X(\tau)\, d\tau + t^3.$$

Find the characteristics of the random function $Y(t)$: $m_y(t)$, $R_y(t, t')$.

Solution.

$$m_y(t) = \int_0^t \tau(\tau^2+3)\, d\tau + t^3 = \frac{t^4}{4} + \frac{3}{2} t^2 + t^3.$$

The homogeneous part of the linear transformation being considered is

$$L_t^{(0)}\{X(t)\} = \int_0^t \tau X(\tau)\, dt.$$

Consequently,

$$R_y(t, t') = \int_0^t d\tau \int_0^{t'} \tau\tau' R_x(\tau, \tau')\, d\tau' = 5 \int_0^t \tau\tau \left(\int_0^{t'} \tau'\tau'\, d\tau' \right) d\tau = \frac{5}{9} t^3 (t')^3.$$

9.50. A random function $X(t)$, with characteristics $m_x(t) = 0$ and $R_x(t, t') = 3e^{-(t+t')}$ is subjected to a linear transformation of the form

$$Y(t) = -t \frac{d}{dt} X(t) + \int_0^t \tau X(\tau)\, d\tau + \sin \omega t.$$

Find the covariance of the random variables $X(0)$ and $Y(1)$ (i.e. of two sections of the random functions: $X(t)$ for $t = 0$ and $Y(t')$ for $t' = 1$).

Solution. On the basis of the solution of the preceding problem

$$R_{xy}(t, t') = L_{t'}^{(0)}\{R_x(t, t')\},$$

where $L_{t'}^{(0)}$ is the homogeneous part of the linear transformation applied to the argument t'. In this case

$$R_{xy}(t,\ t') = -3t' \frac{\partial e^{-(t+t')}}{\partial t'} + 3 \int_0^{t'} \tau' e^{-(t+t')} d\tau'$$

$$= 3t' e^{-(t+t')} + 3e^{-t}[e^{-t'}(-t'-1)+1] = 3e^{-t}(1-e^{-t'}).$$

Setting $t=0,\ t'=1$, we obtain

$$\mathrm{Cov}_{X(0),\,Y(1)} = R_{xy}(0,\ 1) = 3(1-e^{-1}) \approx 1.90.$$

9.51. A random input signal $X(t)$ is transformed by a relay into a random output signal $Y(t)$ which is in a nonlinear relationship with $X(t)$: $Y(t) = \mathrm{sign}\, X(t)$, i.e.

$$Y(t) = \begin{cases} 1 & \text{for} \quad X(t) > 0, \\ 0 & \text{for} \quad X(t) = 0, \\ -1 & \text{for} \quad X(t) < 0. \end{cases}$$

The input signal is the random function $X(t)$ considered in Problem 9.15. Find the distribution of a section of the random function $Y(t)$ and its characteristics $m_y(t)$, $R_y(t,\ t')$.

Solution. The random function $Y(t)$ can assume only two values $+1$ and -1; the value 0 can be neglected since $\mathrm{P}\{X(t)=0\}=0$. The probability that $X(t)>0$ is $p = \int_0^{\infty} \varphi(x)\,dx$. The ordered series of the random variable $Y(t)$ has the form

$$Y(t): \left| \begin{array}{c|c} -1 & +1 \\ \hline 1-p & p \end{array} \right| .$$

Hence $m_y = 2p - 1$, $\mathrm{Var}_y = 1 - (2p-1)^2 = 4p(1-p)$.

Assume $\tau = t' - t$ and $t' > t$. If no event occurred in the Poisson flow during the time τ (and the probability of this is $e^{-\lambda\tau}$), then the values of the random function $Y(t)$ and $Y(t')$ are equal to each other and the conditional correlation function $R_y(t,\ t') = \mathrm{Var}_y = 4p(1-p)$. Now if at least one event occurred during the time τ, then $Y(t)$ and $Y(t')$ are uncorrelated and the conditional correlation function $R_y(t,\ t')$ is zero. Hence, for $t' > t$

$$R_y(t,\ t') = e^{-\lambda\tau} 4p(1-p),$$

and in a general case (for any t and t')

$$R_y(t,\ t') = R_y(\tau) = e^{-\lambda|\tau|} 4p(1-p).$$

9.52. The random input signal $X(t)$ considered in Problem 9.15 is transformed into a random output signal $Y(t)$ by means of a relay

with a dead band:

$$Y(t)=\begin{cases}\operatorname{sign} X(t) & \text{for} \quad |X(t)|>\varepsilon,\\ 0 & \text{for} \quad |X(t)|<\varepsilon,\end{cases}$$

where ε is the dead band of the relay.

Find the distribution of the section of the random function $Y(t)$ and its characteristics: the mean value and the correlation function.

Solution. The random variable $Y(t)$ may assume one of the three values, -1, 0, 1, for any t and has an ordered series

$$Y(t):\left|\begin{array}{c|c|c} -1 & 0 & +1 \\ \hline p_1 & p_2 & p_3 \end{array}\right|,$$

where

$$p_1=\mathrm{P}\{X(t)<-\varepsilon\}=\int_{-\infty}^{-\varepsilon}\varphi(x)\,dx,$$

$$p_2=\mathrm{P}\{|X(t)|<\varepsilon\}=\int_{-\varepsilon}^{\varepsilon}\varphi(x)\,dx, \quad p_3=1-p_1-p_2$$

Hence $m_y=p_3-p_1$, $\operatorname{Var}_y=p_1+p_3-(p_3-p_1)^2$.

Reasoning as we did in the preceding problem, we find the correlation function

$$R_y(\tau)=e^{-\lambda|\tau|}\cdot[p_1+p_3-(p_3-p_1)^2].$$

9.53. A random function $X(t)$ is transformed into a random function

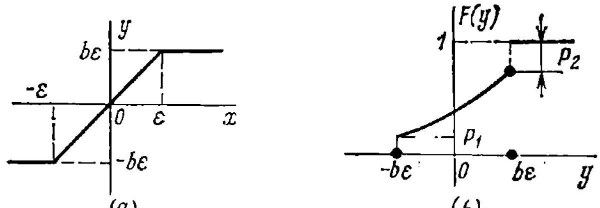


Fig. 9.53

$Y(t)$ by means of a nonlinear device whose operation is described by the formula

$$Y(t)=\begin{cases}-b\varepsilon & \text{for} \quad X(t)<-\varepsilon,\\ bX(t) & \text{for} \quad |X(t)|<\varepsilon,\\ b\varepsilon & \text{for} \quad X(t)>\varepsilon.\end{cases}$$

The graph of the relationship of $y(x)$ is shown in Fig. 9.53*a*.

The random function $X(t)$ considered in Problem 9.15 arrives at the input of the device. Find the univariate distribution of the random

function $Y(t)$ and its characteristics: mean value and correlation function.

Solution. The random variable $Y(t)$, a section of the random function $Y(t)$, has a continuous distribution in the open interval $(-b\varepsilon, +b\varepsilon)$ and, in addition, discrete possible values $-b\varepsilon$ and $+b\varepsilon$ with a nonzero probability. Thus the section $Y(t)$ is a mixed random variable whose distribution function $F(y)$ is continuous on the interval $(-b\varepsilon, +b\varepsilon)$ and suffers discontinuities at the end-points of the interval (points $-b\varepsilon$ and $+b\varepsilon$). The jumps of $F(t)$ at the points of discontinuity are

$$\mathrm{P}\{Y(t)=-b\varepsilon\}=\mathrm{P}\{X(t)<-\varepsilon\}=\int_{-\infty}^{-\varepsilon}\varphi(x)\,dx=p_1,$$

$$\mathrm{P}\{Y(t)=+b\varepsilon\}=\mathrm{P}\{X(t)>\varepsilon\}=\int_{\varepsilon}^{\infty}\varphi(x)\,dx=p_2.$$

Let us find the distribution function of the random variable $Y(t)$ in the interval $(-b\varepsilon;\ +b\varepsilon)$:

$$F(y)=\mathrm{P}\left\{Y(t)<y\right\}=\mathrm{P}\left\{X(t)<\frac{y}{b}\right\}=\int_{-\infty}^{y/b}\varphi(x)\,dx$$

$$=p_1+\int_{-\varepsilon}^{y/b}\varphi(x)\,dx\ (-b\varepsilon<y<b\varepsilon).$$

The graph of the distribution function $F(y)$ is shown in Fig. 9.53b.

The distribution density of the mixed random variable $Y(t)$ in the interval $(-b\varepsilon, +b\varepsilon)$ is equal to the derivative of $F(y)$ on that interval:

$$f(y)=F'(y)=\frac{1}{b}\varphi(y/b)\quad \text{for}\quad -b\varepsilon<y<+b\varepsilon.$$

The characteristics of the random function $Y(t)$ are

$$m_y(t)=m_y=-b\varepsilon p_1+b\varepsilon p_2+\frac{1}{b}\int_{-b\varepsilon}^{b\varepsilon}y\varphi\left(\frac{y}{b}\right)dy$$

$$=b\varepsilon(p_2-p_1)+b\int_{-\varepsilon}^{\varepsilon}x\varphi(x)\,dx,$$

$$\mathrm{Var}_y(t)=\alpha_2[Y(t)]-m_y^2=(\varepsilon b)^2(p_1+p_2)$$

$$+\frac{1}{b}\int_{-b\varepsilon}^{b\varepsilon}y^2\varphi\left(\frac{y}{b}\right)dy-m_y^2=\mathrm{Var}_y.$$

By analogy with the preceding problems $R_y(\tau)=\mathrm{Var}_y\, e^{-\lambda|\tau|}$.

9.54. Consider a random function $X(t)=W\cos(\omega_1 t-\Theta)$, where W is a centred random variable with variance Var_w, Θ is a random

variable distributed with constant density in the interval $(0, 2\pi)$, and ω_1 is a nonrandom parameter ($\omega_1 > 0$). The random variables W and Θ are mutually independent. Find the mean value and the correlation function of the random function $X(t)$. Determine whether the random function $X(t)$ is stationary and ergodic. If it is stationary, then find its spectral density $S_x(\omega)$.

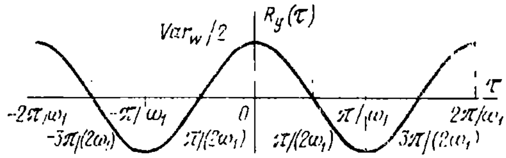


Fig. 9.54

Solution. We represent the random function $X(t)$ in the form

$$X(t) = W\cos(\omega_1 t - \Theta) = W\cos\Theta\cos\omega_1 t + W\sin\Theta\sin\omega_1 t.$$

We designate $W\cos\Theta = U$, $W\sin\Theta = V$ and find the main characteristics of the system of random variables U and V:

$$\mathrm{M}[U] = \mathrm{M}[W\cos\Theta] = \mathrm{M}[W]\,\mathrm{M}[\cos\Theta] = 0,$$
$$\mathrm{M}[V] = \mathrm{M}[W\sin\Theta] = \mathrm{M}[W]\,\mathrm{M}[\sin\Theta] = 0,$$
$$\mathrm{Var}[U] = \mathrm{M}[(W\cos\Theta)^2] = \mathrm{M}[W^2]\,\mathrm{M}[\cos^2\Theta] = \mathrm{Var}_w\,\mathrm{M}[\cos^2\Theta],$$
$$\mathrm{Var}[V] = \mathrm{M}[(W\sin\Theta)^2] = \mathrm{M}[W^2]\,\mathrm{M}[\sin^2\Theta] = \mathrm{Var}_w\,\mathrm{M}[\sin^2\Theta],$$
$$R_{uv} = \mathrm{M}[W\cos\Theta\, W\sin\Theta] = \mathrm{Var}_w\mathrm{M}[\sin\Theta\cos\Theta].$$

Since the value of Θ is distributed uniformly in the interval $(0, 2\pi)$, we have

$$\mathrm{M}[\sin^2\Theta] = \mathrm{M}[\cos^2\Theta] = \int_0^{2\pi}\cos^2 x\,\frac{1}{2\pi}\,dx = \frac{1}{2},$$

$$\mathrm{M}[\sin\Theta\cos\Theta] = \int_0^{2\pi}\sin x\cos x\,\frac{1}{2\pi}\,dx = 0.$$

Thus $\mathrm{M}[U] = \mathrm{M}[V] = 0$, $\mathrm{Var}[U] = \mathrm{Var}[V] = \mathrm{Var}_w/2$, $R_{uv} = 0$. Consequently, the expression

$$X(t) = W\cos(\omega_1 t - \Theta) = U\cos\omega_1 t + V\sin\omega_1 t$$

is a spectral decomposition of a stationary random function: $m_x = 0$, and the correlation function has the form $R_x(\tau) = (\mathrm{Var}_w\cos\omega_1\tau)/2$. The graph of this function is shown in Fig. 9.54.

The random function $X(t)$ is not ergodic since the characteristics found from one realization do not coincide with those determined from several realizations. Indeed, every realization of the random function $X(t)$ is a harmonic oscillation whose amplitude is a value assumed at

random by the variable W. The time average of each realization is zero and coincides with the mean value of the random function $X(t)$, but the variance and the correlation function, found as time averages for one realization, do not coincide with the corresponding characteristics of the random function $X(t)$. For instance,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X^2(t)\,dt=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}W^2\frac{1}{2}[1+\cos 2(\omega_1 t-\Theta)]\,dt=\frac{1}{2}W^2.$$

Let us find the spectral density of the random function $X(t)$. We shall show that it is proportional to the delta-function: $S_x(\omega)=\mathrm{Var}_w\delta(\omega-\omega_1)/2$ $(0<\omega<\infty)$. Indeed, for such a spectral density the correlation function

$$R_x(\tau)=\int_0^\infty S_x(\omega)\cos\omega\tau\,d\omega$$

$$=\int_0^\infty \frac{\mathrm{Var}_w}{2}\delta(\omega-\omega_1)\cos\omega\tau\,d\omega=\frac{\mathrm{Var}_w}{2}\cos\omega_1\tau,$$

which coincides with the correlation function for $X(t)$. And since the direct and inverse Fourier transformations define the spectral density and the correlation function one-to-one, the expression for $S_x(\omega)$ written above yields the spectral density of the random function $X(t)$.

If we use the complex form of the Fourier transformations rather than a real one, we get the spectral density $S_x^*(\omega)$ in the form

$$S_x^*(\omega)=\mathrm{Var}_w[\delta(\omega+\omega_1)+\delta(\omega-\omega_1)]/4 \quad (-\infty<\omega<\infty).$$

Note that by analogy we could also write $S_x(\omega)=\mathrm{Var}_w[\delta(\omega+\omega_1)+\delta(\omega-\omega_1)]/2$, but $\delta(\omega+\omega_1)\equiv 0$ for positive ω (since $\omega_1>0$).

9.55. Show that the sum of elementary random functions of the form

$$X(t)=m_x+\sum_{i=0}^{\infty}W_i\cos(\omega_i t+\Theta_i),$$

where W_i are centred random variables with variance Var_i $(i=0,1,2,\ldots)$, Θ_i is a random variable uniformly distributed on the interval $(0, 2\pi)$ $(i=0,1,2,\ldots)$ (all the random variables W_i, Θ_i being independent), is the spectral decomposition (9.0.24) of the random function $X(t)$.

Solution. In accordance with the solution of the preceding problem

$$W_i\cos(\omega_i t-\Theta_i)=U_i\cos\omega_i t+V_i\sin\omega_i t,$$

where U_i and V_i are uncorrelated random variables with zero mean values. Consequently

$$X(t)=m_x+\sum_{i=0}^{\infty}(U_i\cos\omega_i t+V_i\sin\omega_i t),$$

and this is what we wished to prove.

9.56. Consider a stochastic process $Y(t) = \sum_{i=1}^{n} a_i X_i(t) + b$, where $X_i(t)$ are stationary uncorrelated stochastic processes with characteristics m_i, $R_i(\tau)$ and $S_i^*(\omega)$ $(i = 1, 2, \ldots, n)$, a_i, b being real numbers. Find the characteristics of the stochastic process $Y(t)$.

Answer. $m_y = \sum_{i=1}^{n} a_i m_i + b$, $R_y(\tau) = \sum_{i=1}^{n} a_i^2 R_i(\tau)$, $S_y^*(\omega) = \sum_{i=1}^{n} a_i^2 S_i^*(\omega)$. The stochastic process $Y(t)$ is stationary.

9.57. Consider a stochastic process $Y(t) = \prod_{l=1}^{n} X_l(t)$, where $X_l(t)$ are independent stationary stochastic processes with characteristics $m_l = 0$, $R_l(\tau)$ and $S_l^*(\omega)$ $(l = 1, 2, \ldots, n)$. Find these characteristics.

Answer.

$$m_y = \prod_{l=1}^{n} m_l, \quad R_y(\tau) = \prod_{l=1}^{n} R_l(\tau), \quad S_y^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{l=1}^{n} R_l(\tau)\, e^{i\omega\tau}\, d\tau.$$

The process $Y(t)$ is stationary.

9.58. Find the characteristics of the random function $X(t)$ whose spectral decomposition is given in Problem 9.55.

Solution. We designate $W_i(\cos \omega_i t - \Theta) = X_i(t)$, and then

$$X(t) = m_x + \sum_{i=1}^{\infty} X_i(t).$$

In accordance with the solution of problems 9.54 and 9.56 we have

$$\mathrm{M}[X(t)] = m_x, \quad R_x(\tau) = \sum_{i=1}^{\infty} R_{x_i}(\tau) = \frac{1}{2} \sum_{i=1}^{\infty} \mathrm{Var}_i \cos \omega_i \tau,$$

$$S_x^*(\omega) = \sum_{i=1}^{\infty} \frac{\mathrm{Var}_i}{4} [\delta(\omega + \omega_i) + \delta(\omega - \omega_i)] \quad (-\infty < \omega < \infty).$$

The random function $X(t)$ is stationary but not ergodic.

9.59. Consider a product of two uncorrelated stationary random functions $Z(t) = X(t)\, Y(t)$, the random function $X(t)$ being the same as in Problem 9.14 (a random alternation of the values $+1$ and -1 with an elementary flow of sign changes), and the random function $Y(t)$ the same as in Problem 9.54. Find the characteristics of the random function $Z(t)$.

Solution. We have $m_x(t) = m_y(t) = 0$, $m_z(t) = 0$, $R_x(t, t') = e^{-2\lambda|\tau|}$ and $R_y(t, t') = (\mathrm{Var}_w \cos \omega_1 \tau)/2$ $(\tau = t' - t)$. On the basis of Problem 9.57

$$R_z(t, t') = R_x(t, t')\, R_y(t, t') = (\mathrm{Var}_w e^{-2\lambda|\tau|} \cos \omega_1 \tau)/2.$$

Fig. 9.59 shows one of the possible realizations of the random function $Z(t)$ which is obtained by multiplying the corresponding ordinates of the realizations of the random functions $X(t)$ and $Y(t)$.

9.60*. *The criterion of positive definiteness.* We have a function $R_x(\tau)$ with properties

$$(1)\ R_x(-\tau)=R_x(\tau),\quad (2)\ R_x(0)>0,\quad (3)\ |R_x(\tau)|\leqslant R_x(0).$$

Find out whether the function $R_x(\tau)$ may be a correlation function of a stationary random function, i.e. whether it has the property of positive definiteness. Show that the sufficient condition for positive definiteness is the condition that the function

$$S_x(\omega)=\frac{2}{\pi}\int_0^\infty R_x(\tau)\cos\omega\tau\,d\tau \tag{9.60.1}$$

be nonnegative for any value of ω,

$$S_x(\omega)\geqslant 0, \tag{9.60.2}$$

i.e. that when calculating the spectral density by formula (9.60.1), we should not get negative values of this function for any ω.

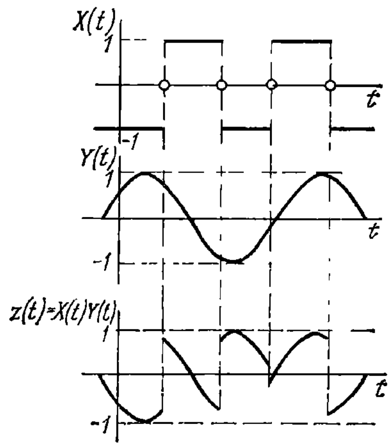


Fig. 9.59

Solution. We assume $S_x(\omega)\geqslant 0$ and prove that in that case the function $R_x(\tau)=R_x(t-t')$ is positive definite. We have

$$R_x(\tau)=\int_0^\infty S_x(\omega)\cos\omega\tau\,d\omega$$

$$=\int_0^\infty S_x(\omega)\cos\omega t\cos\omega t'\,d\omega+\int_0^\infty S_x(\omega)\sin\omega t\sin\omega t'\,d\omega. \tag{9.60.3}$$

The positive definiteness of the function $R_x(t-t')$ means that for the function $\varphi(t)$ and any domain of integration (B) the following condition must be fulfilled:

$$\int_{(B)}\int_{(B)} R_x(t-t')\,\varphi(t)\,\varphi(t')\,dt\,dt'\geqslant 0.$$

Let us verify this inequality with respect to function (9.60.3):

$$\int_{(B)}\int_{(B)}\left\{\int_0^\infty S_x(\omega)\cos\omega t\cos\omega t'\,\varphi(t)\,\varphi(t')\,d\omega\right.$$

$$+\int_0^\infty S_x(\omega)\sin\omega t\sin\omega t'\,\varphi(t)\,\varphi(t')\,d\omega\Big\}\,dt\,dt'$$

$$=\int_0^\infty S_x(\omega)\Big\{\int_{(B)}\cos\omega t\,\varphi(t)\,dt\int_{(B)}\cos\omega t'\,\varphi(t')\,dt'$$

$$+\int_{(B)}\sin\omega t\,\varphi(t)\,dt\int_{(B)}\sin\omega t'\,\varphi(t')\,dt'\Big\}\,d\omega.$$

Designating

$$\int_{(B)}\cos\omega t\varphi(t)\,dt=\psi_1(B,\ \omega),\qquad \int_{(B)}\sin\omega t\,\varphi(t)\,dt=\psi_2(B,\omega),$$

we obtain

$$\int_{(B)}\int_{(B)}R_x(t-t')\,\varphi(t)\,\varphi(t')\,dt\,dt'$$

$$=\int_0^\infty S_x(\omega)\left([\psi_1(B,\ \omega)]^2+[\psi_2(B,\ \omega)]^2\right)d\omega\geqslant 0,$$

since by the hypothesis $S_x(\omega)\geqslant 0$.

We can prove that condition (9.60.3) is not only sufficient but also necessary for the correlation function to be positive definite.

9.61*. Find out whether the function

$$R_x(\tau)=e^{-\alpha|\tau|}\left(\cosh\beta\tau+\frac{\alpha}{\beta}\sinh\beta|\tau|\right)\quad(\alpha>0,\ \beta>0)$$

possesses the properties of a correlation function.

Solution. We must verify that it has the following properties:

(1) $R_x(0)>0$, (2) $k_x(-\tau)=R_x(\tau)$, (3) $|R_x(\tau)|\leqslant R_x(0)$,

(4) $S_x^*(\omega)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}R_x(\tau)\,e^{-i\omega\tau}\,d\tau\geqslant 0$ for any ω.

Properties (1) and (2) are obvious. Let us verify the others.

(3) The function $R_x(\tau)$ is even and, therefore, it is sufficient to investigate its behaviour for $\tau\geqslant 0$:

$$R_x(\tau)=\frac{1}{2}e^{-(\alpha-\beta)\tau}\left(\frac{\alpha}{\beta}+1\right)-\frac{1}{2}e^{-(\alpha+\beta)\tau}\left(\frac{\alpha}{\beta}-1\right).$$

Since $R_x(0)=1$, this expression cannot exceed unity in absolute value. For $\alpha<\beta$ this condition is not fulfilled since $e^{-(\alpha-\beta)\tau}$ increases indefinitely as $\tau\to\infty$. If $\alpha=\beta$, we get $R_x(\tau)\equiv 1$, and for $\alpha>\beta$ we

have $-R_x(\tau) \leqslant 1$. Thus property (3) is fulfilled only for $\alpha \geqslant \beta$.

$$(4)\ S_x^*(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi} \operatorname{Re}\left\{\left(1+\frac{\alpha}{\beta}\right)\right.$$

$$\left. \times \int_0^{\infty} e^{-(\alpha-\beta+i\omega)\tau}\, d\tau + \left(1-\frac{\alpha}{\beta}\right) \int_0^{\infty} e^{-(\alpha+\beta+i\omega)\tau}\, d\tau \right\}$$

$$= \frac{1}{2\pi} \operatorname{Re}\left\{ \frac{\beta+\alpha}{\alpha-\beta+i\omega} + \frac{\beta-\alpha}{\alpha+\beta+i\omega} \right\}$$

$$= \frac{\alpha^2-\beta^2}{2\pi\beta} \left\{ \frac{1}{(\alpha-\beta)^2+\omega^2} - \frac{1}{(\alpha+\beta)^2+\omega^2} \right\}$$

$$= \frac{\alpha^2-\beta^2}{\pi} \cdot \frac{2\alpha}{[(\alpha-\beta)^2+\omega^2]\,[(\alpha+\beta)^2+\omega^2]}$$

for $\alpha \geqslant \beta$ (Re is the real part of a complex number). For $\alpha = \beta$ we have $S_x^*(\omega) = \delta(\omega)$.

Thus for $\alpha \geqslant \beta$ the function $R_x(\tau) = e^{-\alpha|\tau|}\left(\cosh \beta\tau + \frac{\alpha}{\beta} \times \right.$ $\left. \times \sinh \beta\,|\tau|\right)$ possesses all the properties of a correlation function. The graphs of $R_x(\tau)$ and $S_x^*(\omega)$ for $\alpha > \beta$ are shown in Fig. 9.61*a*, *b*.

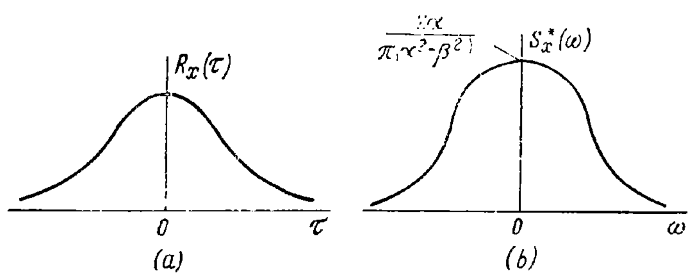


Fig. 9.61

9.62. Show that there is no stationary random function $X(t)$ whose correlation function $R_x(\tau)$ is constant in some interval $(-\tau_1, \tau_1)$ and equal to zero in its exterior.

Solution. Let us assume the contrary, i.e. that there is a random function $X(t)$ for which the correlation function is equal to $b > 0$ for $|\tau| < \tau_1$ and to zero for $|\tau| > \tau_1$. We shall try to find the spectral density of the random function $X(t)$:

$$S_x(\omega) = \frac{1}{\pi} \int_0^{\infty} R_x(\tau) \cos \omega\tau\, d\tau = \frac{1}{\pi} \int_0^{\tau_1} b \cos \omega\tau\, d\tau = \frac{b}{\pi} \frac{\sin \omega\tau_1}{\omega}.$$

It can be seen from this expression that the function $S_x(\omega)$ is negative for some values of ω, and this contradicts the properties of the spectral

density and, consequently, there can be no correlation function of the kind described.

9.63. Does the function $R_x(\tau) = \mathrm{Var}_x\, e^{-\alpha|\tau|} \frac{\alpha}{\beta} \sin \beta\,|\tau|$ possess the properties of a correlation function?

Answer. No, it does not, since the following two conditions are not fulfilled: $R_x(0) > 0$ and $R_x(0) \geqslant |R_x(\tau)|$.

9.64. A stationary random function $X(t)$ has the characteristics m_x and $R_x(\tau)$. Find the crosscorrelation function $R_{xy}(t, t')$ of the random function $X(t)$ and of the random function $Y(t) = 1 - X(t)$.

Solution.

$$R_{xy}(t, t') = \mathrm{M}[\overset{\circ}{X}(t)\overset{\circ}{Y}(t')] = \mathrm{M}[-\overset{\circ}{X}(t)\overset{\circ}{X}(t')]$$
$$= -R_x(t, t') = -R_x(\tau).$$

9.65. A random function $X(t)$ has the characteristics m_x, $R_x(\tau) = \mathrm{Var}_x\, e^{-\alpha|\tau|}$. Find its spectral density.

Solution.

$$S_x^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{\mathrm{Var}_x}{\pi}\,\mathrm{Re}\int_0^{\infty} e^{-(\alpha+i\omega)\tau}\, d\tau,$$

where Re is the real part. We have

$$\int_0^{\infty} e^{-(\alpha+i\omega)\tau}\, d\tau = \int_0^{\infty} \frac{1}{\alpha+i\omega}\, e^{-(\alpha+i\omega)\tau}\, d(\alpha+i\omega)\tau$$

$$= \left|\begin{array}{l} (\alpha+i\omega)\tau = y \\ \text{for } \tau = 0;\ y = 0 \\ \text{for } \tau = \infty;\ y = \infty \end{array}\right| = \frac{1}{\alpha+i\omega}\int_0^{\infty} e^{-y}\, dy$$

$$= \frac{1}{\alpha+i\omega},\quad \mathrm{Re}\int_0^{\infty} e^{-(\alpha+i\omega)\tau}\, d\tau = \mathrm{Re}\,\frac{1}{\alpha+i\omega}$$

$$= \mathrm{Re}\,\frac{1}{\alpha+i\omega}\,\frac{\alpha-i\omega}{\alpha-i\omega} = \mathrm{Re}\,\frac{\alpha-i\omega}{\alpha^2+\omega^2}$$

$$= \mathrm{Re}\left(\frac{\alpha}{\alpha^2+\omega^2} - i\,\frac{\omega}{\alpha^2+\omega^2}\right) = \frac{\alpha}{\alpha^2+\omega^2}.$$

Consequently, $S_x^*(\omega) = \frac{\mathrm{Var}_x}{\pi}\,\frac{\alpha}{\alpha^2+\omega^2}$.

9.66. Find the spectral density of the random function $X(t)$ if its correlation function

$$R_x(\tau) = \mathrm{Var}_x\, e^{-\alpha|\tau|}\cos\beta\tau.$$

Answer.

$$S_x^*(\omega) = \frac{\mathrm{Var}_x\alpha}{\pi}\,\frac{\alpha^2+\beta^2+\omega^2}{[\alpha^2+(\beta-\omega)^2]\,[\alpha^2+(\beta+\omega)^2]}.$$

Hint. Represent $\cos \beta\tau = (e^{i\beta\tau} + e^{-i\beta\tau})/2$ and use the solution of Problem 9.65.

9.67. Find the spectral density of a stationary random function whose correlation function

$$R_x(\tau) = \mathrm{Var}_x\, e^{-(\lambda\tau)^2}.$$

Solution. We have

$$S_x^*(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Var}_x\, e^{-(\lambda\tau)^2} e^{-i\omega\tau}\, d\tau = \frac{\mathrm{Var}_x}{2\pi}\int_{-\infty}^{\infty} e^{-(\lambda\tau)^2 - i\omega\tau}\, d\tau.$$

Using the familiar formula

$$\int_{-\infty}^{\infty} e^{-Ax^2 \pm 2Bx - C}\, dx = \sqrt{\frac{\pi}{A}} \exp\left[-\frac{AC-B^2}{A}\right] \quad (A > 0)$$

and bearing in mind that $i^2 = -1$, we get

$$S_x^*(\omega) = \frac{\mathrm{Var}_x}{2\pi}\sqrt{\frac{\pi}{\lambda^2}} \exp\left[-\frac{\omega^2}{4\lambda^2}\right] = \frac{\mathrm{Var}_x}{4\lambda\sqrt{\pi}} \exp\left[-\frac{\omega^2}{4\lambda^2}\right].$$

The graph of this function is like that of a normal distribution.

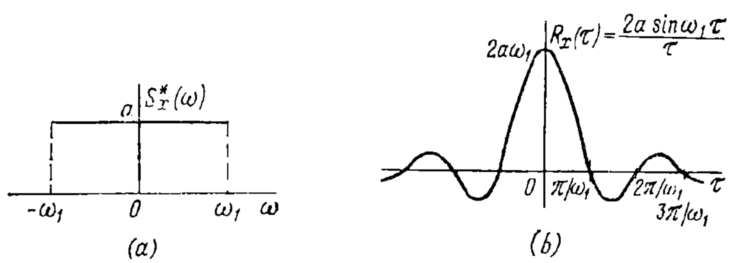


Fig. 9.68

9.68. The spectral density of a stationary random function $X(t)$ is constant on the interval from $-\omega_1$ to $+\omega_1$ and zero outside it, i.e. has the form shown in Fig. 9.68*a*:

$$S_x^*(\omega) = \begin{cases} a & \text{for} \quad |\omega| < \omega_1 \\ 0 & \text{for} \quad |\omega| > \omega_1 \end{cases} = a1\left(1 - \frac{|\omega|}{\omega_1}\right).$$

Find the correlation function $R_x(\tau)$ of the random function $X(t)$.

Solution.

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x^*(\omega)\, e^{i\omega\tau}\, d\omega = 2a\int_0^{\omega_1} \cos \omega\tau\, d\omega = \frac{2a \sin \omega_1\tau}{\tau},$$

$$\mathrm{Var}_x = R_x(0) = 2a\omega_1.$$

The graph of the correlation function is shown in Fig. 9.68*b*.

9.69. *The derivative of a stationary random function.* Given a stationary random function with $m_x(t) = m_x$, $R_x(t, t') = R_x(\tau)$, where $\tau = t' - t$, find the characteristics of its derivative $Y(t) = dX(t)/dt$ and show that it is stationary too.

Solution. Since $Y(t)$ is related to $X(t)$ via a homogeneous linear transformation, it follows that

$$m_y(t) = \frac{d}{dt} m_x(t) = \frac{d}{dt} m_x = 0 = \text{const},$$

$$R_y(t, t') = \frac{\partial^2}{\partial t\, \partial t'} R_x(t, t') = \frac{\partial^2}{\partial t\, \partial t'} R_x(\tau)$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t'} R_x(\tau)\right) = \frac{\partial}{\partial t}\left(\frac{d}{d\tau} R_x(\tau) \frac{\partial \tau}{\partial t'}\right).$$

But $\partial\tau/\partial t' = 1$ and $\partial\tau/\partial t = -1$ and, therefore,

$$R_y(t, t') = \frac{\partial}{\partial t}\left[\frac{d}{d\tau} R_x(\tau)\right] = \frac{\partial^2}{\partial \tau^2} R_x(\tau) \frac{\partial \tau}{\partial t} = -\frac{d^2}{d\tau^2} R_x(\tau).$$

Since the right-hand side of the equation depends only on τ we have

$$R_y(t, t') = R_y(\tau) = -\frac{d^2}{d\tau^2} R_x(\tau),$$

and the random function $Y(t)$ is stationary.

9.70. A stationary random function $X(t)$ has a correlation function $R_x(\tau)$. A random function $Y(t)$ results from its differentiation: $Y(t) = dX(t)/dt$. Find the correlation function $R_y(\tau)$ if

$$\text{(a)}\ R_x(\tau) = e^{-\alpha|\tau|}; \qquad \text{(b)}\ R_x(\tau) = e^{-\alpha|\tau|}(1 + \alpha|\tau|);$$

$$\text{(c)}\ R_x(\tau) = e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta|\tau|\right) \quad (\alpha > 0,\ \beta > 0).$$

Solution. To solve the problem, we shall use the apparatus of generalized functions, the rules for which are given in Appendix 6.

$$\text{(a)}\ R_y(\tau) = -\frac{d^2}{d\tau^2} e^{-\alpha|\tau|} = -\frac{d}{d\tau}\left[-\alpha e^{-\alpha|\tau|} \frac{d|\tau|}{d\tau}\right]$$

$$= \alpha\left[-\alpha e^{-\alpha|\tau|}\left(\frac{d|\tau|}{d\tau}\right)^2 + \frac{d^2|\tau|}{d\tau^2} e^{-\alpha|\tau|}\right]$$

$$= \alpha e^{-\alpha|\tau|}[2\delta(\tau) - \alpha(\text{sign}\,\tau)^2].$$

The presence of the term $2\delta(\tau)$ shows that there is white noise in the function $Y(t)$. Next we write the answers and invite the reader to verify them:

$$\text{(b)}\ R_y(\tau) = \alpha^2 e^{-\alpha|\tau|}(1 - \alpha|\tau|),$$

$$\text{(c)}\ R_y(\tau) = (\alpha^2 + \beta^2)e^{-\alpha|\tau|}\left(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta|\tau|\right).$$

9.71. Find the spectral density of a stationary random function with a correlation function

$$R_y(\tau) = \alpha e^{-\alpha|\tau|}[2\delta(\tau) - \alpha(\text{sign}\,\tau)^2].$$

Solution.

$$S_y^*(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} R_y(\tau)\,e^{-i\omega\tau}\,d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha e^{-\alpha|\tau|}2\delta(\tau)\,e^{-i\omega\tau}\,d\tau-\frac{1}{2\pi}\int_{-\infty}^{\infty}(\operatorname{sign}\tau)^2 e^{-\alpha|\tau|}e^{-i\omega\tau}\,d\tau.$$

Since

$$(\operatorname{sign}\tau)^2=\begin{cases}1 & \text{for } \tau\neq 0,\\ 0 & \text{for } \tau=0\end{cases}$$

and since the integrand of the second integral has no singularities at the point $\tau=0$, we can neglect $\tau=0$ in the second integral. We get

$$S_y^*(\omega)=\frac{\alpha}{\pi}-\frac{\alpha^2}{2\pi}\,\frac{2\alpha}{\alpha^2+\omega^2}=\frac{\alpha}{\pi}\,\frac{\omega^2}{\alpha^2+\omega^2}.$$

The graph of the spectral density $S_y^*(\omega)$ is shown in Fig. 9.71.

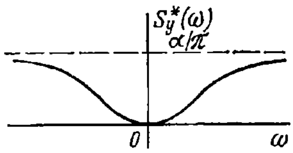


Fig. 9.71

We could have obtained the spectral density $S_y^*(\omega)$ in a simpler way. We represent the random function $Y(t)$ as the derivative of the random function $X(t)$ from Problem 9.70 (a). We have

$$R_x(\tau)=e^{-\alpha|\tau|},\quad S_x^*(\omega)=\frac{1}{\pi}\,\frac{\alpha}{\alpha^2+\omega^2}.$$

The amplitude-frequency characteristic of the differentiation operator is $\Phi(i\omega)=i\omega$, and, consequently,

$$S_y^*(\omega)=S_x^*(\omega)\,|\Phi(i\omega)|^2=\frac{1}{\pi}\,\frac{\alpha}{\alpha^2+\omega^2}\,|i\omega|^2=\frac{\alpha}{\pi}\,\frac{\omega^2}{\alpha^2+\omega^2}.$$

9.72. Given a stationary random function $X(t)$ with a correlation function $R_x(\tau)=(\sin\tau)/\tau$, find the correlation function, variance and spectral density of its derivative: $Y(t)=dX(t)/dt$.

Solution. According to the solution of Problem 9.69,

$$R_y(\tau)=-\frac{d^2R_x(\tau)}{d\tau^2}=-\frac{d^2}{d\tau^2}\left(\frac{\sin\tau}{\tau}\right).$$

Expanding $(\sin\tau)/\tau$ into a Maclaurin series, we get

$$\frac{\sin\tau}{\tau}=1-\frac{\tau^2}{3!}+\frac{\tau^4}{5!}-\frac{\tau^6}{7!}+\dots,$$

$$R_y(\tau)=-\frac{d^2}{d\tau^2}R_x(\tau)=\frac{1}{3}-\frac{\tau^2}{2!\cdot 5}+\frac{\tau^4}{4!\cdot 7}-\dots.$$

Hence $\mathrm{Var}_y=R_y(0)=1/3$.

The spectral density $S_x^*(\omega)=\frac{1}{2}\cdot 1\,(1-|\omega|)$ (see item 15 of Appendix 6). Consequently, $S_y^*(\omega)=S_x^*(\omega)\,\omega^2=\frac{1}{2}\,\omega^2\cdot 1\,(1-|\omega|)$.

9.73. A random function $X(t)$ has a correlation function

$$R_x(\tau)=e^{-\alpha|\tau|}\left(\cosh\beta\tau+\frac{\alpha}{\beta}\sinh\beta|\tau|\right)\quad(\alpha\geqslant\beta>0).$$

A random function $Y(t)=dX(t)/dt$. Find its correlation function $R_y(\tau)$ and spectral density $S_y^*(\omega)$.

Solution. To find $R_y(\tau)$, we use the properties of generalized functions (see [10]):

$$R_y(\tau)=-\frac{d^2}{d\tau^2}R_x(\tau)=-\frac{d}{d\tau}\left[-e^{-\alpha|\tau|}\alpha\frac{d|\tau|}{d\tau}\left(\cosh\beta\tau\right.\right.$$

$$\left.\left.+\frac{\alpha}{\beta}\sinh\beta|\tau|\right)+e^{-\alpha|\tau|}\left(\beta\sinh\beta\tau+\alpha\cosh\beta|\tau|\frac{d|\tau|}{d\tau}\right)\right]$$

$$=\frac{d}{d\tau}\left(\frac{\beta^2-\alpha^2}{\beta}e^{-\alpha|\tau|}\sinh\beta\tau\right)$$

$$=\frac{\alpha^2-\beta^2}{e^{\alpha|\tau|}}\left[\cosh\beta|\tau|-\frac{\alpha}{\beta}\sinh\beta|\tau|\right],$$

$$S_y^*(\omega)=\frac{2\alpha\omega^2}{\pi}\,\frac{\alpha^2-\beta^2}{[(\alpha-\beta)^2+\omega^2]\,[(\alpha+\beta)^2+\omega^2]}.$$

Since the limit $\lim\limits_{\tau\to 0}R_y(\tau)$ exists (it is equal to $R_y(0)=\alpha^2-\beta^2$), the random function $X(t)$ is differentiable.

9.74. Show that the crosscorrelation function $R_{xy}(t,t')$ of a stationary random function $X(t)$ and its derivative $Y(t)=dX(t)/dt$ satisfies the condition $R_{xy}(t,t')=-R_{xy}(t',t)$, i.e. changes when the arguments change places.

Solution. Assume $R_x(t,t')=R_x(\tau)$, where $\tau=t'-t$.

$$R_{xy}(t,t')=\mathrm{M}\left[\mathring{X}(t)\frac{d}{dt'}\mathring{X}(t')\right]=\frac{\partial}{\partial t'}\mathrm{M}\,[\mathring{X}(t)\,\mathring{X}(t')]=\frac{\partial}{\partial t'}R_x(\tau).$$

But $\tau=t'-t$ and, consequently,

$$R_{xy}(t,t')=\frac{d}{d\tau}R_x(\tau)\frac{\partial\tau}{\partial t'}=\frac{d}{d\tau}R_x(\tau).$$

On the other hand,

$$R_{xy}(t',t)=\frac{\partial}{\partial t}R_x(t,t')=\frac{d}{d\tau}R_x(\tau)\frac{\partial\tau}{\partial t}=-\frac{d}{d\tau}R_x(\tau)=-R_{xy}(t,t').$$

and that is what we had to prove.

Thus the crosscorrelation function of a stationary random function and its derivative depends only on $\tau=t'-t$:

$$R_{xy}(\tau)=-\frac{d}{d\tau}R_x(\tau),$$

i.e. the function $X(t)$ and its derivative are "stationary correlated".

9.75. Prove that any stationary random function $X(t)$ and the value of its derivative $Y(t) = dX(t)/dt$ at the same point t are uncorrelated, and if the random function $X(t)$ has a normal distribution, then they are, in addition, independent.

Solution. It was shown in the preceding problem that $R_{xy}(\tau) = -\frac{d}{d\tau} R_x(\tau)$. For $t' = t$, $\tau = 0$

$$R_{xy}(0) = -\frac{d}{d\tau} R_x(0) = -\frac{d}{d\tau} \operatorname{Var}_x = 0.$$

Thus a stationary random function at any point is not correlated with its derivative at the same point.

For a normally distributed random function, the lack of correlation between $X(t)$ and its derivative at the same point implies their independence.

9.76. Find the characteristics of a random function $Y(t) = X(t) \times \frac{dX(t)}{dt}$ if the normal random function $X(t)$ is stationary and has characteristics m_x, $R_x(\tau) = \operatorname{Var} e^{-\alpha|\tau|}\left(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau\right)$.

Solution. $Y(t) = \frac{1}{2}\frac{d[X(t)]^2}{dt}$. We designate $X^2(t) = Z(t)$. In accordance with (9.0.35) and (9.0.36), we have

$$m_z = m_x^2 + R_x(0) = m_x^2 + \operatorname{Var},$$

$$S_z^*(\omega) = 2\int_{-\infty}^{\infty} S_x^*(\omega - v)\, S_x^*(v)\, dv + 4m_x^2 S_x^*(\omega)$$

$$= \frac{\operatorname{Var}\alpha}{\pi}(\alpha^2+\beta^2)\frac{\omega^2(\omega^2+20\alpha^2+4\beta^2)}{[(\omega^2+4\alpha^2+4\beta^2)^2-16\beta^2\omega^2](\omega^2+4\alpha^2)}$$

$$+ 4m_x^2\frac{\operatorname{Var}\alpha}{\pi}\frac{2(\alpha^2+\beta^2)}{(\omega^2+\alpha^2-\beta^2)^2+4\alpha^2\beta^2} = S_{x^2}^*(\omega).$$

We can find the correlation function $R_z(\tau)$ from the expression

$$R_z(\tau) = \int_{-\infty}^{\infty} S_z^*(\omega)\, e^{i\omega\tau}\, d\omega.$$

Hence

$$m_y(t) = \frac{1}{2}\frac{d}{dt}m_z = 0, \quad S_y^*(\omega) = \frac{1}{2}\omega^2 S_z^*(\omega).$$

9.77. *Quadratic rectification of a stationary random function.* A quadratic rectification of a random function $X(t)$ is a nonlinear transformation of the form $Z(t) = a^2X^2(t)$, where a is a constant and $X(t)$ is a random function. Find the characteristics of the random function at the output of a quadratic rectifier with parameter $a = 2$ if a normal

stationary function with characteristics $m_x = 3$, $R_x(\tau) = 4e^{-3|\tau|}$ arrives at its input.

Solution. We designate $X_1(t) = aX(t)$. The characteristics of the random function $X_1(t)$ are $m_{x_1} = am_x = 6$, $R_{x_1}(\tau) = a^2R_x(\tau) = 16e^{-3|\tau|}$. Then $Z(t) = X_1^2(t)$. We set $\mathrm{Var}_1 = 16$, $\alpha_1 = 3$. From formula (9.0.35) we find that $m_z = m_{x_1}^2 + R_{x_1}(0) = 36 + 16 = 52$. From formula (9.0.36)

$$S_z^*(\omega) = 2\int_{-\infty}^{\infty} S_{x_1}^*(\omega - v)\, S_{x_1}^*(v)\, dv + 4m_{x_1}^2 S_{x_1}^*(\omega)$$

$$= 2\int_{-\infty}^{\infty} \frac{\mathrm{Var}_1\,\alpha_1}{\pi}\,\frac{1}{\alpha_1^2+(\omega-v)^2}\,\frac{\mathrm{Var}_1\,\alpha_1}{\pi}\,\frac{1}{\alpha_1^2+v^2}\,dv$$

$$+ 4m_{x_1}^2\,\frac{\mathrm{Var}_1\,\alpha_1}{\pi}\,\frac{1}{\alpha_1^2+\omega^2} = \frac{4\,\mathrm{Var}_1^2\,\alpha_1}{\pi}\,\frac{1}{(2\alpha_1)^2+\omega^2} + \frac{4\,\mathrm{Var}_1\,\alpha_1 m_{x_1}^2}{\pi\,(\alpha_1^2+\omega^2)}.$$

Consequently (see item 6 in Appendix 7),

$$R_z(\tau) = R_1(\tau) + R_2(\tau) = \widetilde{\mathrm{Var}}_1\, e^{-\tilde{\alpha}_1|\tau|} + \widetilde{\mathrm{Var}}_2\, e^{-\tilde{\alpha}_2|\tau|},$$

where $\widetilde{\mathrm{Var}}_1 = 2\mathrm{Var}_1^2 = 2 \times 16^2 = 512$, $\tilde{\alpha}_1 = 2\alpha_1 = 6$, $\widetilde{\mathrm{Var}}_2 = 4\mathrm{Var}_1\, m_{x_1}^2 = 4 \times 16 \times 6^2 = 2\,304$, $\tilde{\alpha}_2 = 3$.

Thus the signal at the output of a quadratic rectifier can be represented as the sum of two uncorrelated stationary stochastic processes $Z(t) = X_1(t) + X_2(t)$ with the characteristics indicated above. It should be emphasized that the stochastic process $Z(t)$ is not normal.

9.78. A normal random function $X(t)$ has characteristics m_x, $R_x(\tau) = \mathrm{Var}_x e^{-\alpha|\tau|}$. Find the characteristics of the random function $W(t) = X(t)\dfrac{dX(t)}{dt}$.

Solution.

$$W(t) = X(t)\frac{dX(t)}{dt} = \frac{1}{2}\frac{d}{dt}X^2(t),$$

$$S_w^*(\omega) = 0.5\omega^2 S_{x^2}^*(\omega).$$

Since the differentiation is linear, we have $m_w(t) = \frac{1}{2}\frac{d}{dt}m_{x^2}(t)$. But in accordance with the solution of the preceding problem, $m_{x^2}(t) = \text{const}$ and, consequently, $m_w(t) = 0$. In the preceding problem we found the spectral density $S_{x^2}(\omega)$, and, consequently,

$$S_w^*(\omega) = \frac{1}{2}\omega^2 S_{x^2}^*(\omega) = \frac{\mathrm{Var}_x^2\, 2\alpha}{\pi}\,\frac{\omega^2}{\omega^2+(2\alpha)^2} + \frac{2\,\mathrm{Var}_x\, m_x^2\alpha\omega^2}{\pi\,(\omega^2+\alpha^2)}$$

and in accordance with item 18 of Appendix 7 we have

$$R_w(\tau) = \frac{\mathrm{Var}_x^2}{\pi}\, e^{-2\alpha|\tau|}\,[2\delta(\tau) - 2\alpha\,(\mathrm{sign}\,\tau)^2]$$

$$+ \frac{2\,\mathrm{Var}_x\, m_x^2}{\pi}\, e^{-\alpha|\tau|}\,[2\delta(\tau) - \alpha\,(\mathrm{sign}\,\tau)^2].$$

We infer from the form of the correlation function $R_w(\tau)$ that the random function $W(t)$ contains white noise.

9.79. The angle of displacement of a radar in the horizontal plane is a normal stochastic process $X(t)$ with mean value $m_x = 0$ and correlation function $R_x(\tau) = \mathrm{Var}_x\, e^{-\alpha|\tau|} \cos \beta\tau$, where $\mathrm{Var}_x = 4\ \mathrm{deg}^2$; $\alpha = 10^{-3}$ 1/s; $\beta = 0.1$ 1/s. At the initial moment $t = 0$ the displacement angle is zero: $X(0) = 0$. Find the probability p that at the moment $t' = 0.2$ s the displacement angle will be smaller than 1°.

Solution. We designate $X(0) = X_0$, $X(t_1) = X_1$. Under the condition that the random variable $X_0 = x_0$, we can find the conditional distribution of the random variable X_1 from the expression $f(x_1 \mid x_0) = f(x_1, x_0)/f(x_1)$, where $f(x_1, x_0)$ is a normal distribution of a system of two random variables with characteristics $m_1 = m_0 = 0$;

$$\mathrm{Var}_1 = \mathrm{Var}_0 = \mathrm{Var}_x = 4\ \mathrm{deg}^2,$$
$$R_{x_1, x_0} = R_x(0.2) = \mathrm{Var}_x\, e^{-\alpha|0.2|} \cos \beta \times 0.2 = 3.68\ \mathrm{deg}^2,$$
$$r_{x_1, x_0} = R_{x_1, x_0}/\sqrt{\mathrm{Var}_{x_1},\ \mathrm{Var}_{x_0}} = 0.921.$$

The conditional distribution $f(x_1 \mid x_0 = 0)$ is normal with characteristics

$$m_{x_1 \mid x_0} = 0, \quad \sigma_{x_1 \mid x_0} = \sigma_2 \sqrt{1 - r^2_{x_1, x_0}} = 0.776.$$

Hence

$$p = \int_{-1}^{1} f(x_1 \mid x_0 = 0)\, dx = 2\Phi\left(\frac{1}{0.776}\right) = 0.81.$$

9.80. The input of an oscillatory unit of an automatic control system, whose transfer function has the form

$$\Phi(p) = k/(Tp^2 + \xi p + k) \quad (\xi > 0),$$

picks up white noise with spectral density $S_x^*(\omega) = N$. Find the variance of the output signal (we speak of sufficiently distant time intervals after the termination of transient processes).

Solution.

$$S_y^*(\omega) = S_x^*(\omega)\, |\Phi(i\omega)|^2 = Nk/(|\, T(i\omega)^2 + \xi i\omega + k\, |^2),$$

whence

$$\mathrm{Var}_y = \int_{-\infty}^{\infty} S_y^*(\omega)\, d\omega = \pi k N/\xi.$$

Note that the variance of the output signal does not depend on the time constant T of the oscillatory unit, but depends only on the amplification factor k, the damping factor ξ and the power of the signal N.

9.81. The transfer function of a system to which a signal $X\ (t)$ is sent has the form

$$\Phi\ (p) = (1 + T_1p)/(T_1^2p^2 + p + k),$$

where $k = 25\ \frac{1}{\text{s}}$ and $T_1 = 0.05$ s.

The spectral density of the input signal

$$S_x^*\ (\omega) = 2T\delta_x/[1 + (T\omega)^2],$$

where $T = 1$ s, and $\delta_x = 4\ \text{deg}^2/\text{s}^2$. Find the variance of the output signal.

Solution.

$$S_y^*\ (\omega) = S_x^*\ (\omega)\ |\Phi\ (i\omega)|^2 = \frac{2\delta_x\ (-T_1^2\ (i\omega)^2+1)}{|TT_1\ (i\omega)^3+(T+T_1)\ (i\omega)^2+(1+kT)\ i\omega+k|^2},$$

$$\text{Var}_y = \int_0^\infty S_y^*\ (\omega)\ d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b_0\ (i\omega)^4+b_1\ (i\omega)^2+b_2}{|a_0\ (i\omega)^3+a_1\ (i\omega)^2+a_2\ i\omega+a_3|^2}\ d\omega$$

$$= \frac{-a_2b_0+a_0b_1-a_0a_1b_2/a_3}{2a_0\ (a_0a_3-a_1a_2)},$$

in this case $b_0 = 0$, $b_1 = -T_1^2$, $b_2 = 1$, $a_0 = TT_1$, $a_1 = T + T_1$, $a_2 = 1 + kT$, $a_3 = k$. Then

$$\text{Var}_y = 4\pi T\delta_x \frac{b_1-a_1b_2/a_3}{2\ (a_0a_3-a_1a_2)} \approx 0.0428\ \text{deg}^2.$$

9.82. The output signal $Y\ (t)$ is related to the input signal $X\ (t)$ by the differential equation

$$a_1 \frac{dX\ (t)}{dt} + a_0X\ (t) = b_1 \frac{dY\ (t)}{dt} + b_0Y\ (t). \tag{9.82}$$

The stationary random function $X\ (t)$ is normal with characteristics m_x and $R_x\ (\tau) = \text{Var}_x\ e^{-\alpha|\tau|}$. Find the characteristics of the output signal $Y\ (t)$.

Solution. Since the random function $X\ (t)$ is subjected to a stationary linear transformation, the random function $Y\ (t)$ is stationary in the steady-state condition. The variable m_y can be found from the equation

$$a_1 \frac{dm_x}{dt} + a_0m_x = b_1 \frac{dm_y}{dt} + b_0m_y.$$

Since m_x = const and m_y = const, it follows that $a_0m_x = b_0m_y$, whence $m_y = a_0m_x/b_0$. Equation (9.82) can be written in an operator form, i.e.,

$$(a_1p + a_0)\ X\ (t) = (b_1p + b_0)\ Y\ (t),$$

whence

$$Y(t) = \frac{a_1 p + a_0}{b_1 p + b_0} X(t) = \Phi(p) X(t),$$

where $\Phi(p)$ is a transfer function.

$$S_y^*(\omega) = S_x^*(\omega) \, |\Phi(i\omega)|^2 = S_x^*(\omega) \left| \frac{a_1 i\omega + a_0}{b_1 i\omega + b_0} \right|^2$$

$$= \frac{\mathrm{Var}_x \alpha}{\pi} \frac{1}{\alpha^2+\omega^2} \frac{a_0^2 + a_1^2\omega^2}{b_0^2 + b_1^2\omega^2} = \frac{\mathrm{Var}_x \alpha}{\pi b_1^2} \frac{1}{\alpha^2+\omega^2} \frac{a_0^2 + a_1^2\omega^2}{b_0^2/b_1^2 + \omega^2}.$$

We can rewrite the expression for $S_y^*(\omega)$ in the form

$$S_y^*(\omega) = \frac{\mathrm{Var}_x \alpha}{\pi b_1^2} \frac{a_0^2 + a_1^2\omega^2}{(\alpha^2+\omega^2)(b_0^2/b_1^2+\omega^2)} = \frac{\mathrm{Var}_x \alpha}{\pi b_1^2} \left(\frac{a}{b^2+\omega^2} + \frac{c}{d^2+\omega^2} \right),$$

where

$$a = a_1^2 - c, \quad c = \frac{a_0^2 - a_1^2 b_0^2/b_1^2}{\alpha^2 - b_0^2/b_1^2}, \quad b = \alpha, \quad d = b_0/b_1.$$

We designate $D_1 = \mathrm{Var}_x \, \alpha a/b$, $D_2 = \mathrm{Var}_x \, \alpha c/(b_1^2 d)$. Then (see item 6 of Appendix 7) $R_y(\tau) = D_1 e^{-b|\tau|} D_2 e^{-d|\tau|}$. The variance of the output signal $\mathrm{Var}_y = R_y(0) = D_1 + D_2$.

9.83. An RC-filter is used to lower the level of noise $X(t)$ (Fig. 9.83). The filter picks up stationary white noise with spectral density $S_x^*(\omega) =$

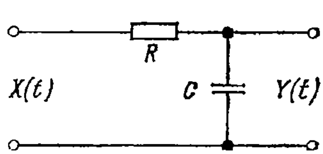


Fig. 9.83

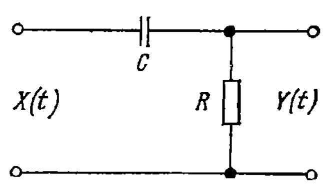


Fig. 9.84

10^{-5} V^{2}s. Find the least time constant $T = RC$ of the filter for which the output signal will exceed 100 mV in absolute value with probability $p = 0.05$ if $m_x = 0$.

Solution. The transfer function of the RC-filter has the form $\Phi(p) = 1/(1 + Tp)$. Consequently,

$$S_y^*(\omega) = \left| \frac{1}{1+Ti\omega} \right|^2 S_x^*(\omega) = \frac{1}{1+T^2\omega^2} \cdot 10^{-5} = \frac{\mathrm{Var}_y \alpha}{\pi} \frac{1}{\alpha^2+\omega^2},$$

where $\alpha = 1/T$ 1/s, $\mathrm{Var}_y = \pi\alpha \times 10^{-5}$ V^2. Hence (see item 5 in Appendix 7) $R_y(\tau) = \mathrm{Var}_y \, e^{-\alpha|\tau|}$.

Since the normal input signal is subjected to a linear transformation, the output signal is also normal with characteristics $m_y = 0$, $\mathrm{Var}_y = (\pi/T) \, 10^{-5}$ V^2. By the hypothesis $p = 1 - 2\Phi(100/\sigma_y) = 0.05$, where $\Phi(x)$ is the error function (Appendix 5). We have $\Phi(100/\sigma_y) = 0.475$, $100/\sigma_y = 1.96$, $\sigma_y = 100/1.96 = 51.1$ mV, $\sigma_y^2 = \mathrm{Var}_y = 26.1 \times 4^{-4}$ V^2.

The smallest time constant $T = (\pi/\mathrm{Var}_y) \, 10^{-5} = 0.012$ s.

9.84. The input of the RC-filter shown in Fig. 9.84 picks up a normal stationary random function $X(t)$ with characteristics m_x, $R_x(\tau) = \text{Var}_x \cos\beta\tau$. Find the characteristics of the output signal $Y(t)$.

Solution. The transfer function of the filter has the form $\Phi(p) = Tp(1+Tp)^{-1}$, where $T = RC$. Consequently, $m_y = 0$ (the constant component does not pass through the capacitor C). The spectral density of the output signal (see item 3 of Appendix 7) has the form

$$S_y^*(\omega) = \left|\frac{Ti\omega}{1+Ti\omega}\right|^2 S_x^*(\omega) = \frac{(T\omega)^2}{1+(T\omega)^2}\,\text{Var}_x\,[\delta(\omega+\beta)+\delta(\omega-\beta)].$$

We can find the variance Var_y of the output signal from the expression (see Appendix 6)

$$\text{Var}_y = \int_{-\infty}^{\infty} S_y^*(\omega)\,d\omega$$

$$= \frac{\text{Var}_x}{2}\int_{-\infty}^{\infty}\frac{(T\omega)^2}{1+(T\omega)^2}\,[\delta(\omega+\beta)+\delta(\omega-\beta)]\,d\omega = \text{Var}_x\,\frac{2(T\beta)^2}{1+(T\beta)^2}.$$

The univariate distribution of the output signal is normal with parameters $m_y = 0$ and Var_y.

9.85. The input of an automatic voltage regulator receives a voltage $X(t)$ which is a normal stationary stochastic process with characteristics $m_x = 220$ V, $R_x(\tau) = \text{Var}_x\, e^{-\alpha|\tau|}(1+\alpha|\tau|)$, where $\text{Var}_x = \sigma_x^2 = 16$ V^2, $\alpha = 200$ 1/s. The voltage regulator operates normally if the voltage $X(t)$ is no lower than 208 V and no higher than 232 V, otherwise it automatically switches off and switches on again when the input voltage $X(t)$ returns to the indicated range. Find the probability p that the regulator will be operating at an arbitrary moment t, the distribution of the duration of the normal operation T_1 and the average duration of the period $\bar{t}_2$ when the operator is switched off.

Solution. In accordance with the solution of Problem 9.70 (*b*),

$$R_y(\tau) = -\frac{\partial^2}{\partial\tau^2}[\text{Var}_x\, e^{-\alpha|\tau|}(1+\alpha|\tau|)]$$

$$= \text{Var}_x\,\alpha^2 e^{-\alpha|\tau|}(1-\alpha|\tau|),\quad \sigma_y = \alpha\sigma_x.$$

The average number of times the level a is crossed is

$$\lambda_a = \frac{1}{2\pi}\left\{\exp\left[-\frac{(a-m_x)^2}{2\sigma_x^2}\right]\right\}\frac{\sigma_y}{\sigma_x} = 0.354 \text{ 1/s}.$$

Consequently, the total average number of times it crosses the permissible level in unit time is $2\lambda_a = \lambda_1 = 0.708$ 1/s. For all practical applications we can consider the distribution of the random variable T_1 to be exponential with expectation $\bar{t} = 1/\lambda_1 = 1.41$ s. The probability that the regulator will be operating at an arbitrary moment is

$$p = \text{P}(208 < X(t) < 232) = \text{P}(|X(t) - m_x| < 12) = 2\Phi(3) = 0.9973.$$

On the other hand, we can find the probability p from the expression $p = \bar{t}_1/(\bar{t}_1 + \bar{t}_2)$, whence $\bar{t}_2 = \bar{t}_1 (1 - p)/p = 1.31 \times 10^{-4}$ s.

9.86. A radio device can function only in a certain temperature range $t_0 \pm 30$ °C, where t_0 is the average rated temperature. When the temperature passes either limit, the device fails. The ambient temperature $X(t)$ is a stationary random function with mean value $m_x = t_0$ and mean square deviation $\sigma_x = 10$ °C, and with a correlation function of the form $R_x(\tau) = 100 (\sin \tau)/\tau$ °C²; the distribution is normal. Find the average number of times the temperature crosses the levels $t_0 \pm 30$ °C during a time period $T = 50$ s. Assuming the number of level crossings to have a Poisson distribution, find the probability that the device will fail during the time T.

Solution. According to the solution of Problem 9.72

$$R_y(\tau) = -\frac{\partial^2 k_x(\tau)}{\partial \tau^2} = -100 \left\{ -\frac{1}{3} + \frac{\tau}{215} - \dots \right\}.$$

Setting $\tau = 0$, we find $\mathrm{Var}_y = R_y(0) = 33.3$, $\sigma_y = 5.76$ °C. According to formula (9.0.42), the average number of times the level $a = m_x + 30$ °C is crossed upwards is

$$\lambda_a = \frac{1}{2\pi} e^{-a^2/200} \frac{\sigma_y}{\sigma_x} = 0.00134 \text{ 1/s}.$$

The average number of downward crossings of $m_x = -30$ °C is the same, hence the total average number of times the permissible limits are crossed is $2\lambda_a = 0.00269$.

For the random variable Y, which is the number of times the level is crossed and which has a Poisson distribution with mean value $2\lambda_a T = 0.00269 \times 50 = 0.1345$, the probability that it exceeds zero (at least one level crossing occurs) is $1 - e^{-0.1345} = 0.125$.

CHAPTER 10

Flows of Events. Markov Stochastic Processes

10.0. A *flow of events* is a sequence of homogeneous events occurring one after another at random moments. Examples are a flow of requests at a telephone exchange; the goals scored during an ice hockey game; the occurrence of faults during the operation of a computer, the job requests at a computing centre, and so on.

A flow of events can be depicted by a series of points with abscissas $\Theta_1, \Theta_2, \ldots, \Theta_n, \ldots$ (Fig. 10.0.1) with intervals $T_1 = \Theta_2 - \Theta_1$, $T_2 = \Theta_3 - \Theta_2, \ldots$, $T_n = \Theta_{n+1} - \Theta_n$. In probability theory a flow of events can be represented as a sequence of random variables: Θ_1, $\Theta_2 = \Theta_1 + T_1$, $\Theta_3 = \Theta_1 + T_1 + T_2, \ldots$.

Note that the term "event" in the concept "flow of events" differs in sense from the term "random event" which we introduced earlier. It is senseless to speak of

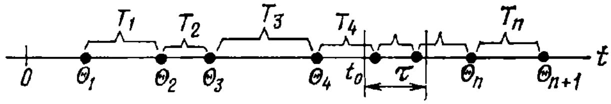


Fig. 10.0.1

the probability of "events" in a flow (say the "probability of a request" at a telephone exchange since sooner or later a request will arrive, and not just one. A "flow of events" can be associated with various random events, for instance:

A = {at least one request will arrive during the interval from t_0 to $t_0 + \tau$}

or

B = {two requests will arrive during the same interval}.

The probabilities of these events can be calculated.

It must be pointed out that we can depict as a series of points not a flow of events itself (it is accidental) but its certain specific realisation.

We mentioned in Chapter 4 flows of events and some of their properties; but we shall now consider them here in more detail. A flow of events is *stationary* if the probability characteristics do not depend on the choice of a reference point or, more specifically, if the probability that a particular number of events will fall on any time interval depends only on the length τ of the interval and does not depend on its place on the t-axis.

A flow of events is said to be *ordinary* if the probability that two or more events may fall in an elementary time interval Δt is negligibly small as compared to the probability that one event may fall on that interval. The ordinariness of a flow means that events in it occur one at a time and not in groups of two, three, etc. (that two events may occur simultaneously is theoretically possible but has zero probability).

An ordinary flow of events can be interpreted as a stochastic process $X(t)$, i.e. the number of events that occurred up to the moment t (Fig. 10.0.2). The stochastic process $X(t)$ increases jumpwise by unity at the points $\Theta_1, \Theta_2, \ldots, \Theta_n$.

A flow of events is known as a *flow without an aftereffect* if the number of events falling on any time interval τ does not depend on how many events fell on any other nonoverlapping interval. In essence the absence of an aftereffect in a flow means that the events which form the flow occur at certain moments independently of one another.

A flow of events is said to be *elementary* if it is stationary, ordinary and has no aftereffects.

The interval of time T between two successive events in an elementary flow has an exponential distribution

$$f(t) = \lambda e^{-\lambda t} \text{ (for } t > 0), \tag{10.0.1}$$

where $\lambda = 1/M[T]$ is an inverse of the mean value of the interval T.

An ordinary flow of events without an aftereffect is a *Poisson flow*. An elementary flow is a special case of a Poisson flow (namely, a stationary Poisson flow).

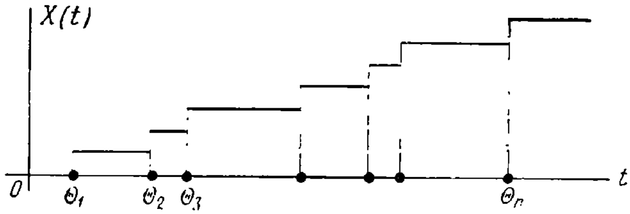


Fig. 10.0.2

The *intensity* λ of a flow of events is the mean number (expectation of the number) of events per unit time. For a stationary flow λ = const; while for a nonstationary flow the intensity is generally a function of time; $\lambda = \lambda(t)$.

The instantaneous intensity of a flow $\lambda(t)$ is defined as the limit of the ratio of the mean number of events which occurred during an elementary time interval $(t, t + \Delta t)$ to the length Δt of the interval, as the length tends to zero. The mean

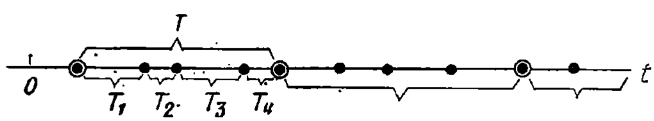


Fig. 10.0.3

number of events occurring during the time interval τ immediately following the moment t_0 (see Fig. 10.0.1) is $a(t_0, \tau) = \int_{t_0}^{t_0+\tau} \lambda(t)\,dt$. If the flow of events is stationary, then $a(t_0, \tau) = a(\tau) = \lambda\tau$.

An ordinary flow of events is a *Palm flow* (or *recurrent* flow, or a flow with a limited aftereffect) if the time intervals $T_1, T_2, \ldots$ between successive events (see Fig. 10.0.1) are independent similarly distributed random variables. Because of the similarity of distributions $T_1, T_2, \ldots$ a Palm flow is always stationary. An elementary flow is a special case of a Palm flow; the intervals of the events in it have an exponential distribution (10.0.1), where λ is the intensity of the flow.

Erlang's flow of order k is a flow of events which results when an elementary flow is "thinned out", i.e. when every kth point (event) in the flow is retained and all the intermediate points are removed (see Fig. 10.0.3, where it is shown how Erlang's fourth-order flow is obtained from an elementary flow).

In Erlang's flow of order k the time interval between two adjacent events is the sum of k independent random variables $T_1, T_2, \ldots, T_k$ which have an exponential distribution with parameter λ:

$$T = \sum_{i=1}^{k} T_i. \tag{10.0.2}$$

The distribution of the random variable T is known as an *Erlang distribution of order* k (see Problem 8.38) and has a density

$$f_k(t) = \frac{\lambda(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \text{ (for } t > 0). \tag{10.0.3}$$

The mean value, variance and mean square deviation of the random variable T (10.0.2) are

$$m_t = k/\lambda, \quad \mathrm{Var}_t = k/\lambda^2, \quad \sigma_t = \sqrt{k}/\lambda \tag{10.0.4}$$

respectively. The coefficient of variation of the random variable (10.0.2) is

$$v_t = \sigma_t/m_t = 1/\sqrt{k}, \tag{10.0.5}$$

hence $v_t \to 0$ as $k \to \infty$, i.e. when the order of Erlang's flow increases, "the degree of randomness" of the interval between the events tends to zero.

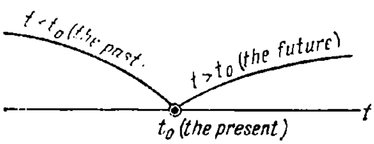


Fig. 10.0.4

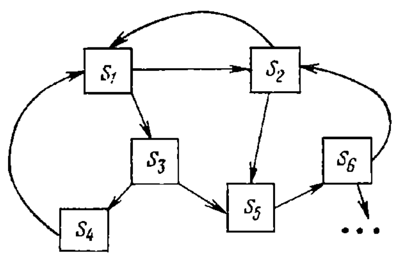


Fig. 10.0.5

Fig. 10.0.6

If, along with "thinning out" of an elementary flow, we also change the scale along the t-axis (by dividing by k), we obtain a *standardized* Erlang flow of order k whose intensity does not depend on k. The time interval $\tilde{T}$ between successive events in a standardized Erlang flow of order k has a density

$$\tilde{f}_k(t) = \frac{k\lambda(k\lambda t)^{k-1}}{(k-1)!} e^{-k\lambda t} \text{ (for } t > 0). \tag{10.0.6}$$

The numerical characteristics of the random variable

$$\tilde{T} = \frac{1}{k} \sum_{i=1}^{k} T_i$$

are

$$M[\tilde{T}] = 1/\lambda, \quad \mathrm{Var}[\tilde{T}] = 1/k\lambda^2, \quad \tilde{\sigma}_t = 1/(\lambda\sqrt{k}), \quad v_t = 1/\sqrt{k}. \tag{10.0.7}$$

As k increases, the standardized Erlang flow approaches indefinitely a *regular flow* with a constant interval $l = 1/\lambda$ between the events.

A stochastic process in a physical system S is known as a *Markov process* (or a process without an aftereffect) if *for any moment* t_0 (Fig. 10.0.4) *the probability of any state of the system in future* (for $t > t_0$) *depends only on its state in the present* (for $t = t_0$) *and does not depend on when and how the system* S *reached that state* (in other words, for a fixed present the future does not depend on the pre-history of the process, i.e. on the past).

In this chapter we shall study only Markov processes with discrete states $s_1, s_2, \ldots, s_n$. It is convenient to illustrate this kind of process by means of a directed *graph of states* (Fig. 10.0.5), where rectangles (or circles) denote the states $s_1, s_2, \ldots$

of system S, and the arrows denote possible transitions from one state to another (only the direct transitions are marked and not passages through other states). Sometimes the possible delays in a state are also marked on a directed graph using an arrow ("loop") directed from the state into itself (Fig. 10.0.6), but this representation can be omitted. A system may have a finite or infinite (but countable) number of states.

A Markov process with discrete states and discrete time is usually called a *Markov chain*. For such a process it is convenient to consider the moments $t_1, t_2, \ldots$ when the system S can change its state as successive steps in the process, and to

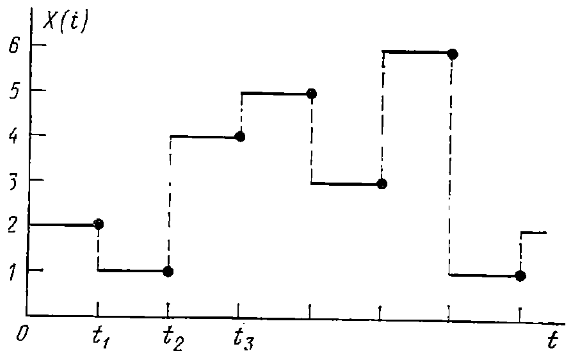


Fig. 10.0.7

consider the ordinal number of a step 1, 2, ..., k, ... rather than time t, to be the argument on which the process depends. In that case the stochastic process is characterized by a sequence of states

$$S(0),\ S(1),\ S(2),\ \ldots,\ S(k),\ \ldots, \tag{10.0.8}$$

where $S(0)$ is the initial state of the system (before the first step), $S(1)$ is the state of the system immediately after the first step, ..., $S(k)$ is the state of the system immediately after the kth step ....

The event $\{S(k) = s_i\}$ = {the system is in state s_i immediately after the kth step} ($i = 1, 2, \ldots$) is a random event and, therefore, the sequence of states (10.0.8) can be regarded as a sequence of random events. The initial state $S(0)$ may be either assigned or accidental. The events of sequence (10.0.8) are said to *form a Markov chain*.

Let us consider a process with n possible states $s_1, s_2, \ldots, s_n$. If we designate as $X(t)$ the number of the state in which the system S is at a moment t, then the process (Markov chain) can be described by an integral-valued random function $X(t) > 0$, whose possible values are 1, 2, ..., n. This function jumps from one integral value to another at given moments $t_1, t_2, \ldots$ and is continuous on the left, as depicted by the points in Fig. 10.0.7.

Let us consider a univariate distribution of the random function $X(t)$. We designate as $p_i(k)$ the probability that the system S will be in the state s_i ($i = 1, 2, \ldots, n$) after the kth step [and before the $(k+1)$th step]. The probabilities $p_i(k)$ are called the *probabilities of the states* of a Markov chain. It is evident that for any k

$$\sum_{i=1}^{n} p_i(k) = 1. \tag{10.0.9}$$

The probability distribution of the states at the beginning of the process

$$p_1(0),\ p_2(0),\ \ldots,\ p_i(0),\ \ldots,\ p_n(0) \tag{10.0.10}$$

is known as the *initial probability distribution* of the Markov chain. In particular, if the exact initial state $S(0)$ of system S is known, say, $S(0) = s_i$, then the initial probability $p_i(0) = 1$ and all the other initial probabilities are zero.

The *probability of transition* at the kth step from the state s_i to a state s_j is the conditional probability that the system S will be in the state s_j after the kth step provided that immediately before this (after $k - 1$ steps) it was in the state s_i.

A Markov chain is said to be *homogeneous* if the transition probabilities do not depend on the ordinal number of the step and only depend on the steps from which and to which the system passes:

$$\mathrm{P}\{S(k) = s_j \mid S(k-1) = s_i\} = P_{ij}. \tag{10.0.11}$$

The transition probabilities P_{ij} of a homogeneous Markov chain form an $n \times n$ square matrix, called a transition matrix.

$$\| P_{ij} \| = \left\| \begin{matrix} P_{11} & P_{12} & \dots & P_{1j} & \dots & P_{1n} \\ P_{21} & P_{22} & \dots & P_{2j} & \dots & P_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{i1} & P_{i2} & \dots & P_{ij} & \dots & P_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{n1} & P_{n2} & \dots & P_{nj} & \dots & P_{nn} \end{matrix} \right\|. \tag{10.0.12}$$

The sum of the transition probabilities in any row of the matrix is equal to unity

$$\sum_{j=1}^{n} P_{ij} = 1 \quad (i = 1, \dots, n). \tag{10.0.13}$$

A matrix which possesses this property is called a *stochastic matrix*. The probability P_{ij} is the probability that a system which is in the state s_i before a given step will remain in that state at a next step.

If the initial probability distribution (10.0.10) and the matrix of transition probabilities (10.0.12) are given for a homogeneous Markov chain, then the probabilities of the states of the system $p_i(k)$ $(i = 1, 2, \dots, n)$ can be found from the recursion formula

$$p_i(k) = \sum_{j=1}^{n} p_j(k-1) \mathrm{P}_{ji} \quad (i = 1, \dots, n; \; j = 1, \dots, n) \tag{10.0.14}$$

For an inhomogeneous Markov chain the transition probabilities in matrix (10.0.12) and formula (10.0.14) depend on the ordinal number of the step k.

In actual calculations using formula (10.0.14) not all the states s_j need be taken into account but only those for which the transition probabilities are nonzero, i.e. those from which arrows lead to the state s_i on the directed graph of states.

A Markov process with discrete states and continuous time is sometimes called a "continuous Markov chain". For such a process the probability of transition from a state s_i to a state s_j is zero for any moment. In that case, instead of the transition probability P_{ij} we consider the *transition probability density* λ_{ij} which is defined as the limit of the ratio of the probability of transition from the state s_i to the state s_j during a small time interval Δt after a moment t to the length of that interval as it tends to zero. The transition probability density can be either constant ($\lambda_{ij} = \text{const}$) or dependent on time $[\lambda_{ij} = \lambda_{ij}(t)]$. In the first case a Markov stochastic process with discrete states and continuous time is said to be *homogeneous*. A stochastic process $X(t)$, which is the number of events in an elementary flow occurring before the moment t, is a typical example of such a process (see Fig. 10.0.2).

When considering stochastic processes with discrete states and continuous time, it is convenient to assume that the transitions of a system S from state to state are affected by some flows of events. In that case the densities of transition probabilities assume the meaning of intensities λ_{ij} of the corresponding flow of events (as soon as the first event occurs in the flow with intensity λ_{ij}, the system

jumps from state s_i to state s_j). If they are all Poisson flows (i.e. ordinary and without aftereffects, with the intensity being constant or dependent on time), the process in the system S becomes Markovian.

When considering Markov stochastic processes with discrete states and continuous time, it is convenient to use a directed graph of states on which the intensity λ_{ij} of the flow of events which shifts the system along a given arrow is marked before each arrow (Fig. 10.0.8). This type of directed graph is known as a *marked graph.**)

The probability that the system S which is in a state s_i, will pass to a state s_j during an elementary time interval $(t, t + dt)$ (the element of the probability of transition from s_i to s_j) is the probability that during the time dt at least one event in the flow which shifts S from s_i to s_j will occur. With an accuracy to infinitesimals of higher orders this probability is $\lambda_{ij}\, dt$.

The *flow of the probabilities of transition* from s_i to s_j is a quantity $\lambda_{ij} p_i(t)$ (here the intensity λ_{ij} may either be dependent on or independent of time).

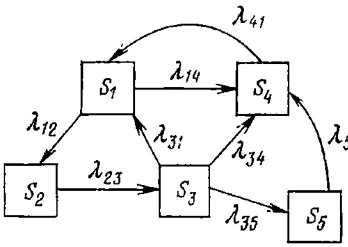


Fig. 10.0.8

Let us consider the case when the system S has a finite number of states $s_1, s_2, \ldots, s_n$. To describe a process in the system, we use the *probabilities of states*

$$p_1(t),\ p_2(t),\ \ldots,\ p_n(t), \tag{10.0.15}$$

where $p_i(t)$ is the probability that at a moment t the system S is in the state s_i:

$$p_i(t) = \mathrm{P}\{S(t) = s_i\}. \tag{10.0.16}$$

Evidently, for any t

$$\sum_{i=1}^{n} p_i(t) = 1. \tag{10.0.17}$$

To find the probabilities (10.0.15), we must solve a system of differential equations (Kolmogorov's equations) which have the form

$$\frac{dp_i(t)}{dt} = \sum_{j=1}^{n} \lambda_{ji} p_j(t) - p_i(t) \sum_{j=1}^{n} \lambda_{ij} \quad (i = 1, 2, \ldots, n),$$

or, omitting the argument t of the variables p_i,

$$\frac{dp_i}{dt} = \sum_{j=1}^{n} \lambda_{ji} p_j - p_i \sum_{j=1}^{n} \lambda_{ij} \ (i = 1, 2, \ldots, n). \tag{10.0.18}$$

Remember that the intensities of flows λ_{ij} may depend on time t (to make the notation briefer, we have omitted the argument t).

It is convenient to derive equations (10.0.18) using a marked graph of states of the system and the following mnemonical rule: *the derivative of the probability of every state is equal to the sum of all the probability flows which pass from other states to the given state minus the sum of all the probability flows which pass from the given state to other states.* For example, for the system S whose marked graph of states

*) We shall not draw loops corresponding to delays of the system in a given state in the directed graph of states since a delay is always possible.

is given in Fig. 10.0.8, the Kolmogorov's equations are

$$\begin{aligned}
dp_1/dt &= \lambda_{31}p_3 + \lambda_{41}p_4 - (\lambda_{12} + \lambda_{14})\, p_1,\\
dp_2/dt &= \lambda_{12}p_1 - \lambda_{23}p_2,\\
dp_3/dt &= \lambda_{23}p_2 - (\lambda_{31} + \lambda_{34} + \lambda_{35})\, p_3,\\
dp_4/dt &= \lambda_{14}p_1 + \lambda_{34}p_3 + \lambda_{54}p_5 - \lambda_{41}p_4,\\
dp_5/dt &= \lambda_{35}p_1 - \lambda_{54}p_5.
\end{aligned} \tag{10.0.19}$$

Since condition (10.0.17) is satisfied for any t, we can express any one of the probabilities (10.0.15) in terms of the other probabilities and thus diminish the number of equations by one.

To solve the system of differential equations (10.0.18) for the probabilities of states $p_1(t), p_2(t), \ldots, p_n(t)$, we must specify the initial probability distribution

$$p_1(0),\ p_2(0),\ \ldots,\ p_i(0),\ \ldots,\ p_n(0), \tag{10.0.20}$$

whose sum is equal to unity:

$$\sum_{j=1}^{n} p_i(0) = 1.$$

If, in a special case, the state of the system S at the initial moment $t = 0$ is exactly known, $S(0) = s_i$, then $p_i(0) = 1$ and the other initial probabilities (10.0.20) are zero.

In many cases, when the process in a system is sufficiently long, the limiting behaviour of the probabilities $p_i(t)$, as $t \to \infty$, becomes of interest. If all the flows

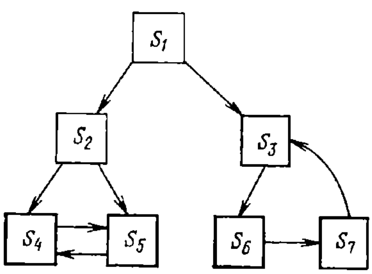


Fig. 10.0.9

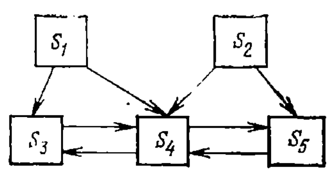


Fig. 10.0.10

of events which shift a system from state to state are elementary (i.e. stationary Poisson processes with constant intensities λ_{ij}), then *limiting* probabilities of states are possible,

$$p_i = \lim_{t\to\infty} p_i(t) \quad (i = 1, \ldots, n), \tag{10.0.21}$$

which do not depend on the state in which the system was at the initial moment. This means that in the course of time a *limiting stationary condition* is established in the system. The system still passes from state to state, but the probabilities of the states no longer change. In this limiting condition each of the limiting probabilities can be interpreted as the *relative mean time* for which the system stays in a given state.

A system for which the limiting probabilities exist is known as an *ergodic* system and the corresponding stochastic process as an *ergodic* process.

The condition $\lambda_{ij} = \text{const}$ alone is insufficient for the existence of limiting probabilities and other conditions must be fulfilled, which can be verified from the graph of states by isolating "essential" and "inessential" states in it. The state s_i is *essential* if there is no state s_j such that having once passed from s_i to s_j in some

way, the system cannot return to s_i. Every state which does not possess this property is *inessential.*

For example, for a system S, whose directed graph of states is shown in Fig. 10.0.9, the states s_1, s_2 are inessential (the system can leave state s_1, say, for s_2 and not return, and leave s_2 for s_4 or s_5 and not return), whereas states s_4, s_5, s_6 and s_7 are essential.

For a finite number of states n, for limiting probabilities to exist, it is necessary and sufficient that *it should be possible to pass from each essential state (in a certain number of steps) to every other essential state.* The graph shown in Fig. 10.0.9 does not satisfy this condition (for instance it is impossible to pass from the essential state s_4 to the essential state s_6). Limiting probabilities do exist for the graph shown in Fig. 10.0.10 (it is possible to pass from every essential state to every other essential state).

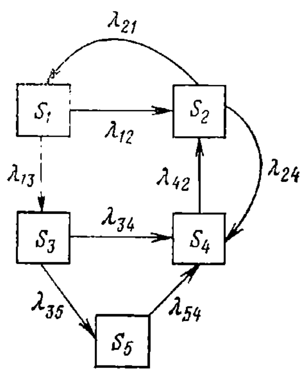


Fig. 10.0.11

Inessential states are thus called because sooner or later the system will leave these states for one of the essential states and will not return. The limiting probabilities for inessential states are naturally zero.

If a system S has a finite number of states $s_1, s_2, \ldots, s_n$, then, for the limiting probabilities to exist, it is *sufficient* that *the system could pass from any state to some other state* in a certain number of steps. If the number of states $s_1, s_2, \ldots, s_n, \ldots$ is infinite, then this condition is no longer sufficient, and the existence of limiting probabilities depends both on the directed graph of states and on the intensities λ_{ij}.

The limiting probabilities of states (provided they exist) can be obtained by solving a *system of algebraic linear equations* which result from the Chapman-Kolmogorov differential equations if their left-hand sides (derivatives) are set zero. But it is more convenient to derive these equations directly from the directed graph of states using the mnemonical rule: *for every state the total outgoing flow of probabilities is equal to the total incoming flow.* For example, for a system S, whose directed marked graph of states is shown in Fig. 10.0.11, the equations for the limiting probabilities of states have the form

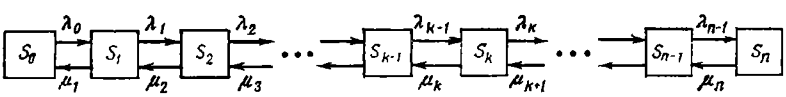


Fig. 10.0.12

$$
\begin{gathered}
(\lambda_{12}+\lambda_{13})\, p_1 = \lambda_{21}p_2, \quad (\lambda_{21}+\lambda_{24})\, p_2 = \lambda_{12}p_1 + \lambda_{42}p_4, \\
(\lambda_{34}+\lambda_{35})\, p_3 = \lambda_{13}p_1, \ \lambda_{42}p_4 = \lambda_{24}p_2 + \lambda_{34}p_3 + \lambda_{54}p_5, \ \lambda_{54}p_5 = \lambda_{35}p_3.
\end{gathered}
\tag{10.0.22}
$$

Thus, for a system S with n states, we obtain a system of n homogeneous algebraic linear equations in n unknowns $p_1, p_2, \ldots, p_n$. This system will yield the unknowns $p_1, p_2, \ldots, p_n$ with an accuracy to within an arbitrary factor. To find the exact values of $p_1, \ldots, p_n$, we must add to the equations the *normalizing condition* $p_1 + p_2 + \ldots + p_n = 1$. We can then express each of the probabilities p_i in terms of the other probabilities (and correspondingly delete one of the equations).

In practical applications we often encounter systems whose directed graphs of states have the form shown in Fig. 10.0.12 (all the states can be extended to form a chain, each state being in a relation, both forward and backward, with the two adjacent states except for the two extreme states, which have only one neighbour).

The chain shown in Fig. 10.0.12 is known as the *birth and death process*. These terms have been borrowed from biological problems, where the state of a population s_k signifies the presence of k units in it. A transition to the right is connected with the "birth" of units and that to the left, with their "death". In Fig. 10.0.12 the "birth rates" $(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ are put beside the arrows leading from left to right, and the "death rates" $(\mu_1, \mu_2, \ldots, \mu_{n-1})$ are put beside the arrows leading from right to left; each of them has an index of the state from which the corresponding arrow leads.

For a birth and death process the limiting probabilities of states are expressed by the formulas

$$p_1=\frac{\lambda_0}{\mu_1}p_0, \quad p_2=\frac{\lambda_0\lambda_1}{\mu_1\mu_2}p_0, \ \ldots,$$

$$p_k=\frac{\lambda_0\lambda_1\ldots\lambda_{k-1}}{\mu_1\mu_2\ldots\mu_k}p_0 \ (k=0, \ \ldots, n), \ \ldots,$$

$$p_n=\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}p_0$$

$$p_0=\left\{1+\frac{\lambda_0}{\mu_1}+\frac{\lambda_0\lambda_1}{\mu_1\mu_2}+\ldots+\frac{\lambda_0\lambda_1\ldots\lambda_{n-1}}{\mu_1\mu_2\ldots\mu_n}\right\}^{-1}. \tag{10.0.23}$$

Pay attention to the rule of calculating the probability of a state (for $k = 1, 2, \ldots, n$)

$$p_k=\frac{\lambda_0\lambda_1\ldots\lambda_{k-1}}{\mu_1\mu_2\ldots\mu_k}p_0,$$

which can be formulated as follows: *the probability of a state in a birth and death process* (see Fig. 10.0.12) *is the product of all the birth rates to the left of s_k divided by the product of all the death rates to the left of s_k, the fraction being multiplied by the probability of the extreme left state p_0.*

If a process is described by the birth and death chain, we can write differential equations for the mean value and variance of the function $X(t)$, i.e. for the number of units in the system at a moment t:

$$\frac{dm_x(t)}{dt}=\sum_{k=0}^{n}(\lambda_k-\mu_k)\,p_k(t), \tag{10.0.24}$$

$$\frac{d\,\mathrm{Var}_x(t)}{dt}=\sum_{k=0}^{n}[\lambda_k+\mu_k+2k(\lambda_k-\mu_k)-2m_x(t)(\lambda_k-\mu_k)]\,p_k(t). \tag{10.0.25}$$

In these formulas we must set $\lambda_n=\mu_0=0$. The intensities λ_k $(0\leqslant k\leqslant n-1)$ and μ_k $(1\leqslant k\leqslant n)$ can be any nonnegative functions of time.

When the values of $m_x(t)$ are sufficiently large (>20) and the condition $0< m_x(t)\pm 3\sqrt{\mathrm{Var}_x(t)}<n$ is satisfied, we can approximately assume that the section of the random function $X(t)$ is a normal random variable with parameters $m_x(t)$, $\sqrt{\mathrm{Var}_x(t)}$, which were obtained by solving equations (10.0.24) and (10.0.25). Formulas (10.0.24) and (10.0.25) remain valid as $n\to\infty$ if the upper limit in the sums is replaced by ∞.

Problems and Exercises

10.1. Two elementary flows are superimposed: flow I with intensity λ_1 and flow II with intensity λ_2 (Fig. 10.1). Is flow III resulting from the superposition elementary and if it is, what is its intensity?

Solution. Yes, it is since the properties of stationarity, ordinariness and absence of an aftereffect are conserved; the intensity of flow III is equal to $\lambda_1 + \lambda_2$.

10.2. An elementary flow of events with intensity λ is thinned out at random. Each event, independently of the other events, may be

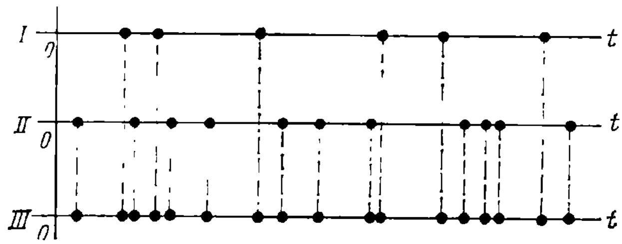


Fig. 10.1

conserved in the flow with probability p or removed with probability $1 - p$ (in what follows we shall call this operation the p-transformation). What is the flow resulting from the p-transformation of an elementary flow?

Solution. The flow is elementary with intensity λp. Indeed, under a p-transformation all the properties of an elementary flow (stationarity, ordinariness, absence of an aftereffect) are conserved and the intensity is multiplied by p.

10.3. The time interval T between events in an ordinary flow has a density

$$f(t) = \begin{cases} \lambda e^{-\lambda(t-t_0)} & \text{for } t > t_0, \\ 0 & \text{for } t < t_0. \end{cases} \qquad (10.3)$$

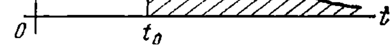


Fig. 10.3

The intervals between the events are independent. (1) Construct a graph of the probability density $f(t)$. (2) Is the flow elementary? (3) Is it a Palm flow? (4) What is its intensity $\tilde{\lambda}$? (5) What is the variation coefficient v_t of the interval between the events?

Solution. (1) See Fig. 10.3. We call a distribution of this kind "exponential, displaced by t_0". (2) No, it is not, since distribution (10.3) is not exponential. (3) Yes, it is, because of the ordinariness of the flow, the independence of the intervals and their similar distribution.

(4) $\tilde{\lambda} = 1/\mathrm{M}[T]$, $\mathrm{M}[T] = 1/\lambda + t_0$, $\tilde{\lambda} = (1/\lambda + t_0)^{-1} = \lambda/(1 + \lambda t_0)$.

$$(5) \quad \mathrm{Var}[T] = \frac{1}{\lambda^2}, \quad \sigma_t = \frac{1}{\lambda}, \quad v_t = \frac{\sigma_t}{\mathrm{M}[T]} = \frac{1/\lambda}{1/\lambda + t_0} = \frac{1}{1 + \lambda t_0}.$$

10.4. There is an elementary flow of events with intensity λ on the t-axis. Another flow is formed from it by inserting another event at the midpoints of the interval between every two adjacent events (see Fig. 10.4, where the circles denote the principal events and the crosses denote the inserted ones). Find the distribution density $f(t)$ of the

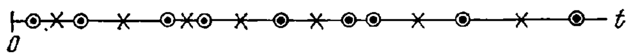


Fig. 10.4

interval T between adjacent events in the new flow. Is the flow elementary? Is it a Palm flow? What is the coefficient of variation of the interval T between the events?

Solution. $T = X/2$, where X has an exponential distribution with parameter λ: $f_1(x) = \lambda e^{-\lambda x}$ $(x > 0)$. Using the solution of Problem 8.1 and setting $a = 1/2$, $b = 0$ in formula (8.1), we obtain

$$f(t) = 2\lambda e^{-2\lambda t} \quad (t > 0). \tag{10.4}$$

This is an exponential distribution with a coefficient of variation $v_t = 1$. Nevertheless, the new flow is not elementary, nor even a Palm flow. We shall first prove that it is not a Palm flow. Though the intervals between the events have the same distribution (10.4), they are not independent. Let us consider two adjacent intervals. They may be independent with probability 1/2 or equal to one another with probability 1/2 and consequently dependent. Thus the new flow of events is not a Palm flow. It is naturally not elementary since an elementary flow is a special case of a Palm flow.

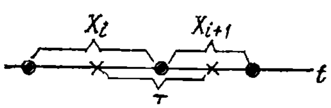


Fig. 10.5

Thus an exponential distribution of an interval between events is not a sufficient condition for a flow to be elementary.

10.5. The hypothesis is the same as in Problem 10.4, except that the new flow consists only of crosses (the midpoints of the intervals). Answer the same questions as in the preceding problem.

Solution. The intensity of the new flow will not evidently change as compared to the intensity of the original flow and will remain equal to λ. The interval between two neighbouring crosses (Fig. 10.5) is

$$T\ (X_i + X_{i+1})/2, \tag{10.5}$$

where X_i and X_{i+1} are two adjacent intervals in the original flow. The variables X_i and X_{i+1} both have an exponential distribution with parameter λ, and half their sum (10.5) will have a standardized Erlang distribution of the 2nd order since the interval T is equal to the sum of two independent exponentially distributed random variables divided by two. Thus the interval T between two crosses has a density $f(t) = 4\lambda^2 t e^{-2\lambda t}$ $(t > 0)$.

In the transformed flow all the neighbouring intervals are dependent since their components are the same random variables. However, this dependence involves only neighbouring intervals. Flows of this kind are sometimes called flows with a weak aftereffect.

The coefficient of variation v_t for the random variable T is [see formula (10.0.7)] $v_t = 1/\sqrt{2} < 1$.

10.6. The traffic on a road in the same direction is an elementary flow with intensity λ. A certain Petrov stands on the road and tries to stop a car. Find the distribution of time T for which he will have to wait, its mean value m_t and mean square deviation σ_t.

Solution. Since an elementary flow has no aftereffects, the "future" does not depend on the "past", in particular on the time when the last car passed. The distribution of time T is the same as the distribution of the time intervals between successive cars, i.e. exponential with parameter λ: $f(t) = \lambda e^{-\lambda t}$ $(t > 0)$, hence $m_t = 1/\lambda$, $\mathrm{Var}_t = 1/\lambda^2$, $\sigma_t = 1/\lambda = m_t$, $v_t = 1$.

Remark. If the traffic on a highway is in more than one lane, it can be considered to be a superposition of several flows, each corresponding to one lane. If each flow is elementary, then the result of the superposition is also an elementary flow since the properties of stationarity, ordinariness and absence of an aftereffect are conserved under a superposition (see Problem 10.1).

10.7. A passenger arrives at a bus stop irrespective of the time-table. The buses arrive regularly with an interval l. Find the distribution of time T for which the passenger will have to wait for a bus and express its characteristics m_t and σ_t in terms of the intensity of the traffic λ.

Solution. The moment the passenger arrives at the bus stop is distributed with a constant density within the interval l between two buses. The distribution density of the waiting time T is also constant [a uniform distribution on the interval $(0, l)$]: $f(t) = 1/l$ $(0 < t < l)$, or, designating $1/l = \lambda$, $f(t) = \lambda$ $(0 < t < 1/\lambda)$. For a uniform distribution on an interval of length $1/\lambda$ we have $m_t = 1/(2\lambda)$, $\mathrm{Var}_t = 1/(12\lambda^2)$, $\sigma_t = 1/(2\sqrt{3}\lambda)$, $v_t = 1/\sqrt{3}$.

10.8*. There is a Palm flow of events on the t-axis, the intervals between which are distributed with density $f(t)$. A point t^* is thrown

Fig. 10.8

at random on the t-axis (say, an "inspector" arrives to verify the occurrence of events, or a "passenger" arrives at a bus stop), the moment t^* being in no way connected with the occurrence of the events in the flow (Fig. 10.8). Find the distribution density of the interval T^* on which the point T^* has fallen, its mean value, variance and mean square deviation.

Solution. It seems at first sight that this density is the same as the distribution density $f(t)$ of **any** interval T between the events. It is not, however, the case. That a point t^* thrown at random **has fallen** on the interval T^* changes its distribution. Indeed, if there are a variety of intervals (large and small) on the t-axis, then there is a greater probability that the point t^* will fall on the larger interval.

Let us find the distribution density $f^*(t)$ of the interval T^* on which the point t^* has fallen. We seek the element of probability $f^*(t)\,dt$ equal to the probability that the point t^* will fall on the interval whose length is in the range $(t, t+dt)$. This probability is approximately equal to the ratio of the total length of all such intervals within a very large interval of time Ω to the total length of that interval.

Assume that a large number N of intervals between the events fit in a very large interval Ω. The average number of intervals, whose length is in the range $(t, t+dt)$, is $Nf(t)\,dt$; the average total length of all such intervals is approximately $tNf(t)\,dt$. The average total length of all the N intervals Ω is (approximately) Nm_t, where $m_t = \mathrm{M}[T] = \int_0^\infty tf(t)\,dt$. Dividing one by the other, we get

$$f^*(t)\,dt \approx \frac{Ntf(d)\,dt}{Nm_t} = \frac{tf(t)}{m_t}\,dt.$$

The approximation becomes the more exact the longer the interval of time Ω we consider (the larger N). In the limit, the distribution of the random variable T^* is

$$f^*(t) = \frac{t}{m_t} f(t) \quad (t > 0). \tag{10.8}$$

$$\mathrm{M}[T^*] = \frac{1}{m_t}\int_0^\infty t^2 f(t)\,dt = \frac{1}{m_t}\alpha_2(t) = \frac{1}{m_t}(\mathrm{Var}_t + m_t^2),$$

$$\mathrm{Var}[T^*] = \alpha_2[T^*] - (M[T^*])^2 = \frac{1}{m_t}\int_0^\infty t^3 f(t)\,dt - (M[T^*])^2.$$

10.9. On the hypothesis of Problem 10.8 Palm's flow is an elementary flow with intensity λ, i.e. $f(t) = \lambda e^{-\lambda t}$ $(t > 0)$. Find the distribution density $f^*(t)$ of the interval T^* on which the point t^* will fall.

Solution. Since $m_t = 1/\lambda$, formula (10.8) yields $f^*(t) = \lambda^2 t e^{-\lambda t}$ $(t > 0)$, which is none other than *Erlang's distribution of the 2nd order* [see formula (10.0.3) for $k = 2$].

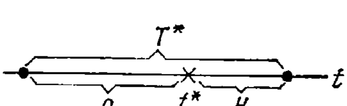


Fig. 10.10

10.10*. There is a Palm flow of events with density $f(t)$ of the interval T between successive events on the t-axis. A random point t^* (an "inspector") falls somewhere within the interval T^* (Fig. 10.10). It divides the interval into two subintervals: Q, from the nearest previous event to t^*, and H, from t^* to the nearest successive event. Find the distributions of the two intervals.

Solution. Let the random variable T^* assume a value s: $T^* = s$, and find the conditional distribution of the interval Q under this condition. We designate its density as $f_Q(t \mid s)$. Since the point t^* is thrown on the t-axis at random (irrespective of the events of the flow), it evidently has a uniform distribution within the interval $T^* = s$:

$$f_Q(t \mid s) = 1/s \quad \text{for} \quad 0 < t < s. \tag{10.10.1}$$

To find the marginal distribution $f_Q(t)$, we must average the density (10.10.1) with the "weight" $f^*(s)$. Using formula (10.8), we obtain

$$f^*(s) = \frac{s}{m_t} f(s), \quad f_Q(t) = \int_0^\infty f_Q(t \mid s) f^*(s)\, ds$$

Taking into account that $f_Q(t \mid s)$ is nonzero only for $s > t$, we can write

$$f_Q(t) = \int_t^\infty \frac{s}{s m_t} f(s)\, ds = \frac{1}{m_t} \int_t^\infty f(t)\, dt = \frac{1}{m_t}[1 - F(t)],$$

where $F(t)$ is the distribution function of the interval T between the events in the Palm flow.

Thus we have

$$f_Q(t) = \frac{1}{m_t}[1 - F(t)]. \tag{10.10.2}$$

It is evident that the time interval $H = T^* - Q$ has the same distribution:

$$f_H(t) = \frac{1}{m_t}[1 - F(t)]. \tag{10.10.3}$$

10.11. Using the results of the preceding problem, verify the solution of Problem 10.6, which we obtained from other considerations.

Solution. We have $f(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$ $(t > 0)$, $m_t = 1/\lambda$. By formula (10.10.3)

$$f_H(t) = \lambda[1 - 1 + e^{-\lambda t}] = \lambda e^{-\lambda t} \quad (t > 0),$$

i.e. the solution of Problem 10.6 is correct.

10.12. A passenger arrives at a bus stop irrespective of the timetable. The flow of buses is a Palm flow with intervals uniformly distributed in the range from 5 to 10 min. Find (1) the distribution density of the interval during which the passenger arrives; (2) the distribution density of time H for which he will have to wait for a bus; (3) the average time he must wait for a bus.

Solution. We have $f(t) = 1/5$ for $t \in (5, 10)$, $m_t = 7.5$.

(1) By formula (10.8) $f^*(t) = t/37.5$ for $t \in (5, 10)$. The graph of the density $f^*(t)$ is shown in Fig. 10.12*a*.

$$(2)\ F(t) = \begin{cases} 0 & \text{for } t \leqslant 5, \\ (t-5)/5 & \text{for } 5 < t \leqslant 10, \\ 1 & \text{for } t > 10. \end{cases}$$

From this, by formula (10.10.3), we get

$$f_H(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ 1/7.5 & \text{for} \quad 0 < t \leqslant 5, \\ (10-t)/37.5 & \text{for} \quad 5 < t \leqslant 10, \\ 0 & \text{for} \quad t > 10. \end{cases}$$

The graph of the density $f_H(t)$ is shown in Fig. 10.12*b*.

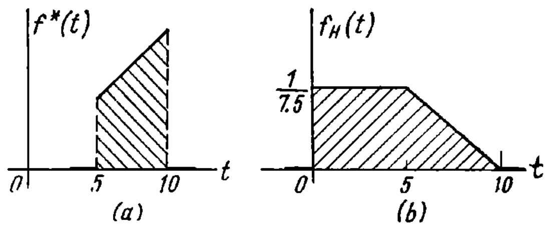


Fig. 10.12

(3) The average waiting time

$$m_H = \int_0^5 \frac{t}{7.5}\,dt + \int_5^{10} t\,\frac{10-t}{37.5}\,dt \approx 6.11 \text{ min.}$$

10.13. We consider Erlang's flow of order k with a distribution density of the interval T between events:

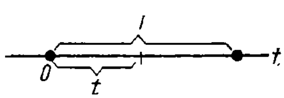


Fig. 10.13

$$f_k(t) = \frac{\lambda(\lambda t)^{k-1}}{(k-1)!} e^{-\lambda t} \quad (t > 0). \qquad (10.13.1)$$

Find the distribution function $F_k(t)$ of this interval.

Solution. We could find the distribution function using the ordinary formula

$$F_k(t) = \int_0^t f_k(t)\,dt,$$

but it is simpler to find it proceeding from the definition $F_k(t) = P\{T < t\}$.

We pass to the opposite event and find $P\{T > t\}$. We associate the origin 0 with one of the events in Erlang's flow and lay off two intervals to the right of it: T (the distance to the next event in Erlang's flow) and $t < T$ (Fig. 10.13).

For the inequality $T > t$ to be satisfied, it is necessary that less than k events in the elementary flow with intensity λ fall on the interval t (either 0 or 1, . . ., or $k-1$). The probability that m events fall on

the interval t is

$$P_m = \frac{(\lambda t)^m}{m!} e^{-\lambda t}.$$

By the probability addition rule

$$P\{T > t\} = \sum_{m=0}^{k-1} \frac{(\lambda t)^m}{m!} e^{-\lambda t},$$

whence

$$F_k(t) = 1 - \sum_{m=0}^{k-1} \frac{(\lambda t)^m}{m!} e^{-\lambda t} = 1 - R(k-1, \lambda t), \qquad (10.13.2)$$

where $1 - R(m, a)$ is a tabulated function (see Appendix 2).

10.14*. The flow of computer failures is Erlang's flow of order k with the density (10.13.1) of the interval between the failures (when the computer goes down, it is brought back into operation instantaneously). An "inspector" arrives at a random moment t^* and waits for the first failure. Find the distribution density of the time H for which he will have to wait for a failure and its mean value m_H.

Solution. By formula (10.10.3)

$$f_H(t) = \frac{1}{m_t} [1 - F_k(t)],$$

where $F_k(t)$ is given by formula (10.13.2) and $m_t = k/\lambda$. Hence

$$f_H(t) = \frac{\lambda}{k} \sum_{m=0}^{k-1} \frac{(\lambda t)^m}{m!} e^{-\lambda t} = \frac{1}{k} \sum_{m=0}^{k-1} \frac{\lambda (\lambda t)^m}{m!} e^{-\lambda t} \quad (t > 0). \quad (10.14.1)$$

We rewrite formula (10.14.1) in the form

$$f_H(t) = \frac{1}{k} \sum_{r=1}^{k} \frac{\lambda (\lambda t)^{r-1}}{(r-1)!} e^{-\lambda t} \quad (t > 0). \qquad (10.14.2)$$

It can be seen from (10.14.2) that the random variable H has a "mixed" distribution consisting of k Erlang's distributions of different orders; it has Erlang's distribution of orders 1, 2, . . ., k with equal probability $1/k$. The mean value of such a random variable can be found from the complete expectation formula

$$m_H = M[H] = \frac{1}{k} \sum_{r=1}^{k} M[H \mid r], \qquad (10.14.3)$$

where $M[H \mid r]$ is the conditional expectation of the random variable H provided that it has Erlang's distribution of the rth order.

From the first formula (10.0.4) we find that $M[H \mid r] = r/\lambda$, whence

$$m_H = \frac{1}{k\lambda}\sum_{r=1}^{k} r = \frac{(k+1)k}{2k\lambda} = \frac{k+1}{2\lambda}. \qquad (10.14.4)$$

10.15*. A Palm flow of events with distribution density $f(t)$ for the interval between the events is subjected to a p-transformation (see Problem 10.2). A random variable V is the interval between the events in the transformed flow. Find the mean value and variance of the random variable V.

Solution. The random variable V is the sum of a random number of independent random variables (see Problem 8.63): $V = \sum_{k=1}^{Y} T_k$, where Y is a discrete random variable which has a geometric distribution $P\{Y = m\} = pq^{m-1}$ $(m = 1, 2, \ldots)$, $q = 1 - p$, and each of the random variables T_k has a distribution $f(t)$.

Then the successive intervals between the events in the p-transformed flow are

$$V_1 = \sum_{k=1} T_k, \quad V_2 = \sum_{j=1}^{Y_2} T_{j+Y_1} \ldots$$

where the random variables $V_1, V_2, \ldots$ are disjoint, and the transformed flow is a Palm flow. In accordance with Problem 8.63

$$m_V = \frac{m_t}{p}, \quad \mathrm{Var}_V = \frac{\mathrm{Var}_t}{p} + m_t^2 \frac{q}{p^2},$$

where

$$m_t = \int_0^\infty t f(t)\,dt \qquad \mathrm{Var}_t = \int_0^\infty (t - m_t)^2 f(t)\,dt$$

Remark. We can prove that a multiple p-transformation of a Palm flow results in a flow which is close to an elementary flow.

10.16. Find the distribution of the interval T between the events in a Palm flow if the random variable T can be determined from the expression $T = \sum_{k=1}^{Y} T_k$, i.e. is the sum of a random number of random terms, where the random variables T_k are independent and have an exponential distribution with parameter λ, and the random variable Y does not depend on them and has a geometric distribution beginning with unity: $p_n = P\{Y = n\} = pq^{n-1}$ $(0 < p < 1;\ n = 1, 2, 3, \ldots)$.

Solution. As shown in Problem 8.62, the random variable T has an exponential distribution with parameter λp, and, consequently, the Palm flow being considered is an elementary flow with intensity λp, which results from a p-transformation of an elementary flow with intensity λ. This confirms the correctness of the solution of Problem 10.2.

10.17. The flow of buses arriving at a bus stop is a Palm flow, and the interval T between each two buses has a distribution density $f_T(t)$. A bus stays at the stop for a nonrandom time τ. A passenger arrives at the stop at a random moment t^* (Fig. 10.17*a*). He boards a bus if it is at the stop and waits for a time τ' if there are no buses, and, if no buses arrive at the stop during that time, leaves the stop and walks

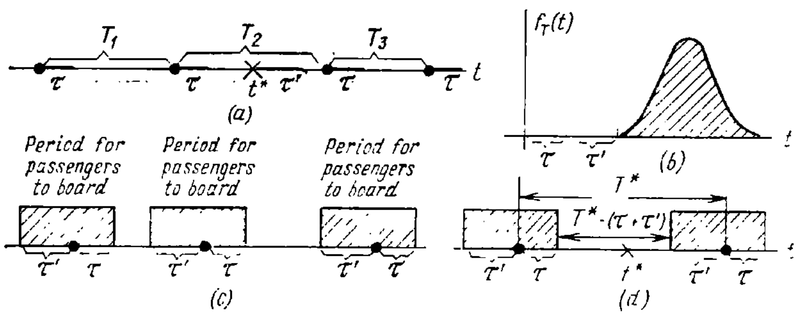


Fig. 10.17

to his destination. The distribution $f_T(t)$ is such that the random variable T cannot be smaller than $\tau + \tau'$ (Fig. 10.17*b*). Find the probability that the passenger will take a bus.

Solution. We consider the opposite event $\bar{A}$ = {the passenger will not catch a bus}. This means that when the passenger arrives at the stop at a moment t^* there are no buses at the stop, and no buses arrive during the period he waits. Each event, i.e. the arrival of a bus at the stop, is followed by a period for the passenger to board, the width of the period being $\tau' + \tau$ (Fig. 10.17*c*).

The event $\bar{A}$ = {the passenger does not catch a bus} corresponds to the point t^* falling outside the boarding period (Fig. 10.17*d*). The point t^* has a uniform distribution over the whole length of the interval T^*. The probability that it will fall on the interval $T^* - (\tau + \tau')$, which is outside the boarding period, is (by the integral total probability formula)

$$P(\bar{A}) = \int_{\tau+\tau'}^{\infty} \frac{t-(\tau+\tau')}{t} f^*(t)\,dt = \int_{\tau+\tau'}^{\infty} \frac{t-(\tau+\tau')}{t} \frac{t}{m_t} f(t)\,dt$$

$$= \frac{1}{m_t} \int_{\tau+\tau'}^{\infty} tf(t)\,dt - \frac{\tau+\tau'}{m_t} \int_{\tau+\tau'}^{\infty} f(t)\,dt,$$

where m_t is the average interval between buses.

The probability that the passenger will catch a bus is $P(A) = 1 - P(\bar{A})$.

10.18. An elementary flow with intensity λ is subjected to the following transformation. If the distance between adjacent events T_i

is smaller than some permissible limit t_0 (a "safety interval"), then the event is displaced by an interval t_0 relative to the preceding event. Now if $T_i > t_0$, then the event remains put (Fig. 10.18). Is the transformed flow formed by the points Θ_1', Θ_2'. . . ., on the t'-axis elementary? Is it a Palm flow?

Solution. The transformed flow is neither elementary nor Palm's since it has an aftereffect that can extend arbitrarily far. For example,

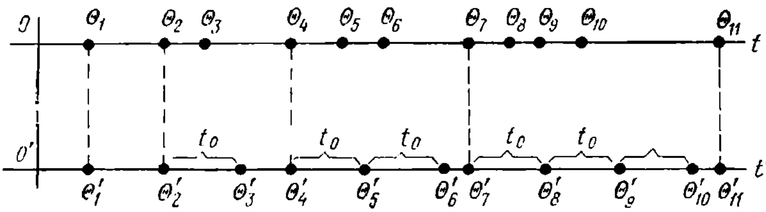


Fig. 10.18

the points Θ_7, Θ_8, Θ_9, Θ_{10} crowd the t'-axis and so every subsequent point on the t'-axis is shifted by a time interval which depends both on when the events occurred and on the intervals between them in the past. If t_0 is much less than the average distance between the events in the original flow, i.e. $t_0 \ll 1/\lambda$, then we can neglect the aftereffect in the transformed flow.

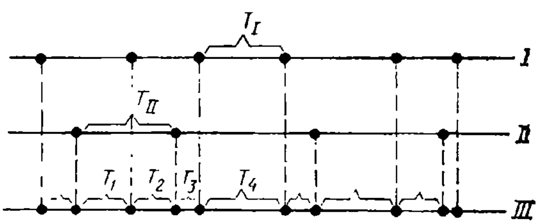


Fig. 10.19

10.19. Two independent Palm flows with distribution densities $f_1(t)$ and $f_2(t)$ for the interval between the events are superimposed. Is the resultant flow a Palm flow?

Solution. The superposition of two Palm flows is shown in Fig. 10.19. It is clear that the intervals T_1, T_2, . . . of the resultant flow III are not independent since their sizes are defined by those of the same interval on the axis I or II. For instance, T_1 and T_2 add together to give T_{II} and, hence, they are not independent. This relationship is rapidly damped, however, as the period between the origins of the intervals increases.

Remark. We can prove that when a sufficiently large number of Palm's flows with comparable intensities are superimposed, a flow results which is close to an elementary flow.

10.20. A computer, when it is running, can be regarded as a physical system S which may be found to be in one of four states: (s_1) the computer is completely sound, (s_2) the computer has a negligible number of faults in the memory and it can continue to solve problems, (s_3) the computer has a considerable number of faults, but it continues to solve a limited range of problems, and (s_4) the computer goes down completely.

Initially the computer is completely sound (state s_1) and is checked at three fixed moments t_1, t_2, t_3. The process in the system S can be regarded as a homogeneous Markov chain with three steps (the first, the second and the third check of the computer). The transition probability matrix has the form

$$\|\mathbf{P}_{ij}\| = \begin{Vmatrix} 0.3 & 0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 1.0 \end{Vmatrix}.$$

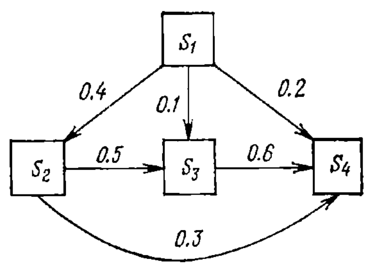


Fig. 10.20

Find the probabilities of the states of the computer after the three checks.

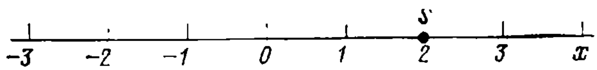
Solution. The directed graph of the states of the computer is illustrated in Fig. 10.20. Each arrow shows the transitional probability. The initial probabilities of states $p_1(0) = 1$ and $p_2(0) = p_3(0) = p_4(0) = 0$.

Using formula (10.0.4) and taking into account in the sum of probabilities only those states from which a direct passage to a given state is possible, we find

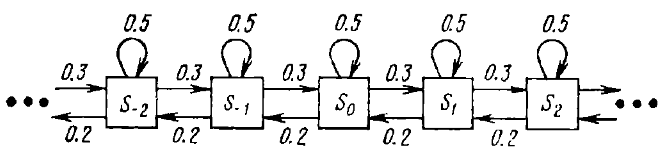
$$p_1(1) = p_1(0)\,\mathrm{P}_{11} = 1 \times 0.3 = 0.3,$$
$$p_2(1) = p_1(0)\,\mathrm{P}_{12} = 1 \times 0.4 = 0.4,$$
$$p_3(1) = p_1(0)\,\mathrm{P}_{13} = 1 \times 0.1 = 0.1,$$
$$p_4(1) = p_1(0)\,\mathrm{P}_{14} = 1 \times 0.2 = 0.2,$$
$$p_1(2) = p_1(1)\,\mathrm{P}_{11} = 0.3 \times 0.3 = 0.09,$$
$$p_2(2) = p_1(1)\,\mathrm{P}_{12} + p_2(1)\,\mathrm{P}_{22} = 0.3 \times 0.4 + 0.4 \times 0.2 = 0.20,$$
$$p_3(2) = p_1(1)\,\mathrm{P}_{13} + p_2(1)\,\mathrm{P}_{23} + p_3(1)\,\mathrm{P}_{33} = 0.27,$$
$$p_4(2) = p_1(1)\,\mathrm{P}_{14} + p_2(1)\,\mathrm{P}_{24} + p_3(1)\,\mathrm{P}_{34} + p_4(1)\,\mathrm{P}_{44} = 0.44,$$
$$p_1(3) = p_1(2)\,\mathrm{P}_{11} = 0.09 \times 0.3 = 0.027,$$
$$p_2(3) = p_1(2)\,\mathrm{P}_{12} + p_2(2)\,\mathrm{P}_{22} = 0.09 \times 0.4 + 0.20 \times 0.2 = 0.076,$$
$$p_3(3) = p_1(2)\,\mathrm{P}_{13} + p_2(2)\,\mathrm{P}_{23} + p_3(2)\,\mathrm{P}_{33} = 0.217,$$
$$p_4(3) = p_1(2)\,\mathrm{P}_{14} + p_2(2)\,\mathrm{P}_{24} + p_3(2)\,\mathrm{P}_{34} + p_4(2)\,\mathrm{P}_{44} = 0.680.$$

Thus the probabilities of states of the computer after three checks are $p_1(3) = 0.027$, $p_2(3) = 0.076$, $p_3(3) = 0.217$, $p_4(3) = 0.680$.

10.21. A point S "walks" along the x-axis (Fig. 10.21*a*) so that at each step it remains put with probability 0.5, jumps to the right by unity with probability 0.3 and jumps to the left by unity with probability 0.2. The state of the system S after k steps is defined by one coordinate (abscissa) of the point S. The initial position of the point is the origin. Considering the sequence of positions of the point S



(a)



(b)

Fig. 10.21

to be a Markov chain, find the probability that after four steps it will be no farther than a unity from the origin.

Solution. We designate the state of the system (point S) as s_i, where i is the coordinate of S on the abscissa axis. The marked graph of states is shown in Fig. 10.21*b*. (For the sake of clarity, loops are shown in the figure which correspond to the delays of S in the previous position.)

The sequence of states forms a Markov chain with an infinite number of states. The transitional probabilities P_{ij} are nonzero only for $j = i$, $j = i - 1$, $j = i + 1$, $P_{i,i} = 0.5$, $P_{i,i+1} = 0.3$, $P_{i,i-1} = 0.2$. All the other transitional probabilities are zero. The required probability P is equal to the sum of probabilities: $p_0(4) + p_1(4) + p_{-1}(4)$. We can find them using the recurrence relations (10.0.14).

We have $p_0(0) = 1$, $p_1(0) = p_{-1}(0) = \ldots = 0$. Furthermore,

$$p_0(1) = p_0(0)\,P_{0,0} = 0.5, \quad p_1(1) = p_0(0)\,P_{0,1} = 0.3,$$

$$p_{-1}(1) = p_0(1)\,P_{0,-1} = 0.2,$$

$$p_0(2) = p_0(1)\,P_{0,0} + p_1(1)\,P_{1,0} + p_{-1}(1)\,P_{-1,0} = 0.5 \times 0.5 + 0.2 \times 0.3 + 0.3 \times 0.2 = 0.37,$$

$$p_1(2) = p_0(1)\,P_{0,1} + p_1(1)\,P_{1,1} = 0.5 \times 0.3 + 0.3 \times 0.5 = 0.30,$$

$$p_2(2) = p_1(1)\,P_{12} = 0.3 \times 0.3 = 0.09,$$

$$p_{-1}(2) = p_0(1)\,P_{0,-1} + p_{-1}(1)\,P_{-1,-1} = 0.5 \times 0.2 + 0.2 \times 0.5 = 0.20,$$

$$p_{-2}(2) = p_{-1}(1)\,P_{-1,-2} = 0.2 \times 0.2 = 0.04,$$

$$p_0(3) = p_0(2)\,P_{0,0} + p_1(2)\,P_{1,0} + p_{-1}(2)\,P_{-1,0} = 0.305,$$

$$p_1(3) = p_0(2)\,P_{0,1} + p_1(2)\,P_{1,1} + p_2(2)\,P_{2,1} = 0.279,$$
$$p_2(3) = p_1(2)\,P_{1,2} + p_2(2)\,P_{2,2} = 0.135,$$
$$p_3(3) = p_2(2)\,P_{2,3} = 0.027,$$
$$p_{-1}(3) = p_{-2}(2)\,P_{-2,-1} + p_{-1}(2)\,P_{-1,-1} + p_0(2)\,P_{0,-1} = 0.186,$$
$$p_{-2}(3) = p_{-2}(2)\,P_{-2,-2} + p_{-1}(2)\,P_{-1,-2} = 0.060,$$
$$p_{-3}(3) = p_{-2}(2)\,P_{-2,-3} = 0.008,$$
$$p_0(4) = p_1(3)\,P_{1,0} + p_0(3)\,P_{0,0} + p_{-1}(3)\,P_{-1,0} \approx 0.264,$$
$$p_1(4) = p_2(3)\,P_{2,1} + p_1(3)\,P_{1,1} + p_0(3)\,P_{0,1} \approx 0.257,$$
$$p_{-1}(4) = p_0(3)\,P_{0,-1} + p_{-1}(3)\,P_{-1,-1} + p_{-2}(3)\,P_{-2,-1} \approx 0.172.$$

The required probability

$$p = p_0(4) + p_1(4) + p_{-1}(4) \approx 0.693.$$

Thus the probability of the event A that after four steps the point S will be no farther than unity from the origin is 0.693.

10.22. Under the conditions of the previous problem find the probability that at no time during the four steps was the point S farther than unity from the origin.

Solution. We designate as $\tilde{p}_0(k)$, $\tilde{p}_1(k)$, $\tilde{p}_{-1}(k)$ the probabilities of states s_0, s_1, s_{-1}, provided that up till the kth step inclusive the point S was never farther than unity from the origin.

At first glance this process is not a Markov chain (since the probabilities of future states depend on the "prehistory", i.e. on whether the point S was at least once farther than unity from the origin), but it can be reduced to a Markov chain by introducing another state, s^*, which corresponds to the point having gone farther than by unity from the origin at least once.

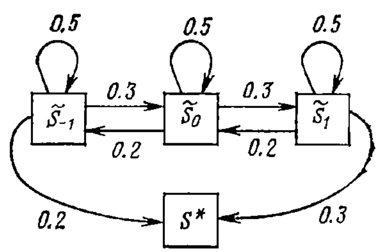


Fig. 10.22

The directed graph of states is illustrated in Fig. 10.22. There is no passage from state s^* to any of the other states; in Markov chain theory such a state is known as an absorbing state. The probability of the event A = {in four stops the point S will never be farther than unity from the origin} can be calculated as the sum of probabilities $\tilde{p}_0(4) + \tilde{p}_{-1}(4) + \tilde{p}_1(4)$ for a system with a graph of states corresponding to that shown in Fig. 10.22. We invite the reader to calculate these probabilities.

10.23. A system S consists of a technical device made up of m units, which from time to time (at moments $t_1, t_2, \ldots, t_k$) undergoes preventive maintenance and repair. After each step (the moment of main-

tenance and repair) the system may be in one of m states: (s_0) all the units are sound (no units are replaced by new ones); (s_1) one unit is replaced by a new unit and the other units are sound; (s_2) two units are replaced by new ones and the other units are sound; . . .; s_i — i units ($i < m$) are replaced by new ones and the other units are sound; . . .; s_m all m units are replaced by new ones.

The probability that at the time of preventive maintenance a unit will be replaced by a new one is p (independently of the states of the other units). Considering the states of system S to be a Markov chain, find the transitional probabilities and calculate the probabilities of system S for $m = 3$ and $p = 0.4$ after three steps (initially all the units are sound).

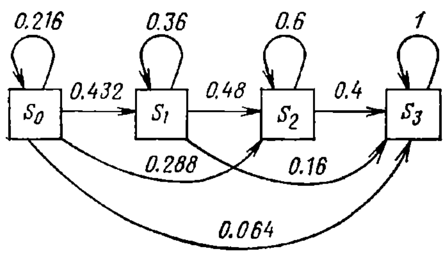


Fig 10.23

Solution. We write the transitional probabilities of the chain designating $q = 1 - p$. For any state s_i of the system the probability P_{ij} is zero for $j < i$; the probability P_{ii} is equal to the probability that no units will have to be replaced by new ones at a given step, i.e. $m - i$ of the old units remain in the device: $P_{ii} = q^{m-i}$. For $i < j$ the probability of the transition P_{ij} is equal to the probability that at a given step $j - i$ units out of the $m - i$ old units will have to be replaced by new ones. Using the binomial distribution, we find $P_{ij} = C_{m-i}^{j-i} p^{j-i} q^{m-j+i}$. The state s_m is absorbing. For $m = 3$, $p = 0.4$ the directed graph of the states of the system has the form shown in Fig. 10.23:

$$\| P_{ij} \| = \begin{Vmatrix} 0.216 & 0.432 & 0.288 & 0.064 \\ 0 & 0.36 & 0.48 & 0.16 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 1.0 \end{Vmatrix}.$$

We have $p_0(0) = 1$ and $p_1(0) = p_2(0) = p_3(0) = 0$. We find the probability of states after one, two, three steps:

$$p_0(1) = 0.216, \quad p_1(1) = 0.432,$$
$$p_2(1) = 0.288, \quad p_3(1) = 0.064,$$
$$p_1(2) = p_1(1)\,P_{11} + p_0(1)\,P_{01} \approx 0.249,$$
$$p_2(2) = p_2(1)\,P_{22} + p_0(1)\,P_{02} + p_1(1)\,P_{12} \approx 0.442,$$
$$p_3(2) = p_3(1)\,P_{33} + p_2(1)\,P_{23} + p_1(1)\,P_{13} + p_0(1)\,P_{03} \approx 0.262,$$
$$p_0(3) = p_0(2)\,P_{00} \approx 0.010,$$
$$p_1(3) = p_1(2)\,P_{11} + p_0(2)\,P_{01} \approx 0.110,$$
$$p_2(3) = p_2(2)\,P_{22} + p_0(2)\,P_{02} + p_1(2)\,P_{12} \approx 0.398,$$
$$p_3(3) = p_3(2)\,P_{33} + p_0(2)\,P_{03} + p_1(2)\,P_{13} + p_2(2)\,P_{23} \approx 0.482.$$

10.24. A computer is inspected at moments t_1, t_2 and t_3. The possible states of the computer are: (s_0) it is completely sound; (s_1) there are a negligible number of faults, which do not prevent the computer from functioning; (s_2) there are a considerable number of faults, which make it possible for the computer to solve only a limited range of problems; and (s_3) the computer goes down.

The transition probability matrix has the form

$$\|P_{ij}\| = \begin{Vmatrix} 0.5 & 0.3 & 0.2 & 0 \\ 0 & 0.4 & 0.4 & 0.2 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

Construct the directed graph of states. Find the probabilities of the states of the computer after one, two, and three inspections, if at the beginning ($t = 0$) the computer was completely sound.

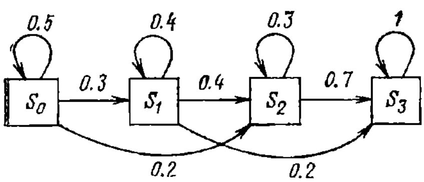


Fig 10.24

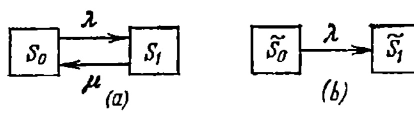


Fig. 10.25

Solution. The graph of states is shown in Fig. 10.24.

$$p_0(1) = p_0(0)\,P_{00} = 1 \times 0.5 = 0.5,$$
$$p_1(1) = p_0(0)\,P_{01} = 1 \times 0.3 = 0.3,$$
$$p_2(1) = p_0(0)\,P_{02} = 1 \times 0.2 = 0.2,$$
$$p_0(2) = p_0(1)\,P_{00} = 0.25, \quad p_1(2) = p_0(1)\,P_{01} + p_1(1)\,P_{11} = 0.27,$$
$$p_2(2) = p_0(1)\,P_{02} + p_1(1)\,P_{12} + p_2(1)\,P_{22} = 0.28,$$
$$p_3(2) = p_2(1)\,P_{23} + p_1(1)\,P_{13} = 0.20, \quad p_0(3) = p_0(2)\,P_{00} = 0.125,$$
$$p_1(3) = p_0(2)\,P_{01} + p_1(2)\,P_{11} = 0.183,$$
$$p_2(3) = p_0(2)\,P_{02} + p_1(2)\,P_{12} + p_2(2)\,P_{22} = 0.242,$$
$$p_3(3) = p_1(2)\,P_{13} + p_2(2)\,P_{23} + p_3(2)\,P_{33} = 0.450.$$

10.25. We consider the operation of a computer. The flow of failures (malfunctions) of an operating computer is an elementary flow with intensity λ. If the computer fails, the malfunction is immediately detected and the computer is repaired. The distribution of the time of repair is exponential with parameter μ: $\varphi(t) = \mu e^{-\mu t}$ ($t > 0$). At the initial moment ($t = 0$) the computer is sound. Find (1) the probability that at a moment t the computer functions; (2) the probability that

during the time $(0, t)$ the computer malfunctions at least once; (3) the limiting probabilities of the states of the computer.

Solution. (1) The states of the system (the computer) are: (s_0) it is sound and functions; (s_1) it is faulty and is repaired. The marked graph of states is shown in Fig. 10.25*a*.

The Chapman-Kolmogorov equations for the probabilities of states $p_0(t)$ and $p_1(t)$ have the form

$$\frac{dp_0}{dt} = \mu p_1 - \lambda p_0, \qquad \frac{dp_1}{dt} = \lambda p_0 - \mu p_1. \tag{10.25.1}$$

Either of these equations can be deleted since for any moment t we have $p_0 + p_1 = 1$. Substituting $p_1 = 1 - p_0$ into the first equation (10.25.1), we obtain one differential equation with respect to p_0:

$$dp_0/dt = \mu - (\lambda + \mu)\, p_0.$$

Solving this equation for the initial condition $p_0(0) = 1$, we obtain

$$p_0(t) = \frac{\mu}{\lambda+\mu}\left[1 + \frac{\lambda}{\mu} e^{-(\lambda+\mu)t}\right], \tag{10.25.2}$$

whence

$$p_1(t) = \frac{\lambda}{\lambda+\mu}\left[1 - e^{-(\lambda+\mu)t}\right]. \tag{10.25.3}$$

(2) To find the probability $\tilde{p}(t)$ that during the time t the computer malfunctions at least once, we introduce new states for the system: ($\tilde{s}_0$) the computer never failed; ($\tilde{s}_1$) the computer malfunctioned at least once. The state $\tilde{s}_1$ is the absorbing one (see Fig. 10.25*b*).

Solving the Chapman-Kolmogorov equation $d\tilde{p}_0/dt = -\lambda\tilde{p}_0$ for the initial condition $\tilde{p}_0(0) = 1$, we get $\tilde{p}_0(t) = e^{-\lambda t}$, whence $\tilde{p}_1(t) = 1 - \tilde{p}_0(t) = 1 - e^{-\lambda t}$. Thus the probability that during the time t the computer malfunctions at least once is $\tilde{p}_1(t) = 1 - e^{-\lambda t}$. In this case the probability could have been calculated more simply, i.e. as the probability that during the time t at least one event (malfunction) will occur in an elementary flow of malfunctions with intensity λ.

(3) As $t \to \infty$, we get from equations (10.25.2) and (10.25.3) the limiting probabilities of states: $p_0 = \mu/(\lambda + \mu)$, $p_1 = \lambda/(\lambda + \mu)$, which could eventually have been obtained directly from the graph of states, using the birth and death process (we invite the reader to do this).

10.26. On the hypothesis of the preceding problem the malfunctioning of the computer is not immediately noticed but is detected after an interval of time which has an exponential distribution with parameter ν. Write and solve the Chapman-Kolmogorov equations for the probabilities of states. Find the limiting probabilities of states (from the directed graph of states).

Solution. The states of the system are: (s_0) the computer is sound and operates; (s_1) the computer is faulty but the malfunction is not detected; (s_2) the computer is being repaired. The graph of states is shown in Fig. 10.26.

The Chapman-Kolmogorov equations for the probabilities of states are

$$\frac{dp_0}{dt}=\mu p_2-\lambda p_0,\quad \frac{dp_1}{dt}=\lambda p_0-\nu p_1,\quad \frac{dp_2}{dt}=\nu p_1-\mu p_2. \quad (10.26.1)$$

We shall solve this system using Laplace transforms. With due regard for the initial conditions for transforms π_i of probabilities p_i, equations (10.26.1) assume the form

$$s\pi_0 = \mu\pi_2 - \lambda\pi_0 + 1,$$
$$s\pi_1 = \lambda\pi_0 - \nu\pi_1,$$
$$s\pi_2 = \nu\pi_1 - \mu\pi_2.$$

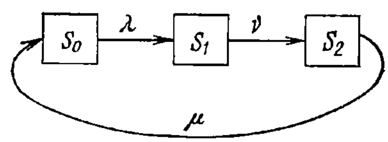


Fig. 10.26

Solving this system of algebraic equations, we get the following equations for the transforms:

$$\pi_1=\frac{\lambda}{s+\nu}\pi_0,\quad \pi_2=\frac{\nu}{s+\mu}\pi_1=\frac{\nu\lambda}{(s+\nu)(s+\mu)}\pi_0,$$
$$\pi_0=\frac{(s+\nu)(s+\mu)}{s\,(s^2+s\,(\mu+\nu+\lambda)+\nu\lambda+\nu\mu+\lambda\mu)}.$$

We introduce the designations

$$a=\frac{\mu+\nu+\lambda}{2}+\sqrt{\frac{(\mu+\nu+\lambda)^2}{4}-\nu\lambda-\nu\mu-\lambda\mu}\,.$$
$$b=-\frac{\mu+\nu+\lambda}{2}-\sqrt{\frac{(\mu+\nu+\lambda)^2}{4}-\nu\lambda-\nu\mu-\lambda\mu}\,.$$

Then the expressions for probabilities assume the form

$$p_0(t)=\frac{ae^{at}-be^{bt}}{a-b}+(\nu+\mu)\frac{e^{at}-e^{bt}}{a-b}+\mu\nu\left[\frac{1}{ab}+\frac{be^{at}-ae^{bt}}{ab\,(a-b)}\right],$$
$$p_1(t)=\lambda\frac{e^{at}-e^{bt}}{a-b}+\lambda\mu\left[\frac{1}{ab}+\frac{be^{at}-ae^{bt}}{ab\,(a-b)}\right],$$
$$p_2(t)=\nu\lambda\left[\frac{1}{ab}+\frac{be^{at}-ae^{bt}}{ab\,(a-b)}\right].$$

To find the limiting probabilities, we can use either the transforms or the probabilities themselves:

$$p_0=\lim_{t\to\infty}p_0(t)=\lim_{s\to 0}s\pi_0(s)=\frac{\mu\nu}{\lambda\mu+\lambda\nu+\nu\mu},$$
$$p_1=\frac{\lambda\mu}{\lambda\mu+\lambda\nu+\nu\mu}\cdot\quad p_2=1-p_0-p_1=\frac{\lambda\nu}{\lambda\mu+\lambda\nu+\nu\mu}. \quad (10.26.2)$$

We can find the final probabilities of states directly from directed graph in Fig. 10.26: $\mu p_2 = \lambda p_0$, $\lambda p_0 = \nu p_1$, $\nu p_1 = \mu p_2$. We can delete one of these equations (say, the last one). Expressing p_2 in terms of p_0 and p_1, i.e. $p_2 = 1 - p_0 - p_1$, we get two equations

$$\mu(1 - p_0 - p_1) = \lambda p_0 \quad \text{and} \quad \lambda p_0 = \nu p_1.$$

The solution of these equations yields the same results (10.26.2).

10.27. An electronic device consists of two identical repeating units. For the device to operate, it is sufficient that at least one unit functions. When one of the units fails, the device continues to function normally due to the other unit. The flow of failures of each unit is elementary with parameter λ. When a unit fails, it is immediately repaired. The time needed to repair the unit is exponential with parameter μ. Initially ($t = 0$) both units operate. Find the following characteristics of the operation of the device:

(1) The probabilities of states (as functions of time): (s_0) both units are sound; (s_1) one unit is sound and the other is being repaired; (s_2) both units are being repaired (the device does not operate).

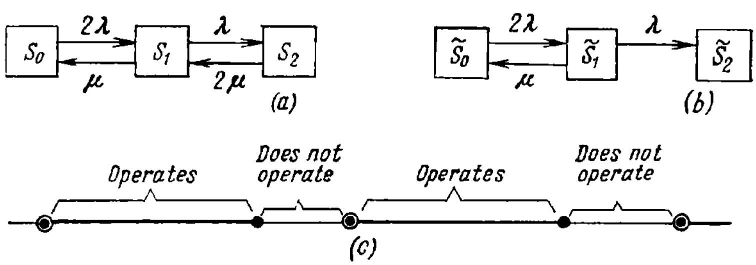


Fig. 10.27

(2) The probability $\tilde{p}(t)$ that during time t the device never fails.

(3) The limiting probabilities of states of the device.

(4) For the limiting (stationary) conditions of the device the average relative time for which the device will operate.

(5) For the same limiting conditions the average time $\bar{t}_{op}$ of failure-free operation of the device (from the moment it is switched on after being repaired till the next malfunction).

Solution. The directed graph of states of the device is shown in Fig. 10.27a (2λ is put before the upper left arrow since two units operate and may fail); for the same reason 2μ is put before the lower right arrow since both units are being repaired).

(1) The Chapman-Kolmogorov equations have the form

$$\frac{dp_0}{dt} = \mu p_1 - 2\lambda p_0, \quad \frac{dp_1}{dt} = 2\lambda p_0 + 2\mu p_2 - (\lambda + \mu) p_1,$$

$$\frac{dp_2}{dt} = \lambda p_1 - 2\mu p_2. \qquad (10.27.1)$$

In this case the following condition must be fulfilled: $p_0 + p_1 + p_2 = 1$.

Solving this system of equations for the initial conditions $p_0(0) = 1$, $p_1(0) = p_2(0) = 0$, we obtain

$$p_0(t) = \frac{ae^{at} - be^{bt}}{a-b} + (\lambda + 3\mu)\frac{e^{at} - e^{bt}}{a+b} + 2\mu^2\left[\frac{1}{ab} + \frac{be^{at} - ae^{bt}}{ab(a-b)}\right],$$

where

$$a = -(\lambda + \mu), \quad b = -2(\lambda + \mu),$$

$$a - b = \lambda + \mu, \quad ab = 2(\lambda + \mu)^2,$$

$$p_1(t) = \frac{a^2 e^{at} - b^2 e^{bt}}{\mu(a-b)} + \frac{(\lambda + 3\mu)(ae^{at} - be^{bt})}{\mu(a-b)} + \frac{2\mu(e^{at} - e^{bt})}{a-b} + \frac{2\lambda}{\mu} p_0(t).$$

From the expressions obtained we get

$$p_2(t) = 1 - p_0(t) - p_1(t) \quad \text{and} \quad p_2(0) = 0.$$

(2) To find the probability $\tilde{p}(t)$, we make the state s_2 (the device stopped operating) absorbing $(\tilde{s}_2)$ (Fig. 10.27*b*). For the probabilities of the states the Chapman-Kolmogorov equations are

$$d\tilde{p}_0/dt = \mu\tilde{p}_1 - 2\lambda\tilde{p}_0, \quad d\tilde{p}_1/dt = 2\lambda\tilde{p}_0 - (\lambda + \mu)\tilde{p}_1, \quad d\tilde{p}_2/dt = \lambda\tilde{p}_1.$$

Solving the first two equations under the initial conditions $\tilde{p}_0(0) = 1$, $\tilde{p}_1(0) = 0$, we obtain

$$\tilde{p}_0(t) = \frac{\alpha e^{\alpha t} - \beta e^{\beta t}}{\alpha - \beta} + (\lambda + \mu)\frac{e^{\alpha t} - e^{\beta t}}{\alpha - \beta},$$

where

$$\alpha = \frac{-(3\lambda + \mu) + \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2}, \quad \beta = \frac{-(3\lambda + \mu) - \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2}$$

(the quantities α and β are negative for any positive values of λ and μ). Furthermore,

$$\tilde{p}_1(t) = \frac{1}{\mu}\frac{d\tilde{p}_0}{dt} + \frac{2\lambda}{\mu}\tilde{p}_0 = \frac{1}{\mu}\Big[\frac{\alpha^2 e^{\alpha t} - \beta^2 e^{\beta t}}{\alpha - \beta}$$

$$+ (\lambda + \mu)\frac{\alpha e^{\alpha t} - \beta e^{\beta t}}{\alpha - \beta}\Big] + \frac{2\lambda}{\mu}\tilde{p}_0(t).$$

The required probabilities $\tilde{p}(t) = \tilde{p}_0(t) + \tilde{p}_1(t)$. Note that the function $\tilde{p}(t)$ is monotone decreasing, and $p(0) = 1$, $p(\infty) = 0$.

(3) The final probabilities of states can be found from the graph in Fig. 10.27*a* and the general formulas (10.0.23) for the birth and death process

$$p_1 = \frac{2\lambda}{\mu} p_0; \quad p_2 = \frac{2\lambda^2}{2\mu^2} p_0 = \left(\frac{\lambda}{\mu}\right)^2 p_0, \quad p_0 + p_1 + p_2 = 1,$$

$$p_0 = [1 + 2\lambda/\mu + (\lambda/\mu)^2]^{-1} = \mu^2/(\lambda + \mu)^2.$$

(4) The average relative time for which the device will operate is

$$1 - p_2 = 1 - \left(\frac{\lambda}{\mu}\right)^2\left(\frac{\mu}{\lambda + \mu}\right)^2 = 1 - \left(\frac{\lambda}{\lambda + \mu}\right)^2.$$

(5) The variable $\bar{t}_{op}$ is the expectation of time T_{op} which passes from the moment the device is switched on to the moment of its next failure.

Let us consider the operation of the device in stationary conditions to consist of a number of cycles: "operates" and "does not operate" (see Fig. 10.27*c*, where the operating sections are shown in bold line). The average time $\bar{t}_{\text{nonop}}$ for which the device does not operate (the expectation of the length of a nonoperating period) is evidently $1/(2\mu)$ (since a flow of transitions to an operating state with intensity 2μ acts on the device which is in the nonoperating state).

Furthermore, the ratio of the average time of trouble-free operation $\bar{t}_{\text{op}}$ to that of "nonoperation" $\bar{t}_{\text{nonop}}$ is equal to the ratio of the final

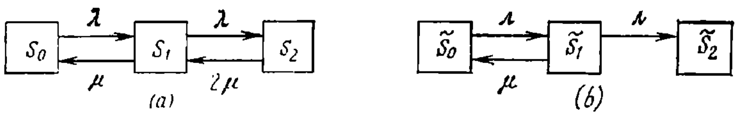


Fig. 10.28

probability $1-p_2$ (that the device is operating) to the probability p_2 (that it does not operate), i.e. $\bar{t}_{\text{op}}/\bar{t}_{\text{nonop}} = (1-p_2)/p_2$. From this, bearing in mind that $\bar{t}_{\text{nonop}} = 1/(2\mu)$, we get

$$\bar{t}_{\text{op}} = \bar{t}_{\text{nonop}} \cdot \frac{1-p_2}{p_2} = \frac{1}{2\mu}\,\frac{1-p_2}{p_2}.$$

10.28. The conditions and questions are the same as in Problem 10.27 except that as long as one of the units operates, the other is kept in reserve and so cannot fail. When the reserve unit is switched on, it begins operating at once and is subject to a flow of failures with intensity λ.

Solution. The directed graph of states of the device has the form shown in Fig. 10.28*a*, and the graph with an absorbing state in Fig. 10.28*b*.

Answers:

$$(1)\quad p_0(t) = \frac{ae^{at}-be^{bt}}{a-b} + (\lambda+3\mu)\,\frac{e^{at}-e^{bt}}{a-b} + 2\mu^2\left[\frac{1}{ab} + \frac{be^{at}-ae^{bt}}{ab(a-b)}\right],$$

$$p_1(t) = \frac{a^2e^{at}-b^2e^{bt}}{\mu(a-b)} + \frac{(\lambda+3\mu)}{\mu(a-b)}(ae^{at}-be^{bt})$$

$$+\frac{2\mu}{a-b}[e^{at}-e^{bt}] + \frac{\lambda}{\mu}\,p_0(t), \quad p_2(t) = 1-p_0(t)-p_1(t),$$

$$a = \frac{-2\lambda-3\mu+\sqrt{4\mu\lambda+\mu^2}}{2}, \qquad b = \frac{-(2\lambda+3\mu)+\sqrt{4\mu\lambda+\mu^2}}{2}.$$

$$(2)\quad \tilde{p}_0(t) = \frac{\alpha e^{\alpha t}-\beta e^{\beta t}}{\alpha-\beta} + (\lambda+\mu)\,\frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta},$$

$$\tilde{p}_1(t) = \frac{1}{\mu}\left[\frac{\alpha^2e^{\alpha t}-\beta^2e^{\beta t}}{\alpha-\beta} + (\lambda+\mu)\,\frac{\alpha e^{\alpha t}-\beta e^{\beta t}}{\alpha-\beta}\right] + \frac{\lambda}{\mu}\,\tilde{p}_0(t),$$

$$\tilde{p}_2(t) = 1-\tilde{p}_0(t)-\tilde{p}_1(t), \quad \alpha = \frac{-(\lambda 2+\mu)+\sqrt{4\lambda\mu+\mu^2}}{2}.$$

$$\beta=\frac{-(\lambda^2+\mu)-\sqrt{4\lambda\mu+\mu^2}}{2}, \quad \tilde{p}(t)=1-\tilde{p}_2(t),$$

(3) $p_1=\frac{\lambda}{\mu}p_0, \quad p_2=\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 p_0, \quad p_0=\left[1+\frac{\lambda}{\mu}+\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right]^{-1}$

(4) $1-p_2$; (5) $\bar{t}_p=\frac{1}{2\mu}\frac{1-p_2}{p_2}$.

10.29. The conditions of Problem 10.27 are changed so that the nonoperating unit is in a reduced reserve and may still fail with intensity $\lambda'<\lambda$. (1) Construct the directed graph of states of the device and write out the Chapman-Kolmogorov equations for the probabilities of states. (2) Without solving the equations, find the limiting probabilities of states; calculate them for $\lambda=1$, $\lambda'=0.5$ and $\mu=2$. (3) Using these numerical data, find the average time $\bar{t}_{op}$ of the trouble-free operation of the device.

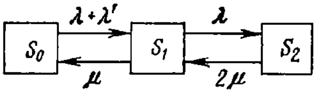


Fig. 10.29

Solution. (1) The directed graph of states is shown in Fig. 10.29 and the Chapman-Kolmogorov equations are

$$\frac{dp_0}{dt}=-(\lambda+\lambda')p_0+\mu p_1, \quad \frac{dp_1}{dt}=-(\lambda+\mu)p_1+(\lambda+\lambda')p_0+2\mu p_2$$

$$p_0+p_1+p_2=1.$$

(2) The limiting probabilities of the states can be found from the general formulas (10.0.23) for the birth and death process:

$$p_1=\frac{\lambda+\lambda'}{\mu}p_0, \quad p_2=\frac{(\lambda+\lambda')\lambda}{2\mu^2}p_0, \quad p_0=\left\{1+\frac{\lambda+\lambda'}{\mu}+\frac{(\lambda+\lambda')\lambda}{2\mu^2}\right\}^{-1}.$$

Substituting here $\lambda=1$, $\lambda'=0.5$, and $\mu=2$, we get

$$p_0=\left\{1+\frac{1.5}{2}+\frac{1.5\times 1}{8}\right\}^{-1}\approx 0.516, \quad p_1\approx 0.387, \quad p_2\approx 0.097.$$

(3) $\bar{t}_p=\frac{1}{4}\frac{1-p_2}{p_2}\approx 2.32.$

10.30. A computer includes four disk storage units. A team of four carries out a preventive inspection on each disk. The overall flow of events, i.e. moments when the inspections are finished by the whole team, is a Poisson flow with intensity $\lambda(t)$. When the inspection is finished, a disk is checked and may prove serviceable with probability p (the validation time is small as compared to the time of inspection and can be neglected). If a disk proves nonserviceable, it is again subjected to inspection (the time needed for this operation does not depend on whether an inspection was carried out before) and so on. At the beginning all the disk units need preventive inspection.

Construct a directed graph of states for the system S (four disk units) and write the differential equations for the probabilities of states. Find

M_τ, the expectation of the number of disks which successfully pass the inspection by the moment τ.

Solution. (s_0) all four disk units need preventive inspection; (s_1) one disk unit successfully passed the inspection and three disk units need repair; (s_2) two disk units successfully passed the inspection and two need repair; (s_3) three disk units successfully passed the inspection and one needs repair; (s_4) all four units successfully passed the inspection.

The fact that each maintenance is successful with probability p is equivalent to a p-transformation of the flow of inspection completions, after which the flow remains a Poisson flow, but with intensity $p\lambda(t)$.

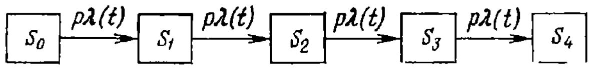


Fig. 10.30

The directed graph of states is shown in Fig. 10.30, and the Chapman-Kolmogorov equations are

$$dp_0/dt = -p\lambda(t)\,p_0, \qquad dp_1/dt = p\lambda(t)\,(p_0 - p_1),$$
$$dp_2/dt = p\lambda(t)\,(p_1 - p_2), \qquad dp_3/dt = p\lambda(t)\,(p_2 - p_3),$$
$$dp_4/dt = p\lambda(t)\,p_3. \tag{10.30.1}$$

The initial conditions are $p_0(0) = 1$ and $p_1(0) = \ldots = p_4(0) = 0$.

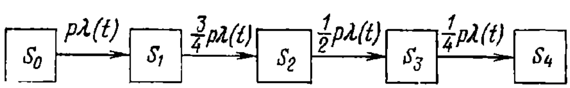


Fig. 10.31

The expectation of the number of disks which successfully passed the inspection by the time τ is

$$M_\tau = \sum_{i=1}^{4} i p_i(\tau). \tag{10.30.2}$$

At a constant intensity λ the solutions of equations (10.30.1) are

$$p_0(t) = e^{-\lambda pt}, \quad p_1(t) = \lambda pt e^{-\lambda pt}, \quad p_2(t) = \frac{(\lambda pt)^2}{2} e^{-\lambda pt},$$

$$p_3(t) = \frac{(\lambda pt)^3}{3!} e^{-\lambda pt}, \quad p_4(t) = 1 - \sum_{i=0}^{3} p_i(t).$$

10.31. On the hypothesis of the preceding problem each member of the team is assigned one disk unit which he must maintain. The flow of the inspection completions per team member has intensity $\lambda(t)/4$. Answer the questions in the preceding problem.

Solution. The directed graph of states is shown in Fig. 10.31 and the Chapman-Kolmogorov equations are

$$dp_0/dt = -p\lambda\,(t)\,p_0, \quad dp_1/dt = p\lambda\,(t)\,(p_0 - (3/4)\,p_1),$$
$$dp_2/dt = p\lambda\,(t)\,((3/4)\,p_1 - (1/2)\,p_2),$$
$$dp_3/dt = p\lambda\,(t)\,((1/2)\,p_2 - (1/4)\,p_3),$$
$$dp_4/dt = (1/4)\,p\lambda\,(t)\,p_3.$$

Expression (10.30.2) for M_τ remains the same for this situation.

10.32. A device is subjected to an elementary flow of faults with intensity λ. A fault is not detected at once but only after a random time interval which has an exponential distribution with parameter ν. As soon as a fault is detected, the device is inspected and is either repaired (with probability p) or rejected and replaced by a new one. The time needed for the inspection is exponential with parameter γ, the time needed for repair is exponential with parameter μ, the time needed for replacing the rejected device by a new one is exponential with parameter $\varkappa$. What are the limiting probabilities of the states of the device? Find (1) the average time for which the device will operate normally; (2) the time for which the device will operate with an undetected failure (produce reject); (3) the average cost of repairing or replacing the device per unit time if the average cost of repair is r and that of a new device is c; (4) the average loss per unit time when the device operates with an undetected refusal if the operation of such a device entails a loss l per unit time.

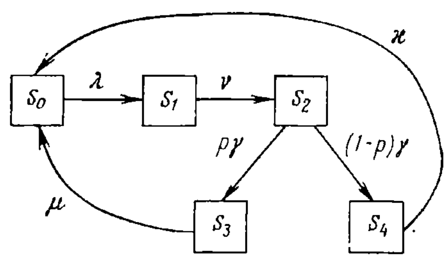


Fig. 10.32

Solution. The states of the device are: (s_0) the device is sound and operates normally; (s_1) the device is faulty but a failure is not detected, the device produces rejects; (s_2) the refusal is detected and the device is inspected; (s_3) the device is repaired; (s_4) the device is replaced. The directed graph of states is shown in Fig. 10.32.

The algebraic linear quations for the final probabilities of states are

$$\lambda p_0 = \mu p_3 + \varkappa p_4, \quad \lambda p_0 = \nu p_1, \quad \nu p_1 = \gamma p_2, \quad p\gamma p_2 = \mu p_3,$$
$$(1 - p)\,\gamma p_2 = \varkappa p_4.$$

The normalizing condition is $p_0 + p_1 + p_2 + p_3 + p_4 = 1$. Solving these equations, we get

$$p_0 = \left[1 + \frac{\lambda}{\nu} + \frac{\lambda}{\gamma} + \frac{p\lambda}{\mu} + \frac{(1-p)\,\lambda}{\varkappa}\right]^{-1},$$

$$p_1 = \frac{\lambda}{\nu}\,p_0, \quad p_2 = \frac{\lambda}{\gamma}\,p_0, \quad p_3 = \frac{p\lambda}{\mu}\,p_0, \quad p_4 = \frac{(1-p)\,\lambda}{\varkappa}\,p_0.$$

(1) The average time of normal operation of the device is p_0. (2) p_1. (3) For an average fraction of time p_3 the device is being repaired; every time the repair lasts for $1/\mu$ on the average; the flow of repaired devices, coming out of the state s_3, has intensity μp_3; the average cost of repairs per unit time is $r\mu p_3$. Similarly, the average cost of new devices per unit time is $c\varkappa p_4$; the total average cost of both is $r\mu p_3 + c\varkappa p_4$. (4) The average losses a faulty device entails per unit time are $l\nu p_1$.

10.33. We now consider how records are stored in a data base of a computer. The intensity of the records to be included in the bases is $\lambda(t)$ and does not depend on how many records have been stored. Every record is stored in the data base for an infinitely long period of time. Find the characteristics $m_x(t)$ and $\mathrm{Var}_x(t)$ of the random function $X(t)$, which

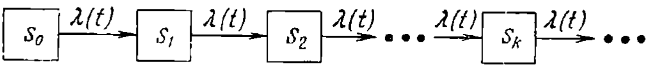


Fig. 10.33

is the number of records stored in the data base, provided that the incoming records arrive in a Poisson process with intensity $\lambda(t)$, and at $t = 0$ the random function $X(0) = 0$.

Solution. In this problem we deal with a process of "pure birth" without any restrictions being imposed on the number of states ($n \to \infty$). All the birth rates $\lambda_k = \lambda(t)$ (Fig. 10.33) and the death rates $\mu_k = \mu_k(t) \equiv 0$.

For this case differential equations (10.0.24) and (10.0.25) assume the form

$$\frac{dm_x(t)}{dt} = \sum_{k=0}^{\infty} \lambda(t)\, p_k(t) = \lambda(t),$$

$$\frac{d\,\mathrm{Var}_x(t)}{dt} = \sum_{k=0}^{\infty} [\lambda(t) + 2k\lambda(t) - 2m_x(t)\,\lambda(t)]\, p_k(t) = \lambda(t),$$

since

$$\sum_{k=0}^{\infty} p_k(t) = 1, \quad \sum_{k=0}^{\infty} k p_k(t) = m_x(t).$$

Solving these equations for the initial conditions $m_x(0) = \mathrm{Var}_x(0) = 0$, we obtain

$$m_x(t) = \mathrm{Var}_x(t) = \int_0^t \lambda(\tau)\, d\tau.$$

This result should have been expected since it can be found directly from the theory of flows. We can prove that for any moment t the random variable $X(t)$ has a Poisson distribution with the characteristics $m_x(t) = \mathrm{Var}_x(t)$ obtained.

10.34. The conditions are the same as in the preceding problem except that a record is stored in the data base for some time after which it is deleted according to some criterion. The flow of deletions for each record is a Poisson process with intensity $\mu(t)$.

Solution. In this problem we have a birth and death chain for the number of records in the data base. The death rate $\mu_k(t) = k\mu(t)$ since k records are stored in the data base in the state s_k and each record is acted upon by a flow of deletions with intensity $\mu(t)$.

The differential equations for the functions $m_x(t)$ and $\mathrm{Var}_x(t)$ have the form

$$\frac{dm_x(t)}{dt} = \sum_{k=0}^{\infty} (\lambda(t) - k\mu(t))\, p_k(t) = \lambda(t) - \mu(t)\, m_x(t), \quad (10.34.1)$$

$$\frac{d\,\mathrm{Var}_x(t)}{dt} = \sum_{k=0}^{\infty} [\lambda(t) + k\mu(t) + 2k\lambda(t) - 2k^2\mu(t) - 2m_x(t)\,\lambda(t)$$

$$+ 2m_x(t)\, k\mu(t)]\, p_k(t) = \lambda(t) + \mu(t)\, m_x(t) - 2\mu(t)\,\mathrm{Var}_x(t), \quad (10.34.2)$$

since

$$\sum_{k=0}^{\infty} p_k(t) = 1, \qquad \sum_{k=0}^{\infty} k p_k(t) = m_x(t),$$

$$\sum_{k=0}^{\infty} k^2 p_k(t) = \alpha_{2x}(t) = \mathrm{Var}_x(t) + m_x^2(t).$$

For the initial conditions $m_x(0) = m_0$ and $\mathrm{Var}_x(0) = \mathrm{Var}_0$; for constant intensities for the addition and deletion of the records, i.e. $\lambda(t) = \lambda$ and $\mu(t) = \mu$, the solutions of these equations have the form

$$m_x(t) = m_0 e^{-\mu t} + \frac{\lambda}{\mu}(1 - e^{-\mu t}) = \frac{\lambda}{\mu} + e^{-\mu t}\left(m_0 - \frac{\lambda}{\mu}\right),$$

$$\mathrm{Var}_x(t) = \lambda\left[\frac{1}{\mu} + \frac{e^{-2\mu t} - 2e^{-\mu t}}{\mu}\right] + (\lambda + \mu\,\mathrm{Var}_0 + \mu m_0)\,\frac{e^{-\mu t} - e^{-2\mu t}}{\mu}$$

$$+ \mathrm{Var}_0(2e^{-2\mu t} - e^{-\mu t}) = m_x(t) + (\mathrm{Var}_0 - m_0)\, e^{-2\mu t}.$$

For the initial conditions $m_0 = \mathrm{Var}_0 = 0$, we get $m_x(t) = \mathrm{Var}_x(t) = \frac{\lambda}{\mu}(1 - e^{-\mu t})$. We can prove that for these initial conditions the process of storing information $X(t)$ has a Poisson distribution for any moment t and for any kind of function $\lambda(t)$ (the intensity of the addition of a record), but for this to take place the deletion intensity μ for the records must be constant.

For constant λ and μ and $t \to \infty$, stationary conditions of information storage will be established, which naturally do not depend on the initial conditions: $m_x = \mathrm{Var}_x = \lambda/\mu$.

10.35. We consider the process of manufacturing a type of computer. The manufacturing intensity of the computer $\lambda(t)$ is shown in Fig. 10.35*a*. It increases linearly during the first year from 0 to 1000 com-

puters a year, for three years the production stays at 1000 computers a year, and then the computer is taken out of production. The average service life of a computer is five years. Considering all the flows of events to be elementary, find the mean value and variance of the number of computers in service at any moment t.

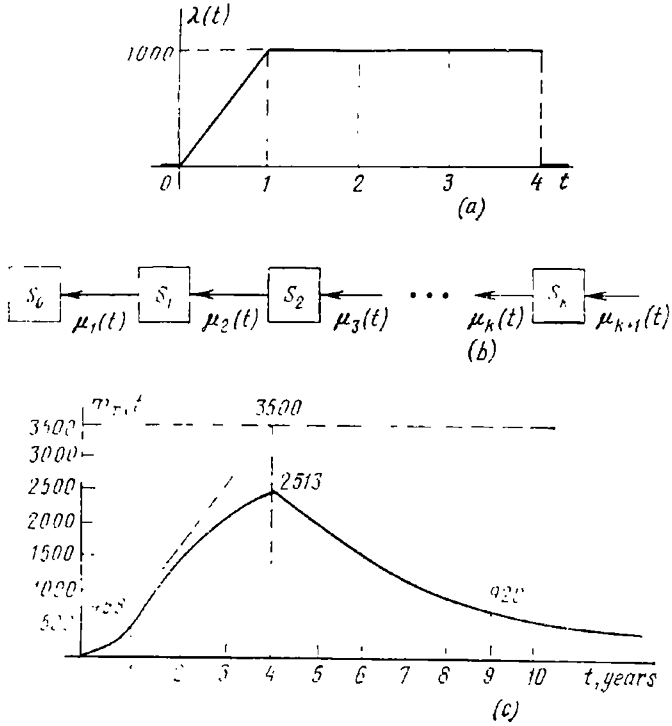


Fig. 10.35

Solution. The manufacturing intensity for the computer

$$
\lambda(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ kt & \text{for} \quad 0 \leqslant t < 1, \\ \lambda & \text{for} \quad 1 \leqslant t < 4, \\ 0 & \text{for} \quad t \geqslant 4, \end{cases}
$$

where $k = 1000$ 1/year^2, $\lambda = 1000$ 1/year.

Let us find the solutions of equations (10.34.1) and (10.34.2) for the time interval $0 \leqslant t < 1$ under the condition that $\mu = 0.2$ 1/year for any time intervals $t > 0$. Under these conditions equation (10.34.1) has the form

$$
\frac{dm_x(t)}{dt} = kt - \mu m_x(t).
$$

Solving this equation for the initial condition $m_x(0) = 0$, we obtain

$$m_x(t) = \frac{k}{\mu^2}(e^{-\mu t} - 1 + \mu t) = 25\,000\left(e^{-t/5} - 1 + \frac{t}{5}\right).$$

In a year's time an average of $m_x(1) = 25\,000\,(e^{-1/5} - 1 + 0.2) = 468$ computers will be in service. Note that if the service life of the computers were infinite, then up to 500 computers would be in service by the end of a year.

Under the same conditions equation (10.34.2) has the form

$$d\mathrm{Var}_x(t)/dt = 1000t + 0.2m_x(t) - 0.4\mathrm{Var}_x(t).$$

Solving this equation for the initial condition $\mathrm{Var}_x(0) = 0$, we get $\mathrm{Var}_x(t) = m_x(t) = 25\,000\,(e^{-1/5} - 1 + t/5)$. In a year the variance of the number of computers in service will be $\mathrm{Var}_x = 468$, $\sigma_x = 21.6$. The number of computers in service in a year's time will be approximately normally distributed with characteristics $m_x = 468$, $\sigma_x = 21.6$.

On the time interval $1 \leqslant t < 4$ the corresponding equations will have the form

$$\frac{dm_x(t)}{dt} = \lambda - \mu m_x(t), \quad \frac{d\,\mathrm{Var}_x(t)}{dt} = \lambda + \mu m_x(t) - 2\mu\,\mathrm{Var}_x(t).$$

We must solve them for the initial conditions $m_x(1) = \mathrm{Var}_x(1) = 468$. We found a solution of these equations in Problem 10.34, whence we have

$$m_x(t) = m_x(1)\,e^{-\mu(t-1)} + \frac{\lambda}{\mu}(1 - e^{-\mu(t-1)}) \quad (1 \leqslant t < 4),$$

$$\mathrm{Var}_x(t) = m_x(t).$$

Let us find the value of $m_x(t)$ for $t = 4$:

$$m_x(4) = m_x(1)\,e^{-3\mu} + \frac{\lambda}{\mu}(1 - e^{-3\mu}) = 2513.$$

Thus the average number of computers in service by the end of the fourth year is 2513. Pay attention to the fact that an average of 3500 computers were manufactured by that time. Consequently, an average of 987 computers were taken out of service during the four years.

A process of "pure death" whose graph is shown in Fig. 10.35*b* will take place on the time interval $t > 4$.

In a general case the differential equations for the mean value and variance of the pure death process have the form

$$\frac{dm_x(t)}{dt} = -\sum_{k=0}^{\infty} \mu_k(t)\,p_k(t),$$

$$\frac{d\,\mathrm{Var}_x(t)}{dt} = \sum_{k=0}^{\infty} [k\mu_k(t) - 2k^2\mu_k(t) + 2m_x(t)\,k\mu_k(t)]\,p_k(t).$$

For our case $\mu_k(t) = k\mu$ and, consequently, we get an equation

$$\frac{dm_x(t)}{dt} = -\sum_{k=0}^{\infty} k\mu p_k(t) = -\mu m_x(t),$$

which we must solve for the initial condition $m_x(4) = 2513$. The solution of this equation has the form

$$m_x(t) = m_x(4)\, e^{-\mu(t-4)} \quad (t > 4).$$

Since $\text{Var}_x(4) = m_x(4)$ and $\mu = \text{const}$, we have $\text{Var}_x(t) = m_x(t)$ on the time interval $t > 4$.

Figure 10.35c shows a relationship between $m_x(t)$, i.e. the average number of computers in service, and time t. The dash line shows a graph of the average number of computers manufactured by a certain time t vesus time t.

10.36. We now consider the process of storing terms in a directory in a data bank. The process is to include the terms in the directory when

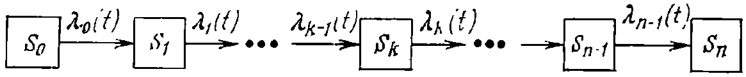


Fig. 10.36

they first appear. For example, the names of organizations for which a factory has production orders are entered into the data bank of the management information system. The entries for the names of the organisations will be accumulated in the data bank of the management information system in the order in which they appear in the input records.

There are an average of $\varkappa$ terms in the directory in every record fed into the data bank, and the intensity of feeding the records into the data bank is $\lambda(t)$. Consequently, the intensity of the flow of terms fed into the data bank is $\lambda(t) = \varkappa\tilde{\lambda}(t)$. We assume that this is a Poisson flow. The number of terms n is finite and nonrandom, although it may not be known beforehand. All the terms in the directory may appear in a record with equal probability while naturally only those terms are fed into the directory that did not yet appear in the records. Find the mean value and variance of the number of terms in the directory.

Solution. We designate the number of terms in the directory as $X(t)$. The random process $X(t)$ is evidently a pure birth process with a finite number of states n, whose graph of states is shown in Fig. 10.36. To find the intensity $\lambda_k(t)$ $(k = 0, 1, \ldots, n-1)$, we assume that the process is in the state s_k. By definition, the probability of this assumption is $p_k(t)$. Provided that the assumption is satisfied, the intensity of the flow of new (not yet in the directory) terms is

$$\lambda_k(t) = \lambda(t)\frac{n-k}{n} = \lambda(t)\left(1 - \frac{k}{n}\right).$$

The differential equations (10.0.24) and (10.0.25) assume the form

$$\frac{dm_x(t)}{dt}=\sum_{k=0}^{n}\lambda(t)\left(1-\frac{k}{n}\right)p_k(t)=\lambda(t)-\frac{m_x(t)\,\lambda(t)}{n}, \quad (10.36.1)$$

$$\frac{d\,\mathrm{Var}_x(t)}{dt}=\sum_{k=0}^{n}\left[\lambda(t)\left(1-\frac{k}{n}\right)+2k\lambda(t)\left(1-\frac{k}{n}\right)\right.$$
$$\left.-2m_x(t)\,\lambda(t)\left(1-\frac{k}{n}\right)\right]p_k(t)$$
$$=\lambda(t)-\lambda(t)\frac{m_x(t)}{n}-2\lambda(t)\frac{\mathrm{Var}_x(t)}{n}. \quad (10.36.2)$$

Let us solve these equations for the simple case when

$$\lambda(t)=\lambda=\text{const},\quad n=\text{const},\quad m_x(0)=\mathrm{Var}_x(0)=0.$$

$$m_x(t)=n\left(1-e^{-\frac{\lambda}{n}t}\right),\quad \lim_{t\to\infty}m_x(t)=n, \quad (10.36.3)$$

$$\mathrm{Var}_x(t)=n\left(1-e^{-\frac{\lambda}{n}t}\right)e^{-\frac{\lambda}{n}t}=m_x(t)\,e^{-\frac{\lambda}{n}t},$$
$$\lim_{t\to\infty}\mathrm{Var}_x(t)=0. \quad (10.36.4)$$

Note that the function $m_x(t)$ increases monotonically, tending to n in the limit, whereas the function $\mathrm{Var}_x(t)$ is zero for $t=0$ and $t\to\infty$ and attains its maximum for a certain value of t_m which can be found from the condition $d\,\mathrm{Var}_x(t)/dt=0$ $(t>0)$.

Hence

$$0.5=e^{-\frac{\lambda}{n}t_m}\to t_m\approx 0.7n/\lambda.$$

For this value of t_m the maximum variance $\max\mathrm{Var}_x(t)\approx n(1-e^{-0.7})\,e^{-0.7}=n\;0.25$, $\sigma_x(t_m)=0.5\sqrt{n}$, and the maximum value of the coefficient of variation $\sigma_x(t_m)/m_x(t_m)=1/\sqrt{n}$.

If we know the intensity λ of the flow of terms to the directory in the records arriving at the data bank and the total number of terms n, then we can determine, with a sufficient accuracy, the average time t_{fill} necessary to fill 95 per cent of the directory, i.e. $1-e^{-\frac{\lambda}{n}t_{\text{fill}}}=0.95$, whence $t_{\text{fill}}\approx 3n/\lambda$.

If n is unknown (which is a frequent occurrence in practice), we can find the estimate n of the quantity n as follows. We determine the actual numbers of the accumulated terms in the directory $m_1, m_2, \ldots, m_l$ at each moment $\tau_1, \tau_2, \tau_3, \ldots, \tau_l$ $(\tau_i<\tau_{i+1})$. We assume these quantities to be approximately equal to the average quantities of accumulated terms: $m_i=n\left(1-e^{-\lambda\tau_i/n}\right)$ $(i=1, 2, \ldots, l)$. Solving this equation for n, we find l values $n_1, n_2, \ldots, n_l$ for the corresponding pairs of values: (m_1, τ_1), (m_2, τ_2), $\ldots$, (m_l, τ_l). We estimate $\bar{n}$ from the formula

$$\bar{n}=\left(\sum_{i=1}^{l}n_i\right)/l.$$

10.37. For the conditions of the preceding problem find the time t_{fill} needed to fill the directory by 95 per cent and the probability that after two years of accumulation the directory will contain less than 90 per cent of all the possible terms if the total number of terms $n = 100\,000$, a total of 100 000 records are fed into the data bank per year and each document contains an average of 1.5 terms.

Solution. We find the intensity of flow of terms per record fed into the data bank per annum:

$$\lambda = 100\,000 \times 1.5 = 150\,000 \ 1/\text{year}.$$

We can find the quantity t_{fill} from the expression $t_{\text{fill}} = 3\ n/\lambda = 3 \times 100\,000/150\,000 = 2$ years. To find the probability that after two years of accumulating terms the directory will contain no less than 90 per cent of all terms, it is first of all necessary to find $m_x\,(2)$ and $\text{Var}_x\,(2)$ [see formulas (10.36.3) and (10.36.4)]: $m_x\,(2) = 100\,000\,(1 - e^{-1.5\times 2}) = 0.95 \times 100\,000 = 95\,000$; $\text{Var}_x\,(2) = 95\,000 \times 0.05 = 4750$; $\sigma_x\,(2) = 68.9$.

Note that the maximum variance $\text{Var}_x\,(t_m) = 0.25n = 25\,000$, $\sigma_x\,(t_m) = 158$.

At the moment $t = 2$ years the number of terms in the directory is a random variable $X\,(2)$, which is approximately normally distributed with the characteristics obtained above. Therefore, $P\,\{X\,(2) > 0.9n\} \approx 1$ since $m_x - 3\sigma_x > 0.9n$.

10.38. We consider a more general case for the operation of a data bank directory. The first complication, as compared to the hypothesis of Problem 10.36, is that the maximum number of terms n in the directory is not constant but a function of time t: $n\,(t)$ (in the case of a directory of the names of organizations this means that the total number of organisations varies with time, increasing or decreasing).

In addition, at some point in time a term fed into the directory is deleted from it because the term becomes obsolete. It is assumed that the flow of deletions is a Poisson process with intensity $\mu\,(t)$, which is the same for all the terms in the directory.

In that case the intensities of the birth and death flows have the form

$$\lambda_k\,(t) = \lambda\,(t)\left(1 - \frac{k}{n\,(t)}\right), \quad \mu_k\,(t) = \mu\,(t)\,k, \tag{10.38.1}$$

and equations (10.0.24) and (10.0.25) assume the form (the relations of the function $m_x\,(t)$, $\text{Var}_x\,(t)$, $n\,(t)$, $\lambda\,(t)$, $\mu\,(t)$, $p_k\,(t)$ and time t are omitted for the sake of brevity)

$$dm_x/dt = \lambda - m_x\,(\lambda/n + \mu),$$
$$d\text{Var}_x/dt = \lambda - m_x\,(\lambda/n - \mu) - 2\,(\lambda/n + \mu)\,\text{Var}_x. \tag{10.38.2}$$

If the quantities λ, n, μ are constant (do not depend on time), then, as $t \to \infty$, stationary conditions are possible, for which $dm_x/dt = d\text{Var}_x/dt = 0$, whence

$$m_x = n\left(1 + \frac{\mu n}{\lambda}\right)^{-1}, \quad \text{Var}_x = m_x\left(1 + \frac{\lambda}{\mu n}\right)^{-1}.$$

CHAPTER 11

Queueing Theory

11.0. A *queueing system* is one designed to serve customers (demands) arriving at random moments. Examples of queueing systems are a telephone exchange, a garage, a booking-office, a hairdresser's, an interactive computer system. Queueing theory deals with the stochastic processes taking place in queueing systems.

A facility which serves customers (demands) is called a *server* (*channel*). There are single-server (one-channel) and multi-server (multi-channel) queueing systems. A booking-office with one man at the window is an example of a single-server system, while a booking-office with several men at the windows is an example of a multi-server system.

We also differentiate between *congestion systems* (*systems with refusals*) and *delay queueing systems*. A customer arriving at a congestion system when all the servers are busy is refused service and departs without taking part in any further proceedings. In a delay queueing system, a customer arriving when all the servers are busy does not leave the system but joins the queue and waits for the server to become free. The number of places m in the queue may either be limited or unlimited. If $m = 0$, a delay system turns into a congestion system. A queue can be bounded both in terms of the number of customers in it (the size or length of the queue) and in terms of the queueing time (systems of this kind are known as "systems with impatient customers").

Delay systems are also subdivided in terms of the *queue discipline*, i.e. customers may be served either in the order of arrival or in a random order, or some customers may be able to obtain service before others (a priority service). A priority service may have several gradations, or ranks of prontos.

The analytic investigation of a queueing system can be performed in a simplest way when all the flows of events transferring it from state to state are elementary (stationary Poisson's). This means that the time intervals between the events in the flow have an exponential distribution with a parameter equal to the intensity of the corresponding flow. For a queueing system this assumption means that both the arrival process and the service process are stationary Poisson's. We use the term *service process* when the arriving customers are served one after another by one continuously busy server. This process is stationary Poisson's only if the service time of a customer T_{ser} is a random variable which has an exponential distribution. The parameter μ of this distribution is an inverse of the average service time: $\mu = 1/\bar{t}_{ser}$, where $\bar{t}_{ser} = M[T_{ser}]$. Instead of saying "the service process is stationary Poisson's" we can say "the service time is exponential". In what follows we shall call, for brevity, every queueing system in which all the processes are stationary Poisson's an *elementary* queueing system. In this chapter we shall mainly deal with elementary queueing systems.

If all the flows of events are stationary Poisson's, then the process in a queueing system is a Markov stochastic process with discrete states and continuous time. When certain conditions are fulfilled, a limit stationary state exists for this process in which neither the probabilities of the states nor the other characteristics of the process depend on time.

The problems the queueing theory deals with include finding the probabilities of various states of a queueing system and establishing a relationship between the parameters (the number of servers n, the intensity of the arrival process λ, the distribution of the service time, and so on) and the *characteristics of the efficiency* of the service system. For example, we can consider:

the average number of customers A served by a queueing system per unit time or the *absolute capacity* of the system for service;

the probability of serving an arriving customer Q or the *relative capacity* of the system for service $Q = A/\lambda$;

the probability of a refusal P_{ref}, i.e. the probability that an arriving customer will not be served, $P_{ref} = 1 - Q$;

the average number of customers present in a system (being served or in the queue) $\bar{z}$;

the average number of customers in the queue $\bar{r}$;

the average waiting time of a customer (the average time a customer is present in the system) $\bar{t}_w$;

the average queueing time of a customer (the average time a customer spends in the queue) $\bar{t}_q$;

the average number of busy servers $\bar{k}$.

In a general case all these characteristics depend on time. But many service systems operate under the same conditions for a sufficiently long time and, therefore, a situation close to stationary is established. Without specifying it every time, we shall everywhere calculate limiting probabilities of the states and the limiting characteristics of the efficiency of a queueing system with respect to its limiting steady-state operation.

For any open*) queueing system operating under limiting stationary conditions the average waiting time of a customer $\bar{t}_w$ is expressed in terms of the average number of customers in the system by *Little's formula*:

$$\bar{t}_w = \bar{z}/\lambda, \tag{11.0.1}$$

where λ is the intensity of the arrival process.

A similar formula (also known as Little's formula) relates the average queueing time of a customer $\bar{t}_q$ and the average number $\bar{r}$ of customers in the queue:

$$\bar{t}_q = \bar{r}/\lambda. \tag{11.0.2}$$

Little's formulas are very useful since they make it possible to calculate one of the two characteristics of the efficiency (the average queueing time for a customer and the average number of customers) without necessarily calculating the other.

It should be emphasized that formulas (11.0.1) and (11.0.2) are valid *for any open queueing system* (single-server, multi-server, and for any kind of arrival and service process), the only requirement being that the arrival and service processes must be stationary.

Another very important expression for an open queueing system is a formula which expresses the average number of busy servers $\bar{k}$ in terms of the absolute capacity of the system for service A:

$$\bar{k} = A/\mu, \tag{11.0.3}$$

where $\mu = 1/\bar{t}_{ser}$ is the intensity of the service process.

Many problems in queueing theory concerning elementary queueing systems can be solved using a birth and death chain (see Chapter 10). If the directed graph of states of a queueing system can be represented as in Fig. 11.0.1, then the limiting probabilities for the states can be expressed by formulas (10.0.23), i.e.

$$p_0 = \left\{1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \ldots + \frac{\lambda_0\lambda_1 \ldots \lambda_{k-1}}{\mu_1\mu_2 \ldots \mu_k} + \ldots + \frac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n}\right\}^{-1},$$

$$p_1 = \frac{\lambda_0}{\mu_1} p_0, \quad p_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2} p_0, \ \ldots,$$

*) A queueing system is said to be *open* if the intensity of the arrival process does not depend on the state of the system itself.

$$p_k = \frac{\lambda_0\lambda_1 \ldots \lambda_{k-1}}{\mu_1\mu_2 \ldots \mu_k} p_0 \quad (0 \leqslant k \leqslant n), \ \ldots,$$
$$p_n = \frac{\lambda_0\lambda_1 \ldots \lambda_{n-1}}{\mu_1\mu_2 \ldots \mu_n} p_0. \tag{11.0.4}$$

When deriving a formula for the average number of customers (in a queue or in a system), we often use the technique for differentiating a series. Thus if $x < 1$, then

$$\sum_{k=1}^{\infty} kx^k = x \sum_{k=1}^{\infty} \frac{d}{dx} x^k = x \frac{d}{dx} \sum_{k=1}^{\infty} x^k = x \frac{d}{dx} \frac{x}{1-x} = \frac{x}{(1-x)^2},$$

and finally

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}. \tag{11.0.5}$$

Below we shall present without proof a number of formulas for the limiting probabilities of states and the characteristics of the efficiency for certain often encountered queueing systems. Other examples of queueing systems will be considered in the form of problems.

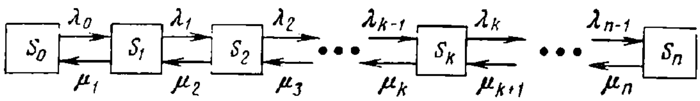


Fig. 11.0.1

1. **An elementary congestion system** (Erlang's problem). Consider an n-server system with refusals at which customers arrive in a stationary Poisson process with intensity λ; the service time is exponential with parameter $\mu = 1/\bar{t}_{\text{ser}}$. The states of the system are numbered in accordance with the number of customers in the system (since there is no queue it coincides with the number of busy servers). Thus we have

s_0—the system is idle;
s_1—one server is busy and the other servers are idle; . . .;
s_k—k servers are busy and the other servers are idle ($1 \leqslant k \leqslant n$); . . . ;
s_n—all n servers are busy.

The limiting probabilities of states are expressed by Erlang's formula:

$$p = \left\{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!} + \ldots + \frac{\rho^n}{n!}\right\}^{-1};$$
$$p_k = \frac{\rho^k}{k!} p_0 \quad (k = 1, 2, \ldots, n), \tag{11.0.6}$$

where $\rho = \lambda/\mu$.

The characteristics of the efficiency are

$$A = \lambda(1 - p_n), \quad Q = 1 - p_n, \quad \mathrm{P_{ref}} = p_n, \quad \bar{k} = \rho(1 - p_n). \tag{11.0.7}$$

For large values of n it is convenient to calculate the probabilities of states given in (11.0.6) in terms of the tabulated functions:

$$P(m, a) = \frac{a^m}{m!} e^{-a} \quad \text{(for a Poisson distribution)} \tag{11.0.8}$$

and

$$R(m, a) = \sum_{k=0}^{m} \frac{a^k}{k!} e^{-a} \tag{11.0.9}$$

(see Appendices 1 and 2), the first of which can be expressed in terms of the second:

$$\mathrm{P}(m, a) = R(m, a) - R(m-1, a). \tag{11.0.10}$$

Using these functions, we can rewrite Erlang's formulas (11.0.6) in the form

$$p_k = \mathrm{P}(k, \rho)/R(n, \rho) \quad (k = 0, 1, \ldots, n). \tag{11.0.11}$$

2. **An elementary single-server system with an unbounded queue.** Consider a single-server system at which customers arrive in a stationary Poisson process with intensity λ. The service time is exponential with parameter $\mu = 1/\bar{t}_{\text{ser}}$. The length of the queue is unlimited. The limiting probabilities exist only for $\rho = \lambda/\mu < 1$ (for $\rho \geqslant 1$ the queue increases indefinitely). The states of the system are numbered in accordance with the number of customers who are in the queue or being served. Thus

s_0—the system is idle;
s_1—the server is busy and there is no queue;
s_2—the server is busy and one customer is waiting to be served; . . . ;
s_k—the server is busy and $k - 1$ customers are in the queue; . . . .

The limiting probabilities of states are expressed by the formulas

$$p = 1 - \rho; \quad p_k = \rho^k (1 - \rho) \quad (k = 1, 2, \ldots), \tag{11.0.12}$$

where $\rho = \lambda/\mu < 1$.

The characteristics of the efficiency of the system are

$$A = \lambda, \quad Q = 1, \quad \mathrm{P}_{\text{ref}} = 0, \tag{11.0.13}$$

$$\bar{z} = \frac{\rho}{1-\rho}, \quad \bar{r} = \frac{\rho^2}{1-\rho}, \quad \bar{t}_{\text{w}} = \frac{\rho}{\lambda(1-\rho)},$$

$$\bar{t}_{\text{q}} = \frac{\rho^2}{\lambda(1-\rho)}, \tag{11.0.14}$$

and the average number of busy servers (or the probability that the server is busy) is

$$\bar{k} = \lambda/\mu = 0. \tag{11.0.15}$$

3. **An elementary single-server system with a bounded queue.** Consider a single-server queueing system at which customers arrive in a stationary Poisson process with intensity λ; the service time is exponential with parameter $\mu = 1/\bar{t}_{\text{ser}}$. There are n places in the queue. If a customer arrives when all the places are occupied, he is refused service and departs. The states of the system are:

s_0—the system is idle;
s_1—the server is busy and there is no queue;
s_2—the server is busy and one customer is in the queue;
s_k—the server is busy and $k - 1$ customers are in the queue;
s_{m+1}—the server is busy and m customers are in the queue.

The limiting probabilities of states exist for any $\rho = \lambda/\mu$ and are

$$p_0 = \frac{1-\rho}{1-\rho^{m+2}}; \quad p_k = \rho^k p_0 \quad (k = 1, \ldots, m+1). \tag{11.0.16}$$

The characteristics of the efficiency of the system are

$$A = \lambda(1 - p_{m+1}); \quad Q = 1 - p_{m+1}; \quad \mathrm{P}_{\text{ref}} = p_{m+1}.$$

The average number of busy servers (the probability that the server is busy) is

$$\bar{k} = 1 - p_0. \tag{11.0.17}$$

The average number of customers in the queue

$$\bar{r} = \frac{\rho^2 [1 - \rho^m (m + 1 - m\rho)]}{(1 - \rho^{m+2})(1 - \rho)}. \tag{11.0.18}$$

The average number of customers in the system

$$\bar{z}=\bar{r}+\bar{k}. \tag{11.0.19}$$

By Little's formula

$$\bar{t}_{\mathrm{w}}=\bar{z}/\lambda; \quad \bar{t}_{\mathrm{q}}=\bar{r}/\lambda. \tag{11.0.20}$$

4. **An elementary multi-server system with an unbounded queue.** Consider an n-server queueing system at which customers arrive in a stationary Poisson process with intensity λ; the service time of a customer is exponential with parameter $\mu = 1/t_{\mathrm{ser}}$. Limiting probabilities exist only for $\rho/n = \varkappa < 1$, where $\rho = \lambda/\mu$. The states of the system are numbered in accordance with the number of customers in the system, hence

s_0—the system is idle;
s_1—one server is busy; ... ;
s_k—k servers are busy ($1 \leqslant k \leqslant n$); ... ;
s_n—all n servers are busy; (there is no queue)
s_{n+1}—all n servers are busy and one customer is in the queue;
s_{n+r}—all the n servers are busy and r customers are in the queue.

The limiting probabilities of states are given by the formulas

$$p_0=\left\{1+\frac{\rho}{1!}+\ldots+\frac{\rho^n}{n!}+\frac{\rho^{n+1}}{n\cdot n!}\,\frac{1}{1-\varkappa}\right\}^{-1},$$

$$p_k=\frac{\rho^k}{k!}\,p_0 \quad (1\leqslant k\leqslant n); \quad p_{n+r}=\frac{\rho^{n+r}}{n^r\cdot n!}\,p_0 \quad (r\geqslant 1). \tag{11.0.21}$$

Using the functions $\mathrm{P}(m, a)$ and $R(m, a)$, we can reduce (11.0.21) to the form

$$p_k=\frac{\mathrm{P}(k,\rho)}{R(n,\rho)+\mathrm{P}(n,\rho)\left(\dfrac{\varkappa}{1-\varkappa}\right)} \quad (k=0, \ldots, n);$$
$$p_{n+r}=\varkappa^r p_n \quad (r=1, 2, \ldots). \tag{11.0.22}$$

The characteristics of the efficiency of the system are

$$\bar{r}=\rho^{n+1}p_0/[n\cdot n!\,(1-\varkappa)^2]=\varkappa p_n(1-\varkappa)^2; \tag{11.0.23}$$

$$\bar{z}=\bar{r}+\bar{k}=\bar{r}+\rho; \tag{11.0.24}$$

$$\bar{t}_{\mathrm{w}}=\bar{z}/\lambda; \quad \bar{t}_{\mathrm{q}}=\bar{r}/\lambda. \tag{11.0.25}$$

5. **An elementary multi-server system with a bounded queue.** The conditions and the numbering of the states are the same as in item 4 except that the number m of places in the queue is limited. Limiting probabilities of states exist for any λ and μ and are expressed by the formulas

$$p_0=\left\{1+\frac{\rho}{1!}+\ldots+\frac{\rho^n}{n!}+\frac{\rho^{n+1}}{n\cdot n!}\,\frac{1-\varkappa^m}{1-\varkappa}\right\}^{-1};$$

$$p_k=\frac{\rho^k}{k!}\,p_0\,(1\leqslant k\leqslant n);$$

$$p_{n+r}=\frac{\rho^{n+r}}{n^r\cdot n!}\,p_0 \quad (1\leqslant r\leqslant m), \tag{11.0.26}$$

where $\varkappa=\rho/n=\lambda/(n\mu)$

The characteristics of the efficiency of the system are

$$A=\lambda(1-p_{n+m});\quad Q=1-p_{n+m};\quad P_{\text{ref}}=p_{n+m};\quad \bar{k}=\rho(1-p_{n+m}); \tag{11.0.27}$$

$$\bar{r}=\frac{\rho^{n+1}p_0}{n\cdot n!}\,\frac{1-(m+1)\varkappa^m+m\varkappa^{m+1}}{(1-\varkappa)^2}; \tag{11.0.28}$$

$$\bar{z}=\bar{r}+\bar{k}; \tag{11.0.29}$$

$$\bar{t}_{\text{q}}=\bar{r}/\lambda;\quad \bar{t}_{\text{w}}=\bar{z}/\lambda. \tag{11.0.30}$$

6. **A multi-server congestion system for a stationary Poisson arrival and an arbitrary service time.** Erlang's formula (11.0.6) remains valid in the case when the arrival process is stationary Poisson's and the service time T_{ser} has an arbitrary distribution with expectation $\bar{t}_{\text{ser}} = 1/\mu$.

7. **A single-server system with an unbounded queue for a stationary Poisson arrival and an arbitrary service time.** If the arrivals at a single-server queueing system come in a stationary Poisson flow with intensity λ, and the service time T_{ser} has an exponential distribution with expectation $1/\mu$ and the coefficient of variation v_μ, then the average number of customers in the queue is expressed by the *Pollaczek-Khinchine formula*

$$\bar{r} = \rho^2 (1 + v_\mu)^2/[2 (1 - \rho)], \tag{11.0.31}$$

where $\rho = \lambda/\mu$, and the average number of customers in the system

$$\bar{z} = \{\rho^2 (1 + v_\mu)^2/[2 (1 - \rho)]\} + \rho. \tag{11.0.32}$$

Using Little's formula, we get from (11.0.31) and (11.0.32),

$$\bar{t}_{\text{q}}=\frac{\rho^2(1+v_\mu)^2}{2\lambda(1-\rho)},\quad \bar{t}_{\text{w}}=\frac{\rho^2(1+v_\mu^2)}{2\lambda(1-\rho)}+\frac{1}{\mu}. \tag{11.0.33}$$

8. **A single-server queueing system for an arbitrary (Palm) arrival process and an arbitrary service time.** There are no exact formulas for this case, and an approximation for the length of the queue is given by the formula

$$\bar{r} \approx \rho^2 (v_\lambda^2 + v_\mu^2)/[2 (1 - \rho)], \tag{11.0.34}$$

where v_λ is the coefficient of variation of the interarrival time; $\rho = \lambda/\mu$; λ is the inverse of the expectation of that time; $\mu = 1/\bar{t}_{\text{ser}}$ is the inverse of the average service time; v_μ is the coefficient of variation of the service time. The average number of customers in the queueing system

$$\bar{z} \approx \{\rho^2 (v_\lambda^2 + v_\mu^2)/[2 (1 - \rho)]\} + \rho, \tag{11.0.35}$$

and the average queueing and waiting times of a customer are

$$\bar{t}_{\text{q}} \approx \rho^2 (v_\lambda^2 + v_\mu^2)/[2\lambda (1 - \rho)], \tag{11.0.36}$$

and

$$\bar{t}_{\text{w}} \approx \{\rho^2 (v_\lambda^2 + v_\mu^2)/[2\lambda (1 - \rho)]\} + 1/\mu \tag{11.0.37}$$

respectively.

9. **An elementary multi-phase delay system.** It is difficult to analyse multiphase queueing systems in the general case since the arrival process of each successive phase is the departure process of the preceding phase and there are aftereffects in the general case. However, *if customers arrive in a stationary Poisson process at a queueing system with an unbounded queue and the service time is exponential, then the departure process is stationary Poisson's* with the same intensity λ as the arrival. It follows that a multi-phase queueing system with an unbounded queue before each phase, a stationary Poisson arrival and an exponential service time at each phase can be analysed as a simple sequence of elementary queueing systems.

If the queue before a phase is bounded, then the departure from that phase is no longer stationary Poisson's and the above technique can only be used for approximations.

In what follows we shall use the notations for the efficiency characteristics as given on pp. 363 and 364, but we shall introduce new ones when needed. When we define the density $f(x)$ on the x-axis, we shall not indicate the values of $f(x)$ on the boundaries of those parts.

If the unit of time is not fixed, we shall designate, for brevity, the intensities of the processes simply by letters, e.g. λ, μ, . . . (without indicating the dimensions). The same refers to the time $\bar{t}_w$ and $\bar{t}_q$. Now if the unit of time is fixed (a minute, an hour, a year, etc.), we shall indicate the units of measurement.

Rather than using a distribution function $F(t)$ it will be more convenient in this chapter to write the distribution of a mixed random variable T as a "generalized" density $f(t)$ which is defined as

$$f(t) = F'_c(t) + \sum_i p_i \delta(t - t_i),$$

where $F'_c(t)$ is the derivative of the distribution function on the intervals of its continuity, $p_i = \mathrm{P}\{T = t_i\}$, and $\delta(x)$ is the delta-function whose properties are given in Appendix 6.

Problems and Exercises

11.1. Consider a single-server congestion system at which customers arrive in a stationary Poisson process with intensity λ, the service time being exponential with parameter μ. At the initial moment $t = 0$ the server is idle. Construct the marked graph of states for the system. Write and solve the Kolmogorov differential equations for the probabilities of states of the system. Find the limiting probabilities of the states and (for steady-state conditions) the characteristics of the efficiency of the queueing system, i.e. A, Q, $\mathrm{P}_{\mathrm{ref}}$, $\bar{k}$.

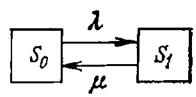


Fig. 11.1

Solution. The states of the queueing system are: s_0, it is idle; s_1, the server is busy. The graph of states is shown in Fig. 11.1. The Kolmogorov equations are

$$dp_0/dt = -\lambda p_0 + \mu p_1, \quad dp_1/dt = \lambda p_0 - \mu p_1. \qquad (11.1.1)$$

Since $p_0 + p_1 = 1$ for any t, we can express p_1 in terms of p_0, i.e. $p_1 = 1 - p_0$, and get one equation for p_0:

$$dp_0/dt = -(\lambda + \mu) p_0 + \mu. \qquad (11.1.2)$$

Solving this equation, we get p_0 as a function of t:

$$p_0(t) = \frac{\mu}{\lambda + \mu}\left[1 + \frac{\lambda}{\mu} e^{-(\lambda+\mu)t}\right],$$

hence

$$p_1(t) = 1 - p_0(t) = \frac{\lambda}{\lambda + \mu}\left[1 - e^{-(\lambda+\mu)t}\right].$$

When $t \to \infty$, we get limiting probabilities, viz.

$$p_0 = \mu/(\lambda + \mu), \quad p_1 = \lambda/(\lambda + \mu). \qquad (11.1.3)$$

We can find them much more easily by solving algebraic linear equations for the limiting probabilities of states:

$$\lambda p_0 = \mu p_1, \quad p_0 + p_1 = 1.$$

We can rewrite formulas (11.1.3) in a more concise form if we introduce a designation $\rho = \lambda/\mu$:

$$p_0 = 1/(1 + \rho), \quad p_1 = \rho/(1 + \rho).$$

The characteristics of the efficiency of the queueing system are

$$A = \lambda p_0 = \frac{\lambda}{1+\rho}, \quad Q = \frac{1}{1+\rho}, \quad \mathrm{P}_{\mathrm{ref}} = p_1 = \frac{\rho}{1+\rho}, \quad \bar{k} = 1 - p_0 = \frac{\rho}{1+\rho}. \tag{11.1.4}$$

11.2. Given a telephone line which is a single-server congestion system, calls arrive at its input in a stationary Poisson process. The traffic intensity is $\lambda = 0.4$ calls per minute. The average duration of a conversation $\bar{t}_{\mathrm{ser}} = 3$ min and has an exponential distribution. Find the limiting probabilities of the states of the system: p_0 and p_1, as well as A, Q, $\mathrm{P}_{\mathrm{ref}}$ and $\bar{k}$. Compare the capacity of the system for service and its rated capacity if each conversation lasts exactly three minutes and the calls arrive regularly, one after another, without a break.

Solution. $\lambda = 0.4$, $\mu = 1/\bar{t}_{\mathrm{ser}} = 1/3$, $\rho = \lambda/\mu = 1.2$. By formulas (11.1.3), $p_0 \approx 1/2.2 \approx 0.455$, $p_1 \approx 0.545$, $Q \approx 0.455$, $A = \lambda Q \approx 0.182$, $\bar{k} = p_1 \approx 0.545$.

Thus, on the average, the telephone line will serve 0.455 calls, i.e. 0.182 calls per minute. The rated capacity of the server would be (if the calls arrived and were served regularly) $A_{\mathrm{rate}} = 1/\bar{t}_{\mathrm{ser}} = 1/3 \approx 0.333$ calls/min, and that is almost twice as large as the actual capacity A.

11.3. We are given a single-server congestion system with customers arriving in a stationary Poisson process with intensity λ. The service

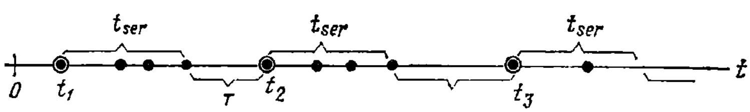


Fig. 11.3

time is nonrandom and is exactly $t_{\mathrm{ser}} = 1/\mu$. Find the relative and absolute capacities of the system for service in the limiting stationary state.

Solution. Let us consider a stationary Poisson arrival process with intensity λ on the t-axis (Fig. 11.3). We shall denote all the customers who are served by circles. Assume that a customer who arrived at the moment t_1 is being served. Then all the customers who arrive after him during the time t_{ser} will not be served. The next

to be served will be the customer who arrives at a moment t_2 such that $t_2 - t_1 > t_{ser}$. Let us consider the interval T between the end of the first customer service period and the moment t_2 the next customer to be served arrives. Since there are no aftereffects in an elementary flow, the distribution of the interval T is the same as that of the interarrival time in the general case, i.e. exponential with parameter λ. The average length of the interval T is $m_t = 1/\lambda$.

Thus nonrandom busy periods of the server ($t_{ser} = 1/\mu$ long) and random idle periods (average length $1/\lambda$) will alternate on the t-axis. A fraction of all the customers

$$\frac{1/\mu}{1/\mu+1/\lambda}=\frac{\lambda}{\lambda+\mu},$$

will fall on the first intervals and a fraction

$$\mu/(\lambda + \mu) = 1/(1 + \rho), \quad \text{where} \quad \rho = \lambda/\mu$$

will fall on the second intervals. The latter quantity is the relative capacity of the queueing system for service

$$Q = 1/(1 + \rho), \tag{11.3.1}$$

whence

$$A = \lambda Q = \lambda/(1 + \rho). \tag{11.3.2}$$

Note that formulas (11.3.1) and (11.3.2) coincide with (11.1.4), which corresponds to the exponential distribution of the service time. And this is natural since Erlang's formulas remain valid for any distribution of the service time with the mean value equal to $1/\mu$.

11.4. Using formula (11.0.5), prove that for an elementary single-server system with an unbounded queue the average number of customers being served is $\bar{z} = \rho/(1 - \rho)$, where $\rho = \lambda/\mu$, and the average number of customers in the queue is $\bar{r} = \rho^2/(1 - \rho)$.

Solution. By formulas (11.0.12) $p_0 = 1 - \rho$, $p_k = \rho^k (1 - \rho)$ $(k = 1, 2, \ldots)$.

We designate the actual (random) number of customers in the system as Z:

$$\bar{z}=\mathrm{M}[Z]=\sum_{k=0}^{\infty} kp_k=\sum_{k=1}^{\infty} k\rho^k(1-\rho)=(1-\rho)\sum_{k=1}^{\infty} k\rho^k.$$

By formula (11.0.5) for $\rho < 1$

$$\sum_{k=1}^{\infty} k\rho^k=\frac{\rho}{(1-\rho)^2},$$

whence $\bar{z} = \rho/(1 - \rho)$. The average number of customers in the queue is $\bar{z}$ minus the average number of busy servers $\bar{k} = A/\mu = \lambda/\mu = \rho$, i.e. $\bar{r} = [\rho/(1 - \rho)] - \rho = \rho^2/(1 - \rho)$.

11.5. A railway shunting yard is a single-server queueing system with an unbounded queue at which trains arrive in a stationary Poisson process. The traffic intensity is $\lambda = 2$ trains per hour. The service time (shunting) of a train at the yard has an exponential distribution with mean value $\bar{t}_{ser} = 20$ min. Find the limiting probabilities of states of the system, the average number z of trains at the yard, the average number $\bar{r}$ of trains in the queue, the average waiting time $\bar{t}_w$ of a train, and the average queueing time $\bar{t}_q$ of a train.

Solution. $\lambda = 2$ trains/h; $\bar{t}_{ser} = 1/3$ h; $\mu = 3$ trains/h; $\rho = \lambda/\mu = 2/3$. By formulas (11.0.12) we have $p_0 = 1 - 2/3 = 1/3$, $p_1 = (2/3)(1/3) = 2/9$, $p_2 = (2/3)^2 (1/3) = 4/27, \ldots, p_k = (2/3)^k (1/3)$, etc. By formulas (11.0.13) and (11.0.14) $\bar{z} = \rho/(1-\rho) = 2$ trains, $\bar{r} = 4/3$ trains, $\bar{t}_w = 1$ h, and $\bar{t}_q = 2/3$ h.

11.6. The hypothesis of the preceding problem is complicated by the condition that no more than three trains can be present simultaneously in the shunting yard (including the train being served). If a train arrives at a moment when there are three trains at the yard, it has to join the queue outside the yard on a side track. The station has to pay a fine of a roubles per hour if a train stays on the side track. Find the average fine per day the station has to pay for trains staying on the side track.

Solution. We calculate the average number of trains z_{side} on the side track:

$$\bar{z}_{side} = 1 \cdot p_4 + 2p_5 + \ldots = \sum_{k=4}^{\infty} kp_k = \sum_{k=4}^{\infty} k\rho^k p_0 = p_0 \sum_{k=4}^{\infty} k\rho^k,$$

$$\sum_{k=4}^{\infty} k\rho^k = p \sum_{k=4}^{\infty} \frac{d}{d\rho} \rho^k = \rho \sum_{k=4}^{\infty} \frac{d}{d\rho} \rho^k = \rho \frac{d}{d\rho} \sum_{k=4}^{\infty} \rho^k$$

$$= \rho \frac{d}{d\rho} \frac{\rho^4}{1-\rho} = \frac{\rho^4 (4-3\rho)}{(1-\rho)^2},$$

$$\bar{z} = p_0 \sum_{k=4}^{\infty} k\rho^k = \frac{\rho^4 (4-3\rho)}{1-\rho} \approx 1.18.$$

By Little's formula the average time a train stays on the side track $\bar{t}_{side} \approx 1.18/\lambda = 1.18/2 = 0.59$ h. On the average $24\,\lambda = 48$ trains arrive at the station every 24 hours. The average daily fine is $48 \times 0.59\,a \approx 28.3a$.

11.7. Using the birth and death scheme, calculate directly from the directed graph of states the limiting probabilities of states for a simple two-server queueing system ($n = 2$) with three places in the queue ($m = 3$) for $\lambda = 0.6$, $\mu = 0.2$ and $\rho = \lambda/\mu = 3$. Find the characteristics of this queueing system, i.e. $\bar{z}$, $\bar{r}$, $\bar{t}_w$, $\bar{t}_q$ without using formulas (11.0.26), proceeding rather from the limiting probabilities, and compare the results with those obtained from (11.0.26).

Solution. The directed graph of states of the queueing system is shown in Fig. 11.7. Introducing the designation $\lambda/\mu = \rho$, we get the

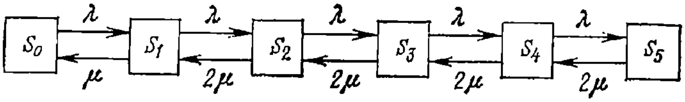


Fig. 11.7

following, using a birth and death chain:

$$p_0 = \left\{1+\rho+\frac{\rho^2}{2}+\frac{\rho^3}{2^2}+\frac{\rho^4}{2^3}+\frac{\rho^5}{2^4}\right\}^{-1} = (40.58)^{-1} \approx 0.025,$$

$$p_1 = \frac{3}{40.58} \approx 0.074, \quad p_2 = \frac{4.5}{40.58} \approx 0.111, \quad p_3 = \frac{6.75}{40.58} \approx 0.165,$$

$$p_4 = \frac{10.15}{40.58} \approx 0.250, \quad p_5 = \frac{15.18}{40.58} \approx 0.375,$$

$$\bar{z} = 1\times 0.074+2\times 0.111+3\times 0.165+4\times 0.250+5\times 0.375 \approx 3.67,$$

$$\bar{r} = 1\times 0.165+2\times 0.250+3\times 0.375 \approx 1.79, \quad \bar{t}_{\mathrm{w}} = \bar{z}/0.6 \approx 6.11,$$

$$\bar{t}_{\mathrm{q}} = \bar{r}/0.6 \approx 2.98.$$

11.8. The formula for $\bar{r}$ (11.0.28) is valid for any $\varkappa < 1$ or $\varkappa > 1$. For $\varkappa = 1$ it is no longer valid, yielding an undetermined value 0/0).

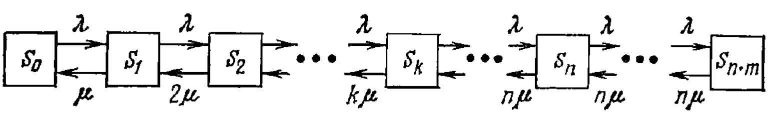


Fig. 11.8

Directly from the birth and death chain derive the probabilities of the states $p_0, p_1, \ldots, p_{n+m}$ for this case and find the efficiency characteristics of the queueing system, i.e. A, Q, P_{ref}, $\bar{k}$, $\bar{r}$, $\bar{z}$, $\bar{t}_{\mathrm{q}}$, $\bar{t}_{\mathrm{w}}$.

Solution. The directed graph of states of the queueing system has the form shown in Fig. 11.8. Using the general formulas for the birth and death scheme and designating $\lambda/\mu = \rho$, we obtain

$$p_0 = \left\{1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\ldots+\frac{\rho^n}{n!}+\frac{\rho^{n+1}}{n\cdot n!}+\ldots+\frac{\rho^{n+m}}{n^m\cdot n!}\right\}^{-1}$$
$$= \left\{1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\ldots+\frac{\rho^n}{n!}\left[\frac{\rho}{n}+\left(\frac{\rho}{n}\right)^2+\ldots+\left(\frac{\rho}{n}\right)^m\right]\right\}^{-1}.$$

For $\varkappa = \rho/n = 1$

$$p_0 = \left\{1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\ldots+\frac{\rho^n}{n!}+\frac{m\rho^n}{n!}\right\}^{-1}, \qquad (11.8.1)$$

$$p_k = \frac{\rho^k}{k!}\,p_0 \ (1 \leqslant k \leqslant n), \quad p_{n+r} = \frac{\rho^n}{n!}\,p_0 \ (1 \leqslant r \leqslant m), \qquad (11.8.2)$$

$$P_{\text{ref}} = p_{n+m}, \quad Q = 1 - p_{n+m} = 1 - \frac{\rho^n}{n!} p_0,$$
$$A = \lambda Q = \lambda\left[1 - \frac{\rho^n}{n!} p_0\right], \quad \bar{k} = A/\mu = \rho\left[1 - \frac{\rho^n}{n!} p_0\right],$$
$$\bar{r} = \sum_{r=1}^{m} r p_{n+r} = \sum_{r=1}^{m} r \frac{\rho^n}{n!} p_0 = \frac{\rho^n}{n!} p_0 \sum_{r=1}^{m} r = \frac{\rho^n}{n!} \frac{m(m+1)}{2} p_0,$$
$$\bar{z} = \bar{r} + \bar{k}, \quad \bar{t}_{\text{w}} = \bar{z}/\lambda, \quad \bar{t}_{\text{q}} = \bar{r}/\lambda. \tag{11.8.3}$$

11.9. A petrol station has two petrol pumps ($n = 2$), but its forecourt can only hold four waiting cars ($m = 4$). The arrival of cars at the station is stationary Poisson's with intensity $\lambda = 1$ car/min. The service time of a car is exponential with mean values $\bar{t}_{\text{ser}} = 2$ min. Find the limiting probabilities of the states of the petrol station and its characteristics A, Q, P_{ref}, $\bar{k}$, $\bar{z}$, $\bar{r}$, $\bar{t}_{\text{w}}$, $\bar{t}_{\text{q}}$.

Solution. $\lambda = 1$, $\mu = 1/2 = 0.5$, $\rho = 2$, $\varkappa = \rho/n = 1$. From formulas (11.8.1)-(11.8.3) we find that

$$p_0 = \left\{1 + 2 + \frac{2^2}{2!} + \frac{2^2}{2!} \cdot 4\right\}^{-1} = \frac{1}{13},$$
$$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{2}{13},$$
$$P_{\text{ref}} = 2/13, \quad Q = 1 - P_{\text{ref}} = 11/13, \quad A = \lambda Q = 11/13$$
$$\approx 0.85 \text{ cars/min},$$
$$\bar{k} = A/\mu = 22/13 \approx 1.69 \text{ pumps},$$
$$\bar{r} = \frac{2^2}{2!} \frac{4(4+1)}{2} \cdot \frac{1}{13} \approx 1.54 \text{ cars}, \quad \bar{z} = \bar{r} + \bar{k} \approx 3.23 \text{ cars}.$$

11.10. Customers arrive at a two-server congestion system at a rate of $\lambda = 4$ customers per hour. The average service time $t_{\text{ser}} = 0.8$ h per customer. The income from every customer served $c = 4$ roubles. The upkeep of every server is 2 roubles/h. Decide whether it is advantageous to increase the number of servers to three.

Solution. By Erlang's formulas (11.0.6)

$$p_0 = \left\{1 + 3.2 + \frac{3.2^2}{2!}\right\}^{-1} = (9.32)^{-1} \approx 0.107, \quad p_2 \approx \frac{5.12}{9.32} \approx 0.550,$$
$$Q = 1 - p_2 \approx 0.450, \quad A = 4Q \approx 1.8 \text{ customers/h}.$$

The income from the customers in the given variant is $D = A \cdot c \approx$ 7.2 roubles/h.

We calculate the same parameters for a three-server queueing system (denoting them by primes):

$$p_0' = \left\{1 + 3.2 + \frac{3.2^2}{2!} + \frac{3.2^3}{3!}\right\}^{-1} \approx 0.0677, \quad p_3' \approx 5.48 \times 0.0677 \approx 0.371,$$
$$Q' = 1 - p_3' \approx 0{,}629, \quad A' = 4Q' \approx 2.52,$$
$$D' = A' \cdot c \approx 10.08 \text{ roubles/h}.$$

The increase in the income is $D' - D = 2.88$ roubles/h, while the increase in the cost of the upkeep is 2 roubles/h. It can be seen that the transition from $n = 2$ to $n = 3$ is economically advantageous.

11.11. We consider an elementary queueing system with a practically unlimited number of channels ($n \to \infty$). The demands arrive with intensity λ while the intensity of the service process (per channel) is μ. Find the limiting probabilities of states of the system and the average number of busy channels $\bar{k}$.

Solution. This queueing system is neither a congestion nor a delay system, but it can be regarded as the limiting case for a system with refusals for $n \to \infty$. Erlang's formulas (11.0.6) yield

$$p_0 = \left\{ \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \right\}^{-1} = e^{-\rho}, \text{ where } \rho = \frac{\lambda}{\mu}, \quad p_k = \frac{\rho^k}{k!} e^{-\rho} = P(k, \rho)$$

(see Appendix 1). For an infinite number of channels $A = \lambda$ and $\bar{k} = \lambda/\mu = \rho$.

11.12. We consider a single-channel congestion system to which demands arrive in a stationary Poisson process with intensity λ. The service time is exponential with parameter $\mu = 1/\bar{t}_{\text{ser}}$. The operating channel can fail from time to time (refuse), the failure process being

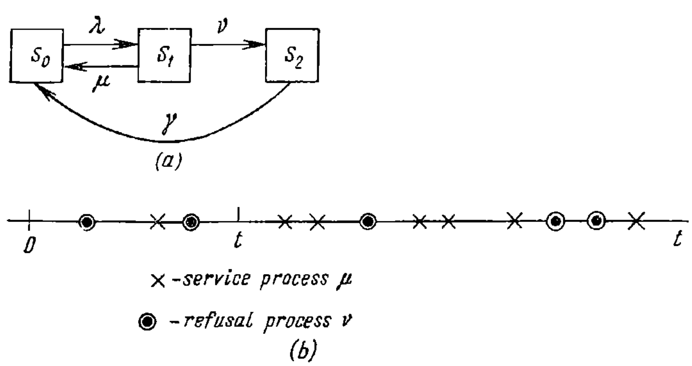


Fig. 11.12

stationary Poisson's with intensity ν. The reconditioning (repair) of a channel begins immediately after it fails, the time of repair T_r being exponential with parameter $\gamma = 1/\bar{t}_r$. The demand which was being served at the moment of failure departs from the system unserved.

Find the limiting probabilities of the following states of the system: s_0, the channel is idle, s_1, the channel is busy and in working order, s_2 the channel is being repaired and the characteristics of the system are A and Q.

Solution. The directed graph of states of the queueing system is given in Fig. 11.12*a*. The algebraic equations for the limiting proba-

bilities of states are

$$\lambda p_0 = \mu p_1 + \gamma p_2, \quad (\mu + \nu)\, p_1 = \lambda p_0, \quad \nu p_1 = \gamma p_2, \tag{11.12.1}$$

they are summed with the normalizing condition

$$p_0 + p_1 + p_2 = 1. \tag{11.12.2}$$

We express the probabilities p_1 and p_2 from (11.12.1) in terms of p_0:

$$p_1 = \frac{\lambda}{\mu+\nu}\, p_0, \quad p_2 = \frac{\nu}{\gamma}\, p_1 = \frac{\lambda\nu}{\gamma(\mu+\nu)}\, p_0.$$

Substituting p_1 and p_2 into (11.12.2), we get

$$p_0 = \{1 + \lambda/(\mu + \nu) + \lambda\nu/[\gamma\,(\mu + \nu)]\}^{-1}. \tag{11.12.3}$$

To find the relative capacity for service Q, the probability p_0 that the demand will be accepted for service must be multiplied by the conditional probability p' that the demand which is accepted for service will actually be served (the channel will not fail during the service time). We shall use the integral total probability formula to find the conditional probability. We advance a hypothesis that the service time of a demand has fallen on the interval from t to $t + dt$, the probability of this hypothesis being approximately $f(t)\,dt$, where $f(t)$ is the distribution density of the service time, i.e. $f(t) = \mu e^{-\mu t}$ $(t > 0)$. The conditional probability that the channel will not fail during the time t is $e^{-\nu t}$; hence

$$p' = \int_0^\infty \mu e^{-\mu t} e^{-\nu t}\, dt = \int_0^\infty \mu e^{-(\mu+\nu)t}\, dt = \frac{\mu}{\mu+\nu}.$$

This conditional probability can be found much more easily: it is equal to the probability that once it has begun, the service procedure will be completed before the channel fails. We superimpose on the t-axis (Fig. 11.12*b*) the two processes, viz. the service process with intensity μ (denoted by crosses) and the refusal process with intensity ν (denoted by circles). We fix a point t on the t-axis and find the probability that the first cross following the fixed point will arrive earlier than a circle. It is evidently equal to the ratio of the intensity of the flow of crosses to the total intensity of the flows of crosses and circles, $\mu/(\mu + \nu)$. Thus

$$Q = p_0 p' = \left(\frac{\mu}{\mu+\nu}\right) \Big/ \left(1 + \frac{\lambda}{\mu+\nu} + \frac{\lambda\nu}{\gamma(\mu+\nu)}\right) = \frac{\mu}{\mu+\nu+\lambda(1+\nu/\gamma)},$$

$$A = \lambda Q. \tag{11.12.4}$$

11.13. The conditions of Problem 11.12 are repeated with the only difference that the channel may fail during an idle period too (with intensity $\nu' < \nu$).

Solution. The directed graph of states of the queueing system is shown in Fig. 11.13. From the equations

$$(\lambda + \nu')\, p_0 = \mu p_1 + \gamma p_2, \quad (\mu+\nu) p_1 = \lambda p_0, \quad \gamma p_2 = \nu p_1 + \nu' p_0,$$

$$p_0 + p_1 + p_2 = 1$$

we find the limiting probabilities

$$p_0=\left\{1+\frac{\lambda}{\mu+\nu}+\frac{\lambda\nu+\mu\nu'+\nu\nu'}{\mu+\nu}\right\}^{-1},$$

$$p_1=\frac{\lambda}{\mu+\nu}p_0,\quad p_2=\frac{\lambda\nu+\mu\nu'+\nu\nu'}{\mu+\nu}p_0,$$

$$Q=p_0\frac{\mu}{\mu+\nu},\quad A=\lambda Q=p_0\frac{\lambda\mu}{\mu+\nu}.$$

11.14. We consider an elementary single-channel queueing system with a limited number of places in the queue, $m = 2$. The operating

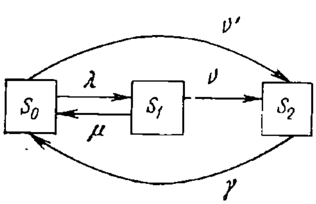


Fig. 11.13

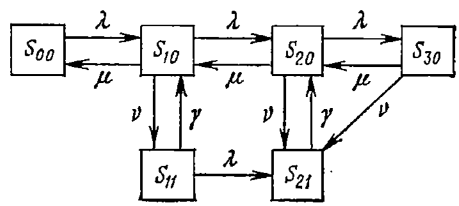


Fig. 11.14

channel may fail (refuse). The demand being served at the moment of failure joins the waiting line if there are unoccupied places, and if there are no places it departs unserved. The intensity of the arrival process is λ, that of the service process is μ, that of the failure process of the channel is ν, and that of the reconditioning (repair) process is γ. Enumerate the states of the system, find their limiting probabilities and determine A, $\bar{k}$, $\bar{r}$, $\bar{z}$, $\bar{t}_w$, $\bar{t}_q$ for $\lambda = 2$, $\mu = 1$, $\nu = 0.5$ and $\gamma = 1$.

Solution. The states of the system are:

s_{00}—the system is idle, the channel is in working order;

s_{10}—the channel is busy and in working order, there is no queue;

s_{11}—the channel has failed and is being repaired; one demand is waiting to be served;

s_{20}—the channel is busy and in working order; one demand is being served and another one is waiting to be served;

s_{21}—the channel has failed and is being repaired; two demands are in the queue;

s_{30}—the channel is busy and in working order; two demands are in the queue and one demand is being served.

The graph of the states of the system is shown in Fig. 11.14. The equations for the limiting probabilities of states are

$$\mu p_{10} = \lambda p_{00},\ \ \mu p_{20} + \gamma p_{11} + \lambda p_{00} = (\mu + \lambda + \nu)\, p_{10},$$

$$\mu p_{30} + \gamma p_{21} + \lambda p_{10} = (\mu + \lambda + \nu)\, p_{20},$$

$$\lambda p_{20} = (\mu + \nu)\, p_{30},\quad \nu p_{20} = (\mu + \nu)\, p_{30},$$

$$\nu p_{10} = (\gamma + \lambda)\, p_{11},\quad \lambda p_{11} + \nu p_{20} + \nu p_{30} = \gamma p_{21},$$

$$p_{00} + p_{10} + p_{11} + p_{20} + p_{21} + p_{30} = 1.$$

Solving these equations at $\lambda = 2$, $\mu = 1$, $\nu = 0.5$ and $\gamma = 1$, we obtain

$$p_{00} = 3/61 \approx 0.049, \quad p_{10} = 6/61 \approx 0.098, \quad p_{20} = 14/61 \approx 0.230,$$

$$p_{30} = 56/183 \approx 0.306, \quad p_{11} = 1/61 \approx 0.016, \quad p_{21} = 55/183 \approx 0.301.$$

Hence

$$\bar{z} = 1\,(p_{10} + p_{11}) + 2\,(p_{20} + p_{21}) + 3p_{30} = 383/183 \approx 2.09,$$

$$\bar{r} = 1\,(p_{20} + p_{11}) + 2\,(p_{21} + p_{30}) = 89/61 \approx 1.46.$$

$$\bar{k} = 1\,(p_{10} + p_{20} + p_{30}) = 116/183 \approx 0.63.$$

The absolute capacity A of a queueing system with nonrefusing channels can be found by multiplying $\bar{k}$ by μ. In our case the productivity of one channel (the number of demands actually served per unit time) can be found by multiplying $\bar{k}$ by the probability $\mu/(\mu + \nu)$ that the service procedure will be completed, i.e. $A = \bar{k}\mu \cdot \mu/(\mu + \nu) = \bar{k}\mu^2/(\mu + \nu) \approx 0.42$.

11.15. There are three chairs ($n = 3$) in an outstation dental surgery and three chairs in the waiting room ($m = 3$). Patients arrive in a stationary Poisson process at a rate of $\lambda = 12$ patients per hour. The service time (for one patient) is exponential with a mean value $\bar{t}_{ser} = 20$ min. If all the three chairs in the waiting room are occupied, an arriving patient does not join the queue. Find the average number of patients that the dentists serve per hour, the average fraction of the arriving patients that are actually served, the average number of occupied chairs in the waiting room, the average waiting time $\bar{t}_w$ of a patient (including both the dental surgery, and waiting room) and the average waiting time given that the patient will be served.

Solution. $\rho = 12/3 = 4$, $\varkappa = \rho/3 = 4/3$, $n = 3$, $m = 3$. From formulas (11.0.26)-(11.0.30) we find that

$$p_0 = \left\{1 + 4 + \frac{4^2}{2} + \frac{4^3}{6} + \frac{4^4}{3 \times 6}\,\frac{1-(4/3)^3}{1-4/3}\right\}^{-1} \approx 0.01218 \approx 0.012,$$

$$p_1 = 4 \times 0.01218 \approx 0.049, \quad p_2 = 8 \times 0.01218 \approx 0.097,$$

$$p_3 = \frac{32}{3} \times 0.01218 \approx 0.130, \quad p_{3+1} = \frac{4^4}{36} \times 0.01218 \approx 0.173,$$

$$p_{3+2} = \frac{4}{3}\,p_{3+1} \approx 0.231, \quad p_{3+3} = \frac{4}{3}\,p_{3+2} \approx 0.307.$$

The average fraction of patients being served is $Q = 1 - R_{ref} = 1 - p_{3+3} \approx 1 - 0.307 = 0.693$.

The average number of patients served by the dentists per hour is $A = \lambda Q \approx 12 \times 0.683 \approx 8.32$.

By formula (11.0.27) the average number of busy servers (dentists) $\bar{k} = 4\,(1 - p_{3+3}) \approx 2.78$.

By formula (11.0.28) the average number of patients in the queue

$$\bar{r} = \frac{4^4 \times 0.01218}{3\times 6} \frac{1-4\times(4/3)^3+3\,(4/3)^4}{(1-4/3)^2} \approx 1.56,$$

$$\bar{z} = \bar{r} + \bar{k} \approx 4.34, \quad \bar{t}_q = \bar{r}/\lambda \approx 0.13 \text{ h}, \quad \bar{t}_w = \bar{z}/\lambda \approx 0.362 \text{ h}.$$

The values of $\bar{t}_w$ and $\bar{t}_q$ are small because some patients do not join the queue and depart unserved. The conditional average waiting time of a patient, provided that he is served, is $\tilde{\bar{t}}_w = \bar{t}_w/Q \approx 0.52$ h, and the conditional average queueing time (under the same condition) $\tilde{\bar{t}}_q = \bar{t}_q/Q \approx 0.19$ h.

11.16. For $\varkappa = 1$ formulas (11.0.26) and (11.0.28) yield an indeterminacy of the form 0/0. Find the value of the indeterminate form and write formulas which are valid for $\varkappa = 1$.

Solution. By L'Hospital's rule

$$\lim_{\varkappa\to 1} \frac{1-\varkappa^m}{1-\varkappa} = \frac{-m\varkappa^{m-1}}{-\varkappa} = m,$$

$$p_0 = \left\{1 + \frac{\rho}{1!} + \ldots + \frac{\rho^n}{n!} + \frac{\rho^{n+1}m}{n\cdot n!}\right\}^{-1}, \tag{11.16.1}$$

$$p_k = \frac{\rho^k}{k!} p_0 \quad (1 \leqslant k \leqslant n), \tag{11.16.2}$$

$$p_{n+1} = p_{n+2} = \ldots = p_{n+m} = \frac{\rho^n}{n!} p_0, \tag{11.16.3}$$

i.e. all the probabilities, beginning with p_n and ending with p_{n+m}, are equal.

The formulas for A, Q, P_{ref} and $\bar{k}$ remain the same.

Finding the value of the indeterminate form in formula (11.0.28), we get

$$\lim_{\varkappa\to 1} \frac{1-(m+1)\varkappa^m + m\varkappa^{m+1}}{(1-\varkappa)^2} = \frac{m\,(m+1)}{2}, \quad \bar{r} = \frac{\rho^{n+1} p_0 m\,(m+1)}{2n\cdot n!}. \tag{11.16.4}$$

Formulas (11.0.29) and (11.0.30) remain the same.

We could derive formulas (11.16.1)-(11.16.3) without finding the value of the indeterminate form but using the birth and death scheme.

11.17. (1) Calculate the efficiency characteristics A, Q, P_{ref}, $\bar{k}$, $\bar{r}$, $\bar{z}$, $\bar{t}_q$, $\bar{t}_w$ for an elementary single-server system with three places in the queue ($m = 3$) given $\lambda = 4$ customers per hour and $\bar{t}_{ser} = 1/\mu = 0.5$. (2) Find out how these characteristics will change when the number of places in the queue is increased to $m = 4$.

Solution. $\mu = 2$, $\rho = \lambda/\mu = 2$. By formulas (11.0.12)-(11.0.16) for $m = 3$ we have $p_0 = 1/31$, $p_4 = 16/31$, $Q \approx 0.484$, $A = \lambda Q \approx 1.93$ customers per hour, $\bar{k} = \rho Q \approx 0.968$, $\bar{r} \approx 2.19$ customers, $\bar{z} \approx 3.16$ customers, $\bar{t}_q \approx 0.55$ h and $\bar{t}_w \approx 0.79$ h.

(2) For $m = 4$ we have $p_0 = 1.63 \approx 0.0158$, $p_5 = 32/63 \approx 0.507$, $Q \approx 0.493$, $A \approx 1.96$ customers per hour, $\bar{r} \approx 3.11$ customers, $\bar{z} \approx 4.09$ customers, $\bar{t}_q \approx 0.78$ h and $\bar{t}_w \approx 1.02$ h.

Thus an increase in the number m of places from three to four leads to a negligible increase in the absolute (and relative) capacity of the system for service. It does increase somewhat the average number of customers in the queue and in the system, and the corresponding average times. This is natural since some of the customers who would be refused in the first variant join the queue in the second.

11.18. How will the efficiency characteristics of the system in the preceding problem change if λ and μ remain the same, $m = 3$, but the number of servers increase to $n = 2$?

Solution. $\varkappa = 1$, from formulas (11.16.1) and (11.16.2) we have $p_0 = 1/11$, $p_1 = \ldots = p_5 = 2/11$, $Q = 1 - 2/11 \approx 0.818$, $A \approx 3.27$ customers/h, $\bar{r} = 12/11 \approx 1.09$ customers, $\bar{k} = A/\mu \approx 1.64$, $\bar{z} = \bar{r} + \bar{k} \approx 2.73$ customers, $\bar{t}_q \approx 0.27$ h and $\bar{t}_w \approx 0.68$ h.

11.19. The queueing system is a railway booking-office with one window ($n = 1$) and an unbounded queue. The man in the window sells tickets to points A and B. On the average, three passengers every 20 minutes arrive to buy a ticket to point A and two passengers every 20 minutes arrive to buy a ticket to B. The arrival process can be considered to be stationary Poisson's. On the average three passengers are served in 10 min. The service time is exponential. Find whether limiting probabilities of states of the system exist and, if they do, calculate the first three, viz p_0, p_1, p_2. Find the efficiency characteristics of the system: $\bar{z}$, $\bar{r}$, $\bar{t}_w$ and $\bar{t}_q$.

Solution. $\lambda_A = 3/20 = 0.15$ customers/min; $\lambda_B = 2/20 = 0.10$ customers/min. The total arrival intensity $\lambda = \lambda_A + \lambda_B = 0.25$ customers/min, $\mu = 3/10 = 0.3$ customers/min and $\rho = \lambda/\mu \approx 0.833 < 1$. Limiting probabilities do exist. By formulas (11.0.12)-(11.0.14) $p_0 \approx 0.167$, $p_1 \approx 0.139$, $p_2 \approx 0.116$, $z \approx \frac{0.833}{0.167} \approx 4.99$ customers, $\bar{r} = 0.833^2/0.167 \approx 4.16$ customers, $\bar{t}_w \approx 4.99/0.25 \approx 20.0$ min and $\bar{t}_q \approx 4.16/0.25 \approx 16.7$ min.

11.20. A single-channel queueing system is a computer which receives calculation jobs. The jobs arrive in a stationary Poisson process, the interarrival time being $\bar{t} = 10$ min. The service time T_{ser} has an Erlang distribution of the third order with expectation $\bar{t}_{ser} = 8$ min. Find the average number $\bar{z}$ of jobs in the system and the average number $\bar{r}$ of jobs in the queue, and also the average waiting time $\bar{t}_w$ and queueing time $\bar{t}_q$ for a job.

Solution. We can find the characteristics of the system from the Pollaczek-Khinchine formulas (11.0.31) and (11.0.32). We have $\lambda = 0.1$ jobs/min, $\mu = 0.125$ jobs/min and $\rho = \lambda/\mu = 0.8$.

The coefficient of variation of the service time for Erlang's distribution of the third order is $1/\sqrt{3}$. By formula (11.0.31) $\bar{r} = 0.64\ (1 +$

$1/3)/(2 \times 0.2) \approx 2.13$. By formula (11.0.32) $\bar{z} = \bar{r} + 0.8 \approx 2.93$. By Little's formula $\bar{t}_q \approx 21.3$ min and $\bar{t}_w \approx 29.3$ min.

11.21. The hypothesis of the preceding problem is changed: jobs arrive now in a Palm flow and not in a stationary Poisson flow. The interarrival time has a generalized Erlang distribution of the second order (see Problem 8.38) with parameters $\lambda_1 = 1/2$ and $\lambda_2 = 1/8$. Using formulas (11.0.34)-(11.0.37), find approximations of the efficiency characteristics of the system.

Solution. A random variable T, which has a generalized Erlang distribution of the second order, is the sum of two random variables T_1 and T_2 which have exponential distributions with parameters $\lambda_1 = 1/2$ and $\lambda_2 = 1/8$. Hence $M[T] = 1/\lambda_1 + 1/\lambda_2 = 10$ min, $\mathrm{Var}[T] = \mathrm{Var}[T_1] + \mathrm{Var}[T_2] = 2^2 + 8^2 = 68$, $v_\lambda^2 = 68/10^2 = 0.68$ and $v_\mu^2 = 1/3$. Consequently $\bar{r} = 1.62$ jobs, $\bar{z} = 2.42$ jobs, $\bar{t}_q = 16.2$ min and $\bar{t}_w = 24.2$ min.

11.22. A device can fail (refuse) from time to time. The refusals occur in a stationary Poisson process with intensity $\lambda = 1.6$ refusals per day.

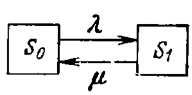


Fig. 11.22

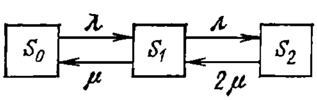


Fig. 11.23

The time needed for reconditioning (repair) T_{rep} has a uniform distribution on the interval from 0 to 1 day. Find (for limiting steady-state conditions) the average fraction R of time for which the device is operating.

Solution. The states of the device are: s_0, the device is operating and s_1, it is being repaired. The directed graph of states of the device is shown in Fig. 11.22, where $\mu = 1/M[T_{rep}] = 1/0.5 = 2$. This graph exactly coincides with the directed graph of states of a single-channel congestion system. We know that if the arrival process is stationary Poisson's and the service time has an arbitrary distribution, then Erlang's formulas (11.0.6) are valid. In this case $\rho = \lambda/\mu = 0.8$, $p_0 = \{1 + \rho/1!\}^{-1} = 1/1.8 \approx 0.556$ and $p_1 = 1 - 0.556 \approx 0.444$. Thus $R \approx 0.556$, i.e. the device will operate a little more than a half of the time and the rest of the time it will be under repair.

11.23. Under the conditions of the preceding problem the device has an exact repeater which can fail only when operating. The parameters λ and μ are the same as in Problem 11.22. Find the value of R and the average number $\bar{k}$ of the faulty devices.

Solution. The states of the system S are: s_0, both devices are sound (one is operating and the other is idle); s_1, one device is operating and the other is under repair; s_2, both devices are under repair. The directed graph of states is shown in Fig. 11.23. The graph exactly coincides with

the graph of states of a two-channel system with refusals. From Erlang's formulas (11.0.6)

$$p_0 = \left\{1 + \frac{\rho}{1!} + \frac{\rho^2}{2!}\right\}^{-1} = \{1 + 0.8 + 0.64/2\}^{-1} \approx 0.472,$$
$$p_1 \approx 0.8 \times 0.472 = 0.378, \quad R = p_0 + p_1 \approx 0.850,$$
$$p_2 = 0.150, \quad \bar{k} = 1 \cdot p_1 + 2 \cdot p_2 = 0.728.$$

The same technique (reduction to a system with refusals) can evidently be used when the number of repeaters is more than one.

11.24. In a very large bookshop the security system is such that having chosen a book in a department a customer takes it to the assistant who writes out an invoice. The customer must then take the invoice to the cash desk and pay for the invoice. Having paid, he can then return to the selling department where the assistant checks the invoice and wraps the book so that the security staff know the book is paid for. The customer must therefore go through three service phases: (1) collection of invoice, (2) payment, (3) book wrapping. Assume that customers arrive at a department in a Poisson flow with $\lambda = 45$ customers per hour.

There are four assistants to write out the invoices and the average service time $\bar{t}_1$ is five minutes. The average service time at the cashier $\bar{t}_2$ (assume for simplicity there is one who only serves the one department and no others) is one minute per customer. Three assistants check the invoice and wrap the books, the average service time $\bar{t}_3$ being two minutes per customer. Assume that the flows are stationary Poisson's and that the system is a three-phase queueing system. Find its efficiency characteristics:

$\bar{r}_1$ ($\bar{r}_2$, $\bar{r}_3$)—the average number of customers in the queue in the first (second and third) service phase;

$\bar{z}_1$, ($\bar{z}_2$, $\bar{z}_3$)—the average number of customers in the first (the second, the third) phase;

$\bar{t}_q^{(1)}$ ($\bar{t}_q^{(2)}$, $\bar{t}_q^{(3)}$)—the average queueing time of a customer in the first (the second, the third) phase;

$\bar{t}_w^{(1)}$ ($\bar{t}_w^{(2)}$, $\bar{t}_w^{(3)}$)—the average waiting time of a customer in the first (the second, the third) phase;

$\bar{r}$—the overall average number of customers in all three queues;

$\bar{z}$—the total average number of customers in the shop;

$\bar{t}_q$—the total average queueing time of a customer;

$\bar{t}_w$—the total average waiting time of a customer.

Answer the following additional questions: (1) In which phase and how should the service be improved to reduce the waiting time of a customer? (2) How can the fact that some customers having got an invoice find they do not have enough money to pay (and so leave the system before the cashier queue) be taken into account? The fraction of such customers is α ($0 < \alpha < 1$).

Solution. Since all the processes are stationary Poisson's, the departure processes from all the three phases are also stationary Poisson's, and three successive phases can be regarded as three individual queueing systems with their own characteristics.

1. **First phase.** Since there are four assistants, the number of servers $n_1 = 4$. Furthermore, $\bar{t}_1 = 1/\mu_1 = 5$ min $= 1/12$ h, $\rho_1 = 45/12 = 15/4 \approx 3.75$ and $\varkappa_1 = \rho_1/n_1 = 15/16 < 1$. From formulas (11.0.21)-(11.0.25) we find that

$$p_0^{(1)} = \left\{1 + 3.75 + \frac{(3.75)^2}{2} + \frac{(3.75)^3}{2\cdot 3} + \frac{(3.75)^4}{2\cdot 3\cdot 4} + \frac{(3.75)^5}{2\cdot 3\cdot 4\cdot 4\,(1-15/16)}\right\}^{-1} \approx (151.58)^{-1} \approx 0.0066, \quad \bar{k}_1 = 3.75,$$

$$\bar{r}_1 = \frac{(3.75)^5}{4\cdot 4!}\,\frac{p_0^{(1)}}{(1-\varkappa_1)^2} \approx 13.01, \quad \bar{z}_1 = 16.76,$$

$$\bar{t}_q^{(1)} = \bar{r}_1/\lambda \approx 0.289 \text{ h} \approx 17.3 \text{ min}, \quad \bar{t}_w^{(1)} = \bar{z}_1/\lambda \approx 0.372 \text{ h} \approx 22.3 \text{ min}.$$

2. **Second phase.** $\lambda = 45$, $n_2 = 1$ and $\rho_2 = 0.75 < 1$. From formulas (11.0.12)-(11.0.14) we find that

$$\bar{r}_2 = \rho_2^2/(1-\rho_2) = 9/4 = 2.25, \quad \bar{z}_2 = \rho_2/(1-\rho_2) = 3,$$

$$\bar{t}_q^{(1)} = \bar{r}_2/\mu = 0.05 \text{ h} = 3 \text{ min}, \quad \bar{t}_w^{(1)} = 1/15 \text{ h} = 4 \text{ min}.$$

3. **Third phase.** $n_3 = 3$, $\lambda = 45$, $\rho_3 = 3/2$ and $\varkappa_3 = 0.5 < 1$. From formulas (11.0.21)-(11.0.25) we find that

$$p_0^{(3)} \approx 0.210, \quad \bar{r}_3 \approx 0.237 \text{ and } \bar{z}_3 \approx 1.737,$$

$$\bar{t}_q^{(3)} \approx 0.316 \text{ min and } \bar{t}_w^{(3)} \approx 2.316 \text{ min}.$$

Adding together the average numbers of customers in each queue, we obtain the total average number of customers in the queues, i.e.

$$\bar{r} = \bar{r}_1 + \bar{r}_2 + \bar{r}_3 \approx 15.5.$$

By analogy we find the average number of customers in the system $\bar{z} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 \approx 21.5$.

The average queueing time per customer

$$\bar{t}_q = \bar{t}_q^{(1)} + \bar{t}_q^{(2)} + \bar{t}_q^{(3)} \approx 20.6 \text{ min}.$$

The average waiting time per customer

$$\bar{t}_w = \bar{t}_w^{(1)} + \bar{t}_w^{(2)} + \bar{t}_w^{(3)} \approx 28.6 \text{ min}.$$

(1) The service can be improved by decreasing the waiting time of a customer in the first phase, which is the weakest phase in the system. The easiest way to do this is to increase the number of assistants, i.e. the number of servers in the first phase. For instance, a simple unit increase in the number of assistants (i.e. a transition from $n_1 = 4$ to $n_1 = 5$) yields an essential gain in time. Indeed, for $n_1 = 5$ we obtain

for the first phase: $\rho_1 = 3.75$, $\varkappa_1 = 3.75/5 = 0.75$, $p_0^{(1)} = \{59.71\}^{-1}$, $\bar{r}_1 \approx 2.08$, $\bar{z}_1 \approx 5.13$, $\bar{t}_q^{(1)} \approx 1.84$ min and $\bar{t}_w^{(1)} \approx 6.84$ min.

(2) We can take into account the fraction of customers α leaving the shop without a purchase by multiplying the intensity of the arrival process of the second and third phases by $(1 - \alpha)$.

11.25. Trains arrive at a railway shunting yard in an Erlang process of order 10 with intensity $\lambda = 1.2$ trains per hour*). The service time of a train T_{ser} is distributed over an interval from 0 to 1 h with density $\varphi(t)$, shown in Fig. 11.25. Using formulas (11.0.34)-(11.0.37), approximate the efficiency characteristics of the yard, i.e. the average number $\bar{z}$ of trains at the yard and in the queue $\bar{r}$, the average waiting time $\bar{t}_w$ of a train and the average queueing time $\bar{t}_q$ of a train.

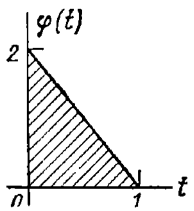


Fig. 11.25

Solution. For the distribution $\varphi(t)$ we have $\bar{t}_{ser} = 1/3$ and $\rho = \lambda/\mu = 0.8 < 1$. For a 10th-order Erlang process $v_\lambda^2 = (1/\sqrt{k})^2 = 0.1$; v_μ^2 can be found by dividing the variance $\mathrm{Var}[T_{ser}]$ by the square of the mean value: $v_\mu^2 = 1/2 = 0.5$.

We find the characteristics of the queueing system:

$$\bar{r} = \rho^2 (v_\lambda^2 + v_\mu^2)/[2(1-\rho)] = 0.8^2 (0.1 + 0.5)/(2\cdot 0.2) = 0.96,$$

$$\bar{z} = \bar{r} + \rho \approx 0.96 + 0.8 = 1.76,$$

$$\bar{t}_q = \bar{r}/\lambda \approx 0.8 \text{ h}, \quad \bar{t}_w = \bar{t}_q + 1/\mu \approx 1.13 \text{ h}.$$

11.26. Show that for an elementary n-server queueing system with an unlimited number of places in the queue the average number of customers in the queue is

$$\frac{\varkappa^{n+1}}{1-\varkappa} \frac{n}{\varkappa^n(n-1)+1} < \bar{r} < \frac{\varkappa^{n+1}}{1-\varkappa}.$$

Solution. We write the expression for $\bar{r}$ in the following form [see (11.0.23) and (11.0.21)]:

$$\bar{r} = \frac{\rho^{n+1}}{n\cdot n!} \frac{p_0}{(1-\varkappa)^2} = \frac{\varkappa}{(1-\varkappa)^2} p_n, \quad \text{where } p_n = \frac{\rho^n}{n!} p_0.$$

Consequently

$$p_n = \frac{\rho^n}{n!}\left(1 + \rho + \frac{\rho^2}{2} + \ldots + \frac{\rho^n}{n!} + \frac{\rho^n}{n!}\frac{\varkappa}{1-\varkappa}\right)^{-1}$$

$$= \left(1 + \frac{n}{\rho} + \frac{n(n-1)}{\rho^2} + \ldots + \frac{n!}{\rho^n} + \frac{\varkappa}{1-\varkappa}\right)^{-1}$$

$$> \left(1 + \frac{n}{\rho} + \frac{n^2}{\rho^2} + \ldots + \frac{n^n}{\rho^n} + \frac{\varkappa}{1-\varkappa}\right)^{-1} = \varkappa^n(1-\varkappa).$$

*) Note that λ here is the intensity of Erlang's flow and not that of the flow which was thinned out to get the Erlang flow.

On the other hand

$$p_n < \left(1 + \frac{1}{\rho} + \frac{n}{\rho^2} + \ldots + \frac{n^{n-1}}{\rho^n} + \frac{\varkappa}{1-\varkappa}\right)^{-1} = \frac{\varkappa^n (1-\varkappa) n}{\varkappa^n (n-1)+1}.$$

Since $[\varkappa^n (1 - \varkappa) n]/[\varkappa^n (n - 1) + 1] < p_n < \varkappa^n (1 - \varkappa)$, the inequality indicated in the problem is satisfied too. Note that the last inequality can be used to approximate all the characteristics of the queueing system in question.

11.27. A railway booking-office has two windows in each of which tickets are sold to two destinations, Leningrad and Kiev. The arrival processes of the passengers who want to buy tickets to Leningrad and Kiev have the same intensity $\lambda_0 = 0.45$ pass/min. The average service time per passenger (selling a ticket to him) $\bar{t}_{\text{ser}} = 2$ min.

A rationalization proposal has been made, i.e. that in order to decrease the queues (in the interests of the passengers), the two booking-offices should specialize, one to sell tickets only to Leningrad and the other only to Kiev. If we consider, as the first approximation, all the arrival processes to be stationary Poisson's verify whether the proposal is reasonable.

Solution. (1) Let us calculate the characteristics of the queue for a two-server queueing system (the existing variant). The intensity of the arrival process $\lambda = 2\lambda_0 = 0.9$ pass/min, $\mu = 1/\bar{t}_{\text{ser}} = 0.5$ pass/min, $\rho = \lambda/\mu = 1.8$ and $\varkappa = \rho/2 = 0.9 < 1$, thus limiting probabilities exist. From formula (11.0.21) we have

$$p_0 = \left\{1 + 1.8 + \frac{1.8^2}{2}\,\frac{1}{1-0.9}\right\}^{-1} \approx 0.0525,$$

and from formula (11.0.23)

$$\bar{r} = \frac{1.8^3 \cdot 0.0525}{2 \cdot 2 \cdot 0.01} \approx 7.7 \text{ passengers}, \quad \bar{t}_{\text{q}} = \frac{7.7}{0.9} \approx 8.56 \text{ min}.$$

(2) In the second (suggested) variant we have two single-server queueing systems with $\rho = \lambda_0/\mu = 0.45/0.5 = 0.9 < 1$.

By formula (11.0.13), the average length of the gueue at a window $\bar{r} = \rho^2/(1 - \rho) = 0.9^2/0.1 = 8.1$ pass, and thus the total length of the queue at the two windows $2\bar{r} = 16.2$ pass.

The queueing time of a passenger, according to (11.0.14), $\bar{t}_{\text{q}} = \bar{r}/\lambda = 8.1/4.5 = 18$ min, and this is almost twice as long a queueing time as in the existing variant, viz. 9.3 min.

Conclusion: the "rationalization" proposal must be rejected since it drastically diminishes the efficiency of the system. The reduction of the efficiency of the system caused by going from a two-server system (the existing variant) to two single-server systems (the proposed variant) is because a division of the booking-office into two specialized offices prevents the men at the windows to repeat each other.

11.28*. *An elementary multi-server queueing system with "impatient" customers and an unbounded queue.* We have an elementary

n-server queueing system; the intensity of the arrival process is λ, and the intensity of the service process is $\mu = 1/\bar{t}_{ser}$. The queueing time of a customer is limited by a certain random time T, which has an exponential distribution with parameter ν (every customer in the queue may depart and the departure process has an intensity ν).

Write the formulas for the limiting probabilities of states, find the relative capacity of the system for service Q, the average length of the queue $\bar{r}$, the average queueing time $\bar{t}_q$ of a customer, the average number $\bar{z}$ of customers in the system and the average waiting time $\bar{t}_w$ per customer.

Solution. As before, we enumerate the states of the system in accordance with the number of customers in the system. The directed graph

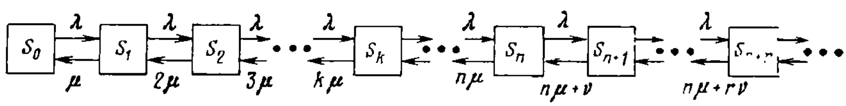


Fig. 11.28

of states is shown in Fig. 11.28. Using the general birth and death formulas and introducing the designations $\rho = \lambda/\mu$ and $\beta = \nu/\mu$, we obtain

$$p_0=\left\{1+\frac{\rho}{1!}+\frac{\rho^2}{2!}+\ldots+\frac{\rho^n}{n!}+\frac{\rho^n}{n!}\left[\frac{\rho}{n+\beta}+\frac{\rho^2}{(n+\beta)(n+2\beta)}\right.\right.$$
$$\left.\left.+\ldots+\frac{\rho^r}{(n+\beta)(n+2\beta)\ldots(n+r\beta)}+\ldots\right]\right\}^{-1},$$
$$p_1=\frac{\rho}{1!}p_0,\ \ldots,\ p_k=\frac{\rho^k}{k!}p_0\ (1\leqslant k\leqslant n),\ \ldots,\ p_n=\frac{\rho^n}{n!}p_0,$$
$$p_{n+1}=\frac{\rho^n}{n!}\cdot\frac{\rho}{n+\beta}p_0,\ \ldots,\ p_{n+r}$$
$$=\frac{\rho^n}{n!}\cdot\frac{\rho^r}{(n+\beta)(n+2\beta)\ldots(n+r\beta)}p_0\,(r\geqslant 1),\ \ldots\ . \quad (11.28.1)$$

The first formula (11.28.1) includes an infinite sum, which is not a geometric progression, but whose terms decrease faster than the terms of a geometric progression. We can prove that the error due to the truncation of the sum to the first $r-1$ terms is smaller than $\frac{\rho^n}{n!}\frac{(\rho/\beta)^r}{r!}e^{-\rho/\beta}$.

We assume that we have calculated the probabilities $p_0, p_1, \ldots, p_n, \ldots, p_{n+r}, \ldots$ and we now show how to find the characteristics of the queueing system: the relative capacity for service Q, the average number of customers in the queue $\bar{r}$ and others. We shall first find Q. All the customers will be served except for those who will depart ahead of time. Let us calculate the average number of customers who depart unserved per unit time. The intensity of the departure

process per customer in the queue is ν and the total average intensity of the departure process per all the customers in the queue is $\nu\bar{r}$. Hence the absolute capacity of the system for service

$$A = \lambda - \nu\bar{r}, \tag{11.28.2}$$

and the relative capacity for service

$$Q = A/\lambda = 1 - \nu\bar{r}/\lambda. \tag{11.28.3}$$

Thus, to find Q we must first of all know $\bar{r}$, which we could find directly from the formula $\bar{r} = 1p_{n+1} + 2p_{n+2} + \ldots + rp_{n+r} + \ldots$. But the drawback of this formula is that it contains an infinite number of terms. This can be avoided by using the expression for the average number of busy servers k in terms of A, i.e. $\bar{k} = A/\mu$ or, taking (11.28.2) into account,

$$\bar{k} = (\lambda - \nu\bar{r})/\mu = \rho - \beta\bar{r}. \tag{11.28.4}$$

From (11.28.4) we get

$$\bar{r} = (\rho - \bar{k})/\beta, \tag{11.28.5}$$

and the average number of busy servers $\bar{k}$ can be calculated as the mean value of the random variable K (the number of busy servers) with the possible values 0, 1, 2, . . ., n and the respective probabilities p_0, p_1, p_2, . . ., p_{n-1}, $[1 - (p_0 + p_1 + \ldots + p_{n-1})]$:

$$\bar{k} = 1p_1 + 2p_2 + \ldots + (n-1)\,p_{n-1} + n\,[1 - (p_0 + p_1 + \ldots + p_{n-1})]. \tag{11.28.6}$$

Furthermore, we can calculate $\bar{r}$ by formula (11.28.5). The quantity $\bar{t}_q$ can be found by Little's formula:

$$\bar{t}_q = \bar{r}/\lambda. \tag{11.28.7}$$

The average number of customers in the system

$$\bar{z} = \bar{r} + \bar{k}, \tag{11.28.8}$$

and the average waiting time per customer

$$\bar{t}_w = \bar{z}/\lambda. \tag{11.28.9}$$

Remark. We can prove that for the queueing system with "impatient" customer we have considered, limiting probabilities always exist when $\beta > 0$. This is confirmed by the fact that the series in the first formula (11.28.1) converges for any positive ρ and β. This means, in essence, that a queue cannot increase indefinitely: the longer the queue, the higher the intensity of the departure process of the customers.

11.29. *An elementary two-server queueing system with "impatient" customers* (see Problem 11.28). The intensity of the arrival process $\lambda = 3$ cust/h, the average service time of one customer $\bar{t}_{ser} = 1/\mu = 1$ h and the average time for which a customer waits for

service "patiently" is 0.5 h. Calculate the limiting probabilities of states, restricting the calculations by the probabilities which are no smaller than 0.001. Find the characteristics of the efficiency of the system, i.e. Q, A, $\bar{k}$, $\bar{r}$, $\bar{t}_q$, $\bar{t}_w$.

Solution. We have $\lambda = 3$, $\mu = 1$, $\nu = 2$, $\rho = 3$, $\beta = 2$, $n = 2$. From the formulas we derived for Problem 11.28 we get

$$p_0 = \left\{1 + 3 + \frac{3^2}{2} + \frac{3^2}{2}\left[\frac{3}{4} + \frac{3^2}{4\cdot 6} + \frac{3^3}{4\cdot 6\cdot 8} + \frac{3^4}{4\cdot 6\cdot 8\cdot 10}\right.\right.$$
$$\left.\left. + \frac{3^5}{4\cdot 6\cdot 8\cdot 10\cdot 12} + \frac{3^6}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} + \frac{3^7}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16}\right]\right\}^{-1} \approx 0.0692,$$

whence

$$p_1 = 3p_0 \approx 0.208, \quad p_2 = \frac{3}{2}p_1 \approx 0.311, \quad p_3 = \frac{3}{4}p_2 \approx 0.234,$$

$$p_4 = \frac{3}{6}p_3 \approx 0.117, \quad p_5 = \frac{3}{8}p_4 \approx 0.044, \quad p_6 = \frac{3}{10}p_5 \approx 0.013,$$

$$p_7 = \frac{3}{12}p_6 \approx 0.003, \quad p_8 = \frac{3}{14}p_7 \approx 0.001.$$

The average number of busy servers (11.28.6) $\bar{k} = 1p_1 + 2(1 - p_0 - p_1) \approx 1.654$, the average size of the queue from (11.28.5) $\bar{r} = (\rho - \bar{k})/\beta = (3 - 1.654)/2 \approx 0.673$, the absolute capacity for service $A = \bar{k}\mu \approx 1.654$ cust/h, the relative capacity for service $Q = A/\lambda \approx 0.551$ and $\bar{t}_q = \bar{r}/\lambda \approx 0.224$ h, $\bar{z} = \bar{r} + \bar{k} \approx 2.327$ and $\bar{t}_w = \bar{z}/\lambda \approx 0.776$ h.

11.30. *An elementary queueing system with "errors"*. Customers arrive at an n-server system with an unbounded queue in a stationary Poisson process with intensity λ, the service time is exponential with parameter μ. Being served is not a guarantee of satisfaction, and the server may satisfy the demands of a customer with probability p or not satisfy them with probability $q = 1 - p$. In the latter case the customer returns to the system either being served immediately, if there is no queue, or joining the queue. Define the states of the system (numbering them in accordance with the number of customers present in the system), find the limiting probabilities of states and the efficiency characteristics of the system and find the average number of reservice claims per unit time if every unsatisfied customer may claim for reservice with probability R.

Solution. The states of the queueing system are

$$\left.\begin{array}{l} s_0\text{—the system is idle;} \\ s_1\text{—one server is busy, . . . ;} \\ s_k\text{—}k\text{ servers are busy } (1 \leqslant k \leqslant n)\text{, . . . ;} \\ s_n\text{—all } n \text{ servers are busy;} \end{array}\right\}\text{there is no queue}$$

s_{n+r}—all n servers are busy, r customers are in the queue ($r = 1, 2, \ldots$).

The directed graph of states is shown in Fig. 11.30, where $\tilde{\mu} = \rho\mu$. It can be seen from this graph that the system is equivalent to another

system with a complete guarantee of satisfaction but with a service intensity of $\tilde{\mu} = p\mu$; for this system $\tilde{\rho} = \lambda/\tilde{\mu} = \lambda/(p\mu)$. Formulas (11.0.21)-(11.0.25) remain valid only when $\tilde{\rho}$ is substituted for ρ and $\tilde{\mu}$ for μ.

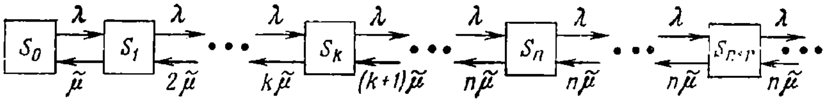


Fig. 11.30

11.31. *An elementary single-server closed queueing system.* One worker serves m machine-tools which fail from time to time (require resetting). The intensity of the failure process of one machine-tool is λ. If the worker is idle when a tool fails, he immediately begins to reset it. If he is not free, the tool joins the queue to be reset. The failure process of the machine-tool is stationary Poisson's, the time needed for reset is exponential with parameter $\mu = 1/\bar{t}_{\text{ser}}$. Define the states of

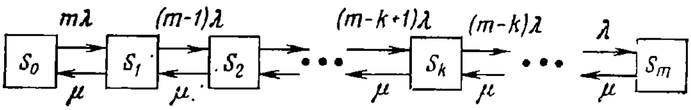


Fig. 11.31

the system, enumerating them in accordance with the number of faulty tools, find the limiting probabilities of states of the system and the efficiency characteristics: A, the average number of tools set up by the worker per unit time, $\bar{w}$, the average number of faulty tools, $\bar{r}$, the average number of tools in the queue, and P_{busy}, the probability that the worker will be busy.

Solution. The states of the system are

s_0—all the machine-tools are in working order (the worker is idle);
s_1—one tool is faulty (the worker is setting it up);
s_k—k tools are faulty, one is being set up, $k-1$ are in the queue;
s_m—all m tools are faulty, one is being set up, $m-1$ are in the queue.

The directed graph of states is shown in Fig. 11.31.

Designating $\rho = \lambda/\mu$, we get the following from the general birth and death formulas:

$$p_0 = \{1 + m\rho + m(m-1)\rho^2 + \ldots + m(m-1)\ldots(m-k+1)\rho^k + \ldots + m!\rho^m\}^{-1},$$

$$p_1 = m\rho p_0, \quad p_2 = m(m-1)\rho^2 p_0, \ \ldots,$$

$$p_k = m(m-1)\ldots(m-k+1)\rho^k p_0 \ (1 \leqslant k \leqslant n), \ \ldots,$$

$$p_m = m!\rho^m p_0. \qquad (11.31.1)$$

In order to find the absolute capacity A, we first find the probability that the worker is busy:

$$P_{\text{busy}} = 1 - p_0. \tag{11.31.2}$$

If the worker is busy, he resets up μ tools per unit time; hence

$$A = (1 - p_0)\,\mu. \tag{11.31.3}$$

We can express the average number of faulty tools $\bar{w}$ in terms of A as follows. Every operating machine-tool generates a failure process with intensity λ with $m - \bar{w}$ tools operating on the average. The failure process the machines generate has an intensity $(m - \bar{w})$ and all the faults are eliminated by the worker. This means that $(1 - p_0)\,\mu = (m - \bar{w})\,\lambda$, whence

$$\bar{w} = m - (1 - p_0)/\rho. \tag{11.31.4}$$

The average number $\bar{r}$ of machine-tools in the queue can be found as follows:

$$\bar{w} = \bar{r} + \bar{k}, \tag{11.31.5}$$

where $\bar{k}$ is the average number of tools being served (or the average number of busy servers). In this case the number of busy servers is zero if the worker is idle and unity if he is busy, hence $\bar{k} = 0 \cdot p_0 + 1\,(1 - p_0) = 1 - p_0$. Consequently

$$\bar{r} = m - (1 - p_0)/\rho - (1 - p_0) \quad \text{or} \quad \bar{r} = m - (1 - p_0)\,(1 + 1/\rho). \tag{11.31.6}$$

11.32*. On the hypothesis of Problem 11.31 find the average time $\bar{t}_q$ for which an arbitrarily chosen machine-tool that has failed will have to wait to be served.

Solution. Little's formula we have used is suitable only for open queueing system, where the intensity of the arrival process does not depend on the state of the system. It cannot be used for closed systems. We can find the time $\bar{t}_q$ by just assuming that a demand arrives (a tool fails) at a moment t. We then find the probability that at that moment the system was in state s_k $(k = 0, \ldots, m - 1)$ (it is clear that it could not be in a state s_m). Let us consider m hypotheses:

H_0—the system was in state s_0 when the demand arrived;
H_1—the system was in state s_1 when the demand arrived;
H_k—the system was in state s_k, . . . when the demand arrived;
H_{m-1}—the system was in state s_{m-1} when the demand arrived.

The a priori probabilities of these hypotheses are $p_0, p_1, \ldots, p_k, \ldots, p_{m-1}$.

We shall now find the a posteriori probabilities of the hypotheses given that the event $A = \{$a tool has failed in an elementary time interval $(t, t + dt)\}$ has occurred. On the hypotheses $H_0, H_1, \ldots,$

H_{m-1} the conditional probabilities of this event are

$$\mathrm{P}(A \mid H_0) = m\lambda\, dt, \quad \mathrm{P}(A \mid H_1) = (m-1)\lambda\, dt, \ \ldots,$$
$$\mathrm{P}(A \mid H_k) = (m-k)\lambda\, dt, \ \ldots, \quad \mathrm{P}(A \mid H_{m-1}) = \lambda\, dt.$$

Using Bayes's formulas, we find the a posteriori probabilities of the hypotheses (provided that the event A occurred). Designating these probabilities as $\tilde{p}_0, \tilde{p}_1, \ldots, \tilde{p}_k, \ldots, \tilde{p}_{m-1}$, we obtain

$$\tilde{p}_0 = \frac{mp_0}{\sum_{k=0}^{m-1}(m-k)p_k}, \quad \tilde{p}_1 = \frac{(m-1)p_1}{\sum_{k=0}^{m-1}(m-k)p_k}, \ \ldots,$$
$$\tilde{p}_k = \frac{(m-k)p_k}{\sum_{k=0}^{m-1}(m-k)p_k}, \ \ldots, \ \tilde{p}_{m-1} = \frac{p_{m-1}}{\sum_{k=0}^{m-1}(m-k)p_k}. \quad (11.32.1)$$

If we know these probabilities, we can find the complete expectation of the queueing time of the faulty tool. If a tool fails when the system is in state s_0, it will not wait for service; if the system is in state s_1, the tool will remain in the system for the time $1/\mu$ on the average; if it is in state s_2, then the time is $2/\mu$ and so on. Multiplying the probabilities (11.32.1) by these numbers and adding them up, we obtain

$$\bar{t}_q = \frac{1}{\mu}\sum_{k=1}^{m-1} k\tilde{p}_k. \quad (11.32.2)$$

11.33. A worker serves four machine-tools ($m = 4$), each of which fails with intensity $\lambda = 0.5$ faults/h. The average time needed for repair $\bar{t}_{rep} = 1/\mu = 0.8$ h. All the processes are stationary Poisson's. Using the formulas in Problems 11.31 and 11.32, find (1) the limiting probabilities of states; (2) the capacity A; (3) the average relative idle period of the worker P_{idle}; (4) the average number of tools in the queue $\bar{r}$; (5) the average number of faulty tools $\bar{w}$; (6) the average queueing time of a faulty tool $\bar{t}_q$; (7) the average output of a group of tools, with due regard for their incomplete reliability, if a working tool produces l articles.

Solution. $\mu = 1/\bar{t}_{rep} = 1.25$ and $\rho = \lambda/\mu = 0.4$.

(1) From formulas (11.31.1) we have $p_0 = \{1 + 1.6 + 1.92 + 1.53 + 0.61\}^{-1} = 6.66^{-1} \approx 0.150$, $p_1 = 1.6p_0 \approx 0.240$, $p_2 = 1.92p_0 \approx 0.288$, $p_3 = 1.53p_0 \approx 0.230$ and $p_4 = 0.061p_0 \approx 0.092$;

(2) $A = 0.850\mu \approx 1.06$ tools per hour;

(3) $\mathrm{P}_{idle} = p_0 = 0.150$;

(4) $\bar{r} \approx 4 - 0.850\,(1 + 2.5) \approx 1.03$;

(5) $\bar{w} \approx 1.03 + \bar{k} \approx 1.03 + 0.85 = 1.88$;

(6) from formulas (11.32.1) and (11.32.2) we have

$$\tilde{p}_0 = \frac{4p_0}{4p_0+3p_1+2p_2+p_3} \approx 0.283;$$

$$\tilde{p}_1 \approx 0.340, \quad \tilde{p}_2 = 0.270, \quad \tilde{p}_3 = 0.108,$$

$$\bar{t}_q = 0.8(\tilde{p}_1 + 2\tilde{p}_2 + 3\tilde{p}_3) \approx 0.964 \text{ h}.$$

(7) the productivity of the group of tools is $(m - \bar{w})\, l \approx 2.12l$.

11.34. *An elementary multi-server closed queueing system.* A team of n workers handle m machine-tools ($n < m$). The failure process of each tool has an intensity λ and the average time for resetting a tool $\bar{t}_{ser} = 1/\mu$. All the processes are stationary Poisson's. Find the limiting probabilities of states of the queueing system, the absolute capacity A and the average number of faulty tools $\bar{w}$.

Solution. We enumerate the states of the system in accordance with the number of faulty tools:

s_0—all the tools are in working order, the workers are idle;

s_1—one tool is faulty, one worker is busy, the other workers are idle; . . . ;

s_k—k tools are faulty, k workers are busy and the other workers are idle ($k < m$); . . . ;

s_n—n tools are faulty and all the workers are busy;

s_{n+1}—$(n+1)$ tools are faulty, n of them are being set up and one tool is waiting for service; . . . ;

s_{n+r}—$(n+r)$ tools are faulty, n of them are being set up and r tools are in the queue ($n + r < m$); . . . ;

s_m—all the m tools are faulty, n of them are being set up and $m - n$ tools are in the queue.

We invite the reader to construct the directed graph of states of the system on his own and we only present the final formulas for the probabilities of states:

$$p_0 = \Big\{1 + \frac{m}{1!}\rho + \frac{m(m-1)}{2!}\rho^2 + \ldots + \frac{m(m-1)\ldots(m-k+1)}{k!}\rho^k$$
$$+ \ldots + \frac{m(m-1)\ldots(m-n)}{n!\,n}\rho^{n+1}$$
$$+ \ldots + \frac{m(m-1)\ldots m-(n+r-1)}{n!\,n^r}\rho^{n+r} + \ldots + \frac{m!}{n!\,n^{m-n}}\rho^m\Big\}^{-1},$$

$$p_1 = \frac{m}{1!}\rho p_0, \quad p_2 = \frac{m(m-1)}{2!}\rho^2 p_0, \ \ldots,$$

$$p_k = \frac{m(m-1)\ldots(m-k+1)}{k!}\rho^k p_0 \ (1 \leqslant k \leqslant n),$$

$$p_n = \frac{m(m-1)\ldots(m-n+1)}{n!}\rho^n p_0, \ \ldots,$$

$$p_{n+r} = \frac{m(m-1)\ldots[m-(n+r-1)]}{n!\,n^r}\rho^{n+r} p_0 \quad (1 \leqslant r \leqslant m-1),$$

$$p_m = \frac{m!}{n!\,n^{m-n}}\rho^m p_0, \quad \text{where} \quad \rho = \lambda/\mu, \qquad (11.34.1)$$

The average number of busy workers can be expressed in terms of these probabilities, viz.

$$\bar{k} = 0\cdot p_0 + 1p_1 + 2p_2 + \ldots + (n-1)\,p_{n-1} + n\,(p_n + p_{n+1} + \ldots + p_m) = p_1 + 2p_2 + \ldots + (n-1)\,p_{n-1} + n\,(1 - p_0 - p_1 - \ldots - p_{n-1}). \quad (11.34.2)$$

The absolute capacity for service

$$A = \bar{k}\mu, \quad (11.34.3)$$

and the average number of faulty tools

$$\bar{w} = m - \bar{k}\mu/\lambda = m - \bar{k}/\rho. \quad (11.34.4)$$

11.35. Two workers ($n = 2$) are handling six machine-tools ($m = 6$). A tool must be reset every half-hour on the average. Ten minutes, on the average, are needed for a worker to reset a tool. All the processes are stationary Poisson's. (1) Find the characteristics of the queueing system, i.e. the average number of busy workers $\bar{k}$, the absolute capacity for service A and the average number of faulty tools $\bar{w}$. (2) Find out whether the characteristics of the system will be improved if the workers were to reset the tools together, spending five minutes on the average to reset a tool when working together.

Solution. (1) We solve the problem in the first variant (the workers work separately). We have $m = 6$, $n = 2$, $\lambda = 2$, $\mu = 6$ and $\rho = \lambda/\mu = 1/3$. By formulas (11.34.1) we have

$$p_0 = \left\{1 + \frac{6}{1}\frac{1}{3} + \frac{6\cdot5}{1\cdot2}\frac{1}{3^2} + \frac{6\cdot5\cdot4}{1\cdot2\cdot2}\frac{1}{3^3} + \frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot2^2}\frac{1}{3^4} + \frac{6\cdot5\cdot4\cdot3\cdot2}{1\cdot2\cdot2^3}\frac{1}{3^5} + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot2^4}\frac{1}{3^6}\right\}^{-1} \approx 0.153,$$

$$p_1 = 6/1 \times 1/3 p_0 \approx 0.306.$$

We use formula (11.34.2) to find the average number of busy workers, i.e. $\bar{k} = 1\cdot p_1 + 2\,(1 - p_0 - p_1) \approx 1.235$. The absolute capacity $A = \bar{k}\mu \approx 7.41$. The average number of faulty tools $\bar{w} = 6 - 7.41/2 \approx 2.30$.

(2) If the workers work together to reset the tools, then the queueing system turns into a one-channel system ($m = 6$, $n = 1$) for $\mu = 12$. We do the calculations for $\rho = \lambda/\mu = 1/6$. By formulas (11.31.1) we get

$$p_0 = \left\{1 + 1 + \frac{6\cdot5}{6^2} + \frac{6\cdot5\cdot4}{6^3} + \frac{6\cdot5\cdot4\cdot3}{6^4} + \frac{6\cdot5\cdot4\cdot3\cdot2}{6^5} + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6^6}\right\}^{-1}$$

$$\approx 0.264,\ p_1 \approx 0.264,\ p_2 \approx 0.220,\ p_3 \approx 0.147,\ p_4 \approx 0.076,$$

$$p_5 \approx 0.024, \quad p_6 \approx 0.004, \quad \bar{w} = 6 - \frac{0.736}{1/6} \approx 1.59.$$

The average number of busy channels $\bar{k} = 1 - p_0 = 0.736$. Bearing in mind, however, that in this case the "channel" consists of two workers, the average number of busy workers is

$$\bar{k} = 2 \times 0.736 \approx 1.47 \text{ and } A = \bar{k}\mu = 0.736 \times 12 \approx 8.8.$$

Thus, in this case, by assisting each other, the workers (channels) increase the average busy period from 1.23 to 1.47, reduce the average number of faulty tools from 2.30 to 1.59, and increase the capacity from 7.4 to 8.8.

11.36. Jobs arrive at an elementary three-channel congestion system in a process with intensity $\lambda = 4$ jobs/min, the service time per job per channel $\bar{t}_{ser} = 1/\mu = 0.5$ min. Is it better, from the point of view of the capacity of the system, to make all three channels serve each job together so as to decrease the service time of a job to a third? How will this affect the average waiting time of a job?

Solution. (1) We find the probabilities of states of the system without any interchannel assistance, using Erlang's formulas (11.0.6), i.e.

$$p_0 = \left\{1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right\}^{-1} \approx 0.158, \quad \mathrm{P_{ref}} = p_3 = \frac{2^3}{3!} p_0 \approx 0.21,$$

$$Q = 1 - \mathrm{P_{ref}} \approx 0.79, \quad A = \lambda Q \approx 3.16.$$

We then calculate the average waiting time of a job as the probability Q that a job will be served multiplied by the average service time, i.e. $\bar{t}_w \approx 0.79 \cdot 0.5 \approx 0.395$ min

(2) If we combine the three channels into one with parameter $\mu = 3 \times 2 = 6$, we obtain

$$p_0 = \frac{1}{(1+2/3)^{-1}} = 0.6, \quad p_1 = (2/3)\,0.6 = 0.4,$$

$$\mathrm{P_{ref}} = p_1 = 0.4, \quad Q = 1 - P_{ref} = 0.6, \quad A = \lambda Q = 4 \times 0.6 = 2.4.$$

Comparing the capacity of this system to that of the first variant, we see that it has not risen (as was the case in the preceding problem), indeed it has fallen! It is easy to understand why this happened. When the two channels were combined, the probability of refusal increased (the probability that an arriving job finds both channels busy and departs unserved).

The average waiting time of a job is smaller in the second variant than in the first: $\bar{t}_w = Q\,(1/6) = 0.1$ min. However, this decrease has a high cost since it comes at the expense of a number of jobs departing unserved and, hence, their waiting time is zero.

Why then in the preceding problem did combining two channels into one increase the efficiency of service? We invite the reader to think this over and explain the seeming contradiction. Is it because in a system with refusals jobs do not join the queue?

11.37. We are given an elementary three-channel system with an unbounded queue. The intensity of the arrival process is $\lambda = 4$ demands/h and the average service time $\bar{t}_{ser} = 1/\mu = 0.5$ h.

Bearing in mind (1) the average length of the queue, (2) the average queueing time of a demand and (3) the average waiting time of a demand, is it advantageous to combine all three channels into one with an average service time three times as small?

Solution. (1) In the original variant (three-channel system) $n = 3$, $\lambda = 4$, $\mu = 1/0.5 = 2$, $\rho = \lambda/\mu = 2$ and $\varkappa = \rho/n = 2/3 < 1$. The limiting probabilities exist. We calculate p_0 by formulas (11.0.21):

$$p_0 = \left\{1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4 \cdot (1/3)}{3!\,(1-2/3)}\right\}^{-1} = \frac{1}{9},$$

$$\bar{r} = \frac{2^4 \cdot 1/9}{3 \cdot 3!\,(1/3)^2} = \frac{8}{9} \approx 0.889, \quad \bar{t}_{\text{q}} = \frac{\bar{r}}{\lambda} = \frac{2}{9} \approx 0.222 \text{ h},$$

$$\bar{t}_{\text{w}} = \bar{t}_{\text{q}} + \bar{t}_{\text{ser}} \approx 0.722.$$

(2) When we combine the three channels $n = 1$, $\lambda = 4$, $\mu = 6$, and $\rho = 2/3$. From formulas (11.0.12)-(11.0.14) we have

$$\bar{r} = \frac{(2/3)^2}{1/3} \approx 1.333, \quad \bar{t}_{\text{q}} = \frac{\bar{r}}{\lambda} \approx 0.333 \text{ h},$$

$$\bar{t}_{\text{w}} = \bar{t}_{\text{q}} + \bar{t}_{\text{ser}} = 1/3 + 1/6 = 0.500.$$

We see that although combining three channels into one slightly reduces the average waiting time of a demand (from 0.722 to 0.500), it increases the average length of the queue and the average queueing time of a demand. This is because the arriving demands must wait in the queue while the three channels serve a demand together.

The increase in the service efficiency when channels are combined in a closed queueing system is due to the fact that the intensity of the arrival process of demands decreases when their sources (machine-tools) fail.

11.38. We consider a queueing system which is a taxi-rank at which passengers arrive in a stationary Poisson process with intensity λ and cars arrive in a stationary Poisson process with intensity μ. The passengers form a queue which decreases by unity when a car arrives at the rank (we take a case when the taxi-driver drives the passenger wherever the latter wants to). If there are no passengers at the rank, the cars form a queue. The number of parking places at the rank is limited (is equal to m) and the number of places in the queue for passengers is also limited (equal to l). All the processes are stationary Poisson's. A passenger boards a taxi instantaneously. Construct the directed graph of states of the system, find the limiting probabilities of states, the average length $\bar{r}_{\text{p}}$ of the queue of passengers, the average length $\bar{r}_{\text{c}}$ of the queue of cars, the average queueing time per passenger $\bar{t}_{\text{q.p}}$, the average queueing time per car $\bar{t}_{\text{q.c}}$ and see how these characteristics will change as $m \to \infty$ and $l \to \infty$.

Solution. We shall enumerate the states of the system in accordance with the numbers of passengers and cars at the taxi-rank, labelling

them by two indices: the first index for the number of passengers and the second for the number of cars. The state $s_{0,0}$ means that there are neither passengers nor cars at the rank; the state $s_{0,k}$ means no cars and k passengers; the state $s_{c,0}$ means c cars and no passengers. The directed graph of states of the system is shown in Fig. 11.38. The graph corresponds to the birth and death process. Employing the general formulas (11.0.4) for this process and designating $\lambda/\mu = \rho$ we obtain

$$p_{l-1,0} = \rho p_{l,0}, \quad p_{l-2,0} = \rho^2 p_{l,0}, \ldots, p_{c,0} = \rho^{l-c} p_{l,0}, \ldots,$$
$$p_{0,0} = \rho^l p_{l,0}, \ldots, p_{0,k} = \rho^{l+k} p_{l,0}, \ldots, p_{0,m} = \rho^{l+m} p_{l,0},$$
$$p_{l,0} = \{1 + \rho + \rho^2 + \ldots + \rho^{l+m}\}^{-1} \qquad (11.38.1)$$

or, summing the geometric progression with a common ratio ρ,

$$p_{l,0} = (1 - \rho)/(1 - \rho^{l+m+1}). \qquad (11.38.2)$$

The probabilities in (11.38.1) form a geometric progression with the first term $p_{l,0}$ and a common ratio ρ. If $\rho > 1$, then the most probable

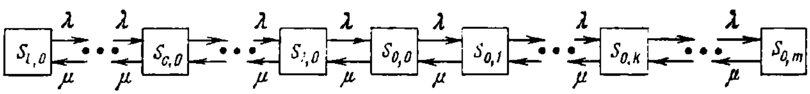


Fig. 11.38

state of the system is $s_{0,m}$, which means that there are no cars and all the places in the queue of passengers are occupied; if $\rho < 1$, then the most probable state is $s_{l,0}$, which means that there are no passengers and all the places in the queue of cars are occupied.

As $m \to \infty$ and $l \to \infty$, no limiting probabilities exist; for $\rho > 1$ the queue of passengers and for $\rho < 1$ the queue of cars tend to increase indefinitely (but this tendency is restricted by the fact that neither the number of passengers nor the number of taxes in the city is infinite).

11.39*. In a self-service canteen there is one distribution counter at which both the first and the second courses are served. Customers arrive at the canteen in a stationary Poisson process with intensity λ; the service of the first as well as the second course takes a random time which has an exponential distribution with the same parameter μ. Some customers take the first and the second course (the fraction of such customers is q), while the others take only the second course (the fraction of them is $1 - q$). Find: (1) the conditions under which there is a stable, stationary state of service at the canteen; (2) the average length of the queue and the average waiting time of a customer if he needs a time τ on the average to eat one course and a time 2τ to eat both courses.

Solution. The service time of one customer is a random variable T which has either an Erlang distribution of order 2 with the mean value $\bar{t}_{ser} = 2/\mu$ with probability q or an exponential distribution

with parameter μ with probability $1 - q$. Let us find the mean value of the random variable T. To do that, we use the complete expectation formula (4.0.20) with two hypotheses: $H_1 = \{$a customer takes only the second course$\}$ and $H_2 = \{$a customer takes both courses$\}$. The probabilities of these hypotheses are $P(H_1) = 1 - q$ and $P(H_2) = q$.
The complete expectation of the random variable T

$$M[T] = P(H_1) M[T \mid H_1] + P(H_2) M[T \mid H_2] = (1 - q)(1/\mu) + q(2/\mu) = (q + 1)/\mu.$$

This means that the canteen can serve $\mu/(q + 1)$ customers per unit time on the average. If $\lambda \geqslant \mu/(q + 1)$, then the system is overloaded and limiting probabilities do not exist, but if $\lambda < \mu/(q + 1)$, then they exist. We assume $\lambda < \mu/(q + 1)$.

To find the average size of the queue $\bar{r}$, the average queueing $\bar{t}_q$ and waiting $\bar{t}_w$ times of a customer, we use the Pollaczek-Khinchine formula (11.0.31). To use this formula, we must know the coefficient of variation of the random variable T, i.e. the service time. We first find the second moment about the origin of that variable $M[T^2]$. From the complete expectation formula (with the same hypotheses H_1 and H_2) we obtain

$$M[T^2] = (1 - q) M[T^2 \mid H_1] + qM[T^2 \mid H_2]. \qquad (11.39.1)$$

On the hypothesis H_1 the random variable T has an exponential distribution with parameter μ:

$$M[T^2 \mid H_1] = \mathrm{Var}[T \mid H_1] + (M[T \mid H_1])^2 = 1/\mu^2 + (1/\mu)^2 = 2/\mu^2.$$

On the hypothesis H_2 we calculate the second moment about the origin of the variable T by the formula

$$M[T^2 \mid H_2] = \mathrm{Var}[T \mid H_2] + (M[T \mid H_2])^2.$$

But $M[T \mid H_2] = 2/\mu$, $(M[T \mid H_2])^2 = 4/\mu^2$; the variance $\mathrm{Var}[T \mid H_2]$ is twice as large as $\mathrm{Var}[T \mid H_1]$ since it is the variance of the sum of two independent random variables T_1 and T_2 which each have the same distribution, i.e. $\mathrm{Var}[T \mid H_2] = 2/\mu^2$. Consequently

$$M[T^2 \mid H_2] = 2/\mu^2 + 4/\mu^2 = 6/\mu^2,$$

whence we get by formula (11.39.1)

$$M[T^2] = (1 - q)\, 2/\mu^2 + q6/\mu^2 = 2(1 + 2q)/\mu^2.$$

The variance of the random variable T

$$\mathrm{Var}[T] = M[T^2] - (M[T])^2 = (1 + 2q - q^2)/\mu^2.$$

From this we can find the coefficient of variation of the random variable T, i.e.

$$v_\mu = \sqrt{1 + 2q - q^2}/(q + 1).$$

Substituting this expression and the expression $\rho = \lambda (q + 1)/\mu$ into the Pollaczek-Khinchine formula (11.0.31), we get

$$\bar{r} = \frac{(\lambda^2/\mu^2)(q+1)^2\left[1+\frac{1+2q-q^2}{(q+1)^2}\right]}{2[1-(\lambda/\mu)(q+1)]} = \frac{(\lambda^2/\mu^2)(1+2q)}{1-(\lambda/\mu)(q+1)}.$$

Furthermore, $\bar{t}_q = \bar{r}/\lambda$ and $\bar{t}_w = \bar{t}_q + (q + 1)/\mu + (q + 1)\tau = \bar{t}_q + (q + 1)(\tau + 1/\mu)$.

11.40. *An example of an elementary congestion system with priority.* Customers arrive at a two-server congestion system in two stationary Poisson processes: process I with intensity λ_1 and process II with intensity λ_2 (for brevity we shall call them "customers I" and "customers II"). Customers I have a priority of service over customers II such that if a customer I arrives when the servers are busy and at least one of them is serving customer II, then the arriving customer I displaces customer II, occupies his place, and the latter departs unserved. If a customer I arrives when the servers are all serving customers I, he is refused and leaves the system. Customer II is refused if he arrives when both servers are busy (no matter which customers are being served).

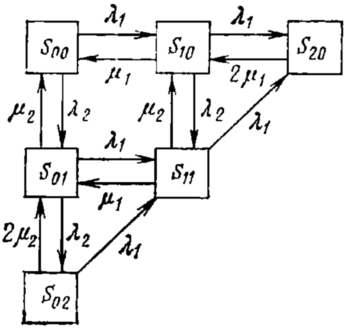


Fig. 11.40

Construct a marked directed graph of states of the queueing system labelling the states by the indices (i, j), the first index showing the number of customers I and the second, the number of customers II present in the system. Write the equations for the limiting probabilities of the states. Solve them at $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$. Express the following efficiency characteristics of the system in terms of p_{ij} $(i + j \leqslant 2)$:

$P^{(1)}_{ref}$ $(P^{(2)}_{ref})$—the probability that customer I (II) will be refused when he arrives;

A_1 (A_2)—the average number of customers I (II) served by the queueing system per unit time;

$\bar{k}_1$ $(\bar{k}_2)$—the average number of servers busy with serving customers I (II);

P_{ref}, A, $\bar{k}$—the same characteristics for the system as a whole, irrespective of the type of customer.

Solution. The states of the system are s_{00}, there are no customers in the system; s_{10}, there is one customer I and no customers II in the system; s_{01}, there are no customers I in the system and one customer II; s_{20}, there are two customers I in the system and no customers II; s_{11}, there is one customer each I and II; s_{02}, there are no customers I and two customers II in the system. The marked graph of states of the system is shown in Fig. 11.40.

The equations for the limiting probabilities of states:

$$(\lambda_1 + \lambda_2) p_{00} = \mu_1 p_{10} + \mu_2 p_{01}, \quad (\lambda_1 + \lambda_2 + \mu_1) p_{10} = \lambda_1 p_{00} + 2\mu_1 \times p_{20} + \mu_2 p_{11}, \quad 2\mu_1 p_{20} = \lambda_1 p_{10} + \lambda_1 p_{11}, \quad (\lambda_1 + \lambda_2 + \mu_2) \times p_{01} = \lambda_2 p_{00} + \mu_1 p_{11} + 2\mu_2 p_{02}, \quad (\lambda_1 + \mu_1 + \mu_2) p_{11} = \lambda_2 p_{10} + \lambda_1 p_{01} + \lambda_1 p_{02}, \quad (\lambda_1 + 2\mu_2) p_{02} = \lambda_2 p_{01}, \quad p_{00} + p_{10} + p_{20} + p_{01} + p_{11} + p_{02} = 1.$$

Solving them at $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, we get

$$p_{00} = 0.20,\ p_{10} = 0.25,\ p_{20} = 0.20,\ p_{01} = 0.15,\ p_{11} = 0.15,\ p_{02} = 0.05,\ \mathrm{P}_{\mathrm{ref}}^{(1)} = p_{20} = 0.2\mathrm{P}_{\mathrm{ref}}^{(2)} = p_{20} + p_{11} + p_{02} = 0.40, \quad A_1 = \lambda_1 (1 - \mathrm{P}_{\mathrm{ref}}^{(1)}) = 0.8.$$

We calculate the variable A_2 taking into account that some customers II that have started to be served are displaced by customers I and depart unserved. The average number of such customers per unit time is $\lambda_{11} (p_{11} + p_{02})$ and, consequently,

$$A_2 = \lambda_2 [1 - (p_{20} + p_{11} + p_{02})] - \lambda_1 (p_{11} + p_{02}) = 0.4,$$
$$\bar{k}_1 = A_1/\mu_1, \quad \bar{k}_2 = A_2/\mu_2.$$

The probability $\mathrm{P}_{\mathrm{ref}}$ that an arbitrarily chosen customer arriving at the system will be refused can be found by the total probability formula with hypotheses $H_1 = \{$a customer I has arrived$\}$ and $H_2 = \{$a customer II has arrived$\}$. The probabilities of these hypotheses are

$$\mathrm{P}(H_1) = \lambda_1/(\lambda_1 + \lambda_2) \quad \text{and} \quad \mathrm{P}(H_2) = \lambda_2/(\lambda_1 + \lambda_2).$$

Consequently

$$\mathrm{P}_{\mathrm{ref}} = \frac{\lambda_1}{\lambda_1 + \lambda_2} \mathrm{P}_{\mathrm{ref}}^{(1)} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \mathrm{P}_{\mathrm{ref}}^{(2)} = 0.3.$$

Note that we could obtain all the characteristics for customers I by completely ignoring customers II and treating the problem as if only customers I arrived at a two-server system with refusals. We invite the reader to verify this by calculating all the characteristics for a two-server system with refusals at which only customers I arrive.

11.41. The conditions of the preceding problem are changed so that the number of servers in the system is $n = 3$. Construct the directed graph of states of the system. Write the equations for the limiting probabilities p_{ij} $(i + j \leqslant 3)$, where i is the number of customers I and j is the number of customers II which are present in the system. Considering the equations to have been solved, express the same efficiency characteristics as enumerated in the preceding problem in terms of p_{ij}.

Solution. The directed graph of states is shown in Fig. 11.41.

The equations for the limiting probabilities of states are

$$(\lambda_1+\lambda_2)\,p_{00}=\mu_1 p_{10}+\mu_2 p_{01},\quad (\lambda_1+\lambda_2+\mu_1)\,p_{10}$$
$$=\lambda_1 p_{00}+2\mu_1 p_{20}+\mu_2 p_{11},\quad (\lambda_1+\lambda_2+2\mu_1)\,p_{20}$$
$$=\lambda_1 p_{10}+3\mu_1 p_{30}+\mu_2 p_{21},$$
$$(\lambda_1+\lambda_2+\mu_2)\,p_{01}=\lambda_2 p_{00}+\mu_2 p_{11}+2\mu_2 p_{02},$$
$$(\lambda_1+\lambda_2+\mu_1+\mu_2)\,p_{11}=\lambda_1 p_{01}+\lambda_2 p_{10}+2\mu_1 p_{21}+2\mu_2 p_{12},$$
$$(\lambda_1+\mu_2+2\mu_1)\,p_{21}=\lambda_1(p_{11}+p_{12})+\lambda_2 p_{20},\quad (\lambda_1+\lambda_2+2\mu_2)\,p_{02}$$
$$=\lambda_2 p_{01}+\mu_1 p_{12}+3\mu_2 p_{03},\quad (\lambda_1+\mu_1+2\mu_2)\,p_{12}=\lambda_1(p_{02}+p_{03})+\lambda_2 p_{11},$$
$$(\lambda_1+3\mu_2)\,p_{03}=\lambda_2 p_{02},$$
$$p_{00}+p_{10}+p_{20}+p_{30}+p_{01}+p_{11}+p_{21}+p_{02}+p_{12}+p_{03}=1,$$
$$\mathrm{P}_{\mathrm{ref}}^{(1)}=p_{30},\quad \mathrm{P}_{\mathrm{ref}}^{(2)}=p_{30}+p_{21}+p_{12}+p_{03},$$
$$A_1=\lambda_1(1-p_{30}),\quad A_2=\lambda_2\,[1-(p_{30}+p_{21}+p_{12}+p_{03})]$$
$$-\lambda_1(p_{03}+p_{12}+p_{03}),$$
$$\bar{k}_1=A_1/\mu_1,\quad \bar{k}_2=A_2/\mu_2,\quad \bar{k}=\bar{k}_1+\bar{k}_2,$$
$$\mathrm{P}_{\mathrm{ref}}=\frac{\lambda_1}{\lambda_1+\lambda_2}\,\mathrm{P}_{\mathrm{ref}}^{(1)}+\frac{\lambda_2}{\lambda_1+\lambda_2}\,\mathrm{P}_{\mathrm{ref}}^{(2)}.$$

11.42. *An example of a queueing system with absolute priority.* Customers I and II arrive at a single-server queueing system with two places in the queue ($m = 2$) in two stationary Poisson processes with intensities λ_1 and λ_2. The service times are exponential with parameters μ_1 and μ_2. A customer I who arrives at the system displaces customer II if he is being served or takes a place in the queue in front of him if he is in the queue. The displaced customer II leaves the system unserved if there are no more places in the queue or joins the queue if there are places. Labelling the states of the system with the indices i and j corresponding to the number of customers I and II present in the system, construct a marked directed graph of states of the system and write the equations for the limiting probabilities of states. Considering these equations to have been solved, express the following efficiency characteristics of the system in terms of p_{ij} ($i + j \leqslant 3$).

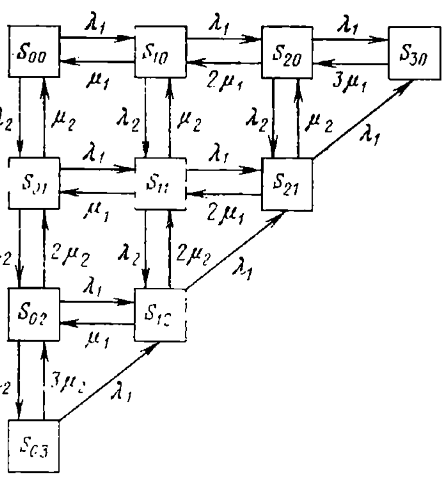


Fig. 11.41

$P_{ref}^{(1)}$ ($P_{ref}^{(2)}$)—the probability that a customer I (II) will be refused immediately after his arrival;

Q_1 (Q_2)—the probability that a customer I (II) will be served;

$\bar{z}_1$ ($\bar{z}_2$)—the average number of customers I (II) in the queueing system;

$\bar{r}_1$ ($\bar{r}_2$)—the average number of customers I (II) in the queue;

$\bar{t}_w^{(1)}$ ($\bar{t}_w^{(2)}$)—the average waiting time of customer I (II);

$\bar{t}_q^{(1)}$ ($\bar{t}_q^{(2)}$)—the average queueing time of customer I (II);

$\bar{t}_w$—the average waiting time of any (arbitrary) customer;

$\bar{t}_q$—the average queueing time of any customer.

Solution. The states of the queueing system are s_{ij}, i customers I and j customers II are in the system ($i + j \leqslant 3$).

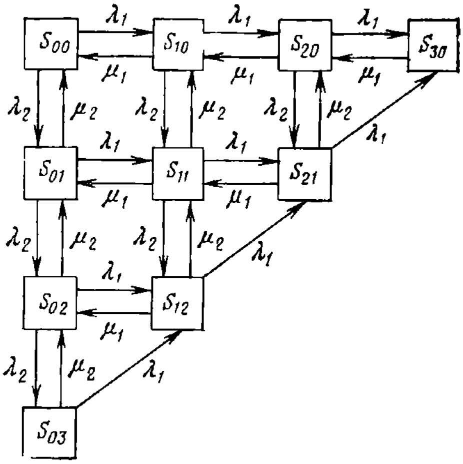


Fig. 11.42

The marked directed graph of states is shown in Fig. 11.42. The equations for the limiting probabilities of states are

$$(\lambda_1 + \lambda_2)\, p_{00} = \mu_1 p_{10} + \mu_2 p_{01}, \quad (\lambda_1 + \lambda_2 + \mu_1)\, p_{10} = \lambda_1 p_{00}$$
$$+ \mu_1 p_{20} + \mu_2 p_{11}, \quad (\lambda_1 + \lambda_2 + \mu_1)\, p_{20} = \lambda_1 p_{10} + \mu_1 p_{30} + \mu_2 p_{21},$$
$$\mu_1 p_{30} = \lambda_1 (p_{20} + p_{21}), \quad (\lambda_1 + \lambda_2 + \mu_2)\, p_{01} = \lambda_2 p_{00} + \mu_1 p_1 + \mu_2 p_{02},$$
$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\, p_{11} = \lambda_2 p_{10} + \lambda_1 p_{01} + \mu_1 p_{21} + \mu_2 p_{12},$$
$$(\lambda_1 + \lambda_2 + \mu_2)\, p_{02} = \lambda_2 p_{01} + \mu_1 p_{12} + \mu_2 p_{03},$$
$$(\lambda_1 + \mu_1 + \mu_2)\, p_{12} = \lambda_1 p_{02} + \lambda_2 p_{11} + \lambda_1 p_{03}, \quad (\lambda_1 + \mu_2)\, p_{03} = \lambda_2 p_{02},$$
$$p_{00} + p_{10} + p_{20} + p_{30} + p_{01} + p_{11} + p_{21} + p_{02} + p_{12} + p_{03} = 1,$$
$$P_{ref}^{(1)} = p_{30}, \quad P_{ref}^{(2)} = p_{30} + p_{21} + p_{12} + p_{03}, \quad Q_1 = 1 - P_{ref}^{(1)} = 1 - p_{30}.$$

To find Q_2, we first seek A_2 which is the average number of customers II served per unit time, i.e.

$$A_2 = \lambda_2 (1 - P_{ref}^{(2)}) - \lambda_1 (p_{03} + p_{12} + p_{21})$$
$$= \lambda_2 [1 - (p_{30} + p_{21} + p_{12} + p_{03})] - \lambda_1 (p_{03} + p_{12} + p_{21}).$$

Dividing this expression by λ_2, we find the average fraction of customers II being served (the probability that a customer II will be served),

i.e. $Q_2 = A_2/\lambda_2$. Then

$$\bar{z}_1 = 1\cdot(p_{10} + p_{11} + p_{12}) + 2\,(p_{20} + p_{21}) + 3p_{30},\ \bar{z}_2 = 1\cdot(p_{01} + p_{11} + p_{21}) + 2\,(p_{02} + p_{12}) + 3p_{03},\quad \bar{r}_1 = 1\cdot(p_{20} + p_{21}) + 2p_{30},$$

$$\bar{r}_2 = 1\,(p_{11} + p_{21}) + 2p_{12}.$$

By Little's formulas

$$\bar{t}_{\text{w}}^{(1)} = \bar{z}_1/\lambda_1,\quad \bar{t}_{\text{w}}^{(2)} = \bar{z}_2/\lambda_2,\quad \bar{t}_{\text{q}}^{(1)} = \bar{r}_1/\lambda_1,\quad \bar{t}_{\text{q}}^{(2)} = \bar{r}_2/\lambda_2,$$

$$\bar{t}_{\text{w}} = \frac{\lambda_1}{\lambda_1+\lambda_2}\,\bar{t}_{\text{w}}^{(1)} + \frac{\lambda_2}{\lambda_1+\lambda_2}\,\bar{t}_{\text{w}}^{(2)},\qquad \bar{t}_{\text{q}} = \frac{\lambda_1}{\lambda_1+\lambda_2}\,\bar{t}_{\text{q}}^{(1)} + \frac{\lambda_2}{\lambda_1+\lambda_2}\,\bar{t}_{\text{q}}^{(2)}.$$

We can calculate all the characteristics relating to customers I without solving equations (11.42), but simply by ignoring the presence of customers II and considering the system to be one with a limited number of places in the queue ($m = 2$) and using the formulas in Sec. 11.0 (item 3) to find its characteristics.

11.43. *An elementary system without a queue and with a "heating up" of the channels.* Demands arrive at the input of an n-channel queueing system in a stationary Poisson process with intensity λ. The service time is exponential with parameter μ. Before the service is begun, the channel must be prepared ("heated up"). The time needed to "heat" the channel T_{heat} has an exponential distribution with parameter ν and does not depend on how long before the channel ceased functioning. A demand which arrives when a channel is idle occupies it and waits for it to be heated up, and then its service commences. A demand which arrives when all the channels are busy (when another demand is being served or is waiting to be served) departs unserved. Find the limiting probabilities of the system and the efficiency characteristics, i.e. the probability of refusal P_{ref}, the relative capacity for service Q, the absolute capacity A and the average number of busy channels $\bar{k}$.

Solution. We assume that the job servicing procedure consists of two phases: waiting for a channel to be heated and the servicing itself: $\tilde{T}_{\text{ser}} = T_{\text{heat}} + T_{\text{ser}}$. The random variable $\tilde{T}_{\text{ser}}$ has a generalized Erlang distribution of order 2 (see Problem 8.39) with parameters μ and ν. We know that Erlang's formulas (11.0.6) are valid not only for exponential but also for any distribution of service time. Let us find the variable $\tilde{\mu} = 1/\text{M}\,[\tilde{T}_{\text{ser}}]$. We have

$$\text{M}\,[\tilde{T}_{\text{ser}}] = \text{M}\,[T_{\text{heat}}] + \text{M}\,[T_{\text{ser}}] = 1/\mu + 1/\nu = (\mu + \nu)/(\mu\nu),$$

whence $\tilde{\mu} = (\mu\nu)/(\mu + \nu)$. Having calculated $\tilde{\rho} = \lambda/\tilde{\mu}$ and substituting this value of $\tilde{\rho}$ into Erlang's formulas (11.0.6), we obtain

$$p_0 = \left\{1 + \frac{\tilde{\rho}}{1!} + \ldots + \frac{\tilde{\rho}^k}{k!} + \ldots + \frac{\tilde{\rho}^n}{n!}\right\}^{-1},\quad p_k = \frac{\tilde{\rho}^k}{k!}\,p_0\quad (1 \leqslant k \leqslant n),$$

$$P_{\text{ref}} = p_n = \frac{\tilde{\rho}^n}{n!}\,p_0,\quad Q = 1 - \frac{\tilde{\rho}^n}{n!}\,p_0,\quad A = \lambda Q = \lambda\left(1 - \frac{\tilde{\rho}^n}{n!}\,p_0\right).$$

To find the average number of busy channels $\bar{k}$, we must divide A by $\tilde{\mu}$:

$$\bar{k}=\frac{\lambda}{\tilde{\mu}}\left(1-\frac{\tilde{\rho}^n}{n!}\right)p_0=\tilde{\rho}\left(1-\frac{\tilde{\rho}^n}{n!}\right)p_0.$$

11.44. *An elementary one-channel queueing system with a "heated" channel.* Demands arrive at a one-channel queueing system with an unbounded queue in a stationary Poisson process with intensity λ. The service time is exponential with parameter μ ($\mu > \lambda$). Before the

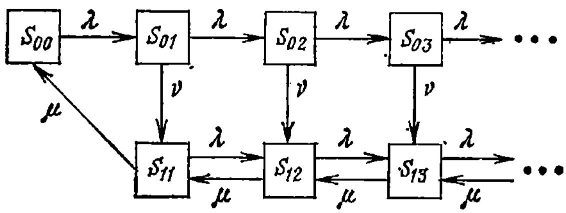


Fig. 11.44

demand is serviced, the idle channel must be heated. The time needed to heat the channel is exponential with parameter ν and does not depend on how long before the channel ceased functioning. If the service begins immediately after the service of the previous demand has ended, no reheating is necessary. Construct the directed graph of states of the queueing system and write the equations for the limiting probabilities of states. Express the efficiency characteristics of the system in terms of these probabilities: the average numbers of demands in the system $\bar{z}$ and in the queue $\bar{r}$ and the average waiting and queueing times of a job $\bar{t}_w$ and $\bar{t}_q$ respectively.

Solution. The states of the queueing system are (Fig. 11.44):

s_{00}—the channel is idle but not hot;

s_{01}—a job has arrived and is waiting for the channel to be heated;

s_{11}—the channel is hot, one demand is being served, there is no queue;

s_{02}—the channel is being heated, there are two demands in the queue, . . . ;

s_{0l}—the channel is being heated up, there are l demands in the queue;

s_{1l}—the channel is serving demands, $l-1$ demands are in the queue, ...

The equations for the limiting probabilities are

$$\lambda p_{00}=\mu p_{11},\quad (\lambda+\nu)\,p_{01}=\lambda p_{00},\quad (\lambda+\mu)\,p_{11}=\nu p_{01}+\mu p_{12},$$
$$(\lambda+\nu)\,p_{02}=\lambda p_{01},\quad (\lambda+\mu)\,p_{12}=\nu p_{02}+\lambda p_{11}+\mu p_{13},\ \ldots$$
$$(\lambda+\nu)\,p_{0,l}=\lambda p_{0,l-1},$$
$$(\lambda+\mu)\,p_{1,l}=\nu p_{0,l}+\lambda p_{1,l-1}+\mu p_{1,l+1},\ \ldots,$$
$$\bar{z}=\sum_{l=0}^{\infty} l\,(p_{0,l}+p_{1,l}),\quad \bar{r}=\sum_{l=1}^{\infty} l\,(p_{0,l}+p_{1,l+1}).$$

By Little's formula

$$\bar{t}_w = \bar{z}/\lambda, \quad \bar{t}_q = \bar{r}/\lambda.$$

11.45*. We are given a single-server system with a limited number of places in the queue $m = 2$. Customers arrive at it in a stationary Poisson process with intensity λ.

The service time has a generalized Erlang distribution with parameters μ_1 and μ_2 (see Problem 8.39). Find the probabilities of the states of the system:

s_0—there are no customers in the system;

s_1—there is one customer in the system (no queue);

s_2—there are two customers in the system (one is being served and the other is waiting to be served);

s_3—there are three customers in the system (one is being served and two are in the queue).

(1) Find the efficiency characteristics of the system: P_{ref}, Q, A, $\bar{z}$, $\bar{r}$, $\bar{t}_w$, $\bar{t}_q$. (1) Calculate them for the values $\lambda = 2$, $\mu_1 = 6$ and $\mu_2 = 12$. (2) Compare them with the characteristics which would result for an elementary queueing system with the same λ and μ equal to $1/\bar{t}_{ser}$, where $\bar{t}_{ser}$ is the average service time of a customer in this system.

Solution. (1) The service process is not Poisson's and, therefore, the system is not Markovian. This means that to find the probabilities of states of the system, we cannot use the ordinary techniques we used for Markov processes with discrete states and continuous time. However, we can artificially reduce the process taking place in the system to a Markov process by applying the so-called method of phases.

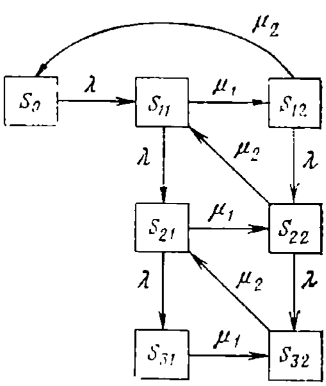


Fig. 11.45

We assume that the service is in two phases (I and II) which last for the times T_1 and T_2 respectively. The total service time $T_{ser} = T_1 + T_2$, where T_1 has an exponential distribution with parameter μ_1 and T_2 has an exponential distribution with parameter μ_2. Then T_{ser} will have a generalized Erlang distribution with parameters μ_1 and μ_2 (see Problem 8.39).

We introduce the following states of the system:

s_0—the system is idle;

s_{11}—there is one customer in the system, the service is in the first phase;

s_{12}—there is one customer in the system, the service is in the second phase;

s_{21}—there are two customers in the system (one is being served and the other is waiting to be served); the service is in the first phase;

s_{22}—there are two customers in the system, the service is in the second phase;

s_{31}—there are three customers in the system, the service is in the first phase;

s_{32}—there are three customers in the system, the service is in the second phase.

There are no other states since ($m = 2$ by the hypothesis) there can be no more than three customers in the system. The marked directed graph of states of the system is shown in Fig. 11.45. Using this approach, we divide the state s_1 into two states: s_{11} and s_{12} (or, in a brief form, $s_1 = s_{11} + s_{12}$). Similarly $s_2 = s_{21} + s_{22}$ and $s_3 = s_{31} + s_{32}$. The equations for the limiting probabilities which correspond to the graph in Fig. 11.45 are

$$\lambda p_0 = \mu_2 p_{12}, \quad (\lambda + \mu_1) p_{11} = \lambda p_0 + \mu_2 p_{22}, \quad (\lambda + \mu_2) p_{12} = \mu_1 p_{11},$$
$$(\lambda + \mu_1) p_{21} = \mu_2 p_{32} + \lambda p_{11}, \qquad (\lambda + \mu_2) p_{22} = \mu_1 p_{21} + \lambda p_{12},$$
$$\mu_1 p_{31} = \lambda p_{21}, \qquad \mu_2 n_{32} = \mu_1 p_{31} + \lambda p_{22},$$

with the normalizing condition $p_0 + p_{11} + p_{12} + p_{21} + p_{22} + p_{31} + p_{32} = 1$.

Solving these equations, we obtain

$$p_0 = \left\{1 + \frac{(\lambda+\mu_2)\lambda}{\mu_1\mu_2} + \frac{\lambda}{\mu_2} + \frac{\lambda^3(\lambda+\mu_1+\mu_2)+\lambda^2\mu_2(\lambda+\mu_2)}{\mu_1^2\mu_2^2}\right.$$
$$+\frac{\lambda^2(\lambda+\mu_1+\mu_2)}{\mu_1\mu_2^2} + \frac{\lambda^4(\lambda+\mu_1+\mu_2)+\lambda^3\mu_2(\lambda+\mu_2)}{\mu_1^2\mu_2^3}$$
$$\left.+\frac{\lambda^3(\lambda+\mu_1+\mu_2)}{\mu_1\mu_2^3} + \frac{\lambda^4(\lambda+\mu_1+\mu_2)+\lambda^3\mu_2(\lambda+\mu_2)}{\mu_1^3\mu_2^2}\right\}^{-1},$$

$$p_{11} = \frac{(\lambda+\mu_2)\lambda}{\mu_1\mu_2} p_0, \quad p_{12} = \frac{\lambda}{\mu_2} p_0, \quad p_{21} = \frac{\lambda^3(\lambda+\mu_1+\mu_2)+\lambda^2\mu_2(\lambda+\mu_2)}{\mu_1^2\mu_2^2} p_0,$$
$$p_{22} = \frac{\lambda^2(\mu_1+\mu_2+\lambda)}{\mu_1\mu_2^2} p_0,$$
$$p_{31} = \frac{\lambda^4(\lambda+\mu_1+\mu_2)+\lambda^3\mu_2(\lambda+\mu_2)}{\mu_1^3\mu_2^2} p_0,$$
$$p_{32} = \left[\frac{\lambda^4(\lambda+\mu_1+\mu_2)+\lambda^3\mu_2(\lambda+\mu_2)}{\mu_1^2\mu_2^3} + \frac{\lambda^3(\lambda+\mu_1+\mu_2)}{\mu_1\mu_2^3}\right] p_0. \tag{11.45.1}$$

Next we find the limiting probabilities of the states s_1, s_2, s_3:

$$p_1 = p_{11} + p_{12} = \frac{\lambda}{\mu_1\mu_2}(\lambda+\mu_1+\mu_2) p_0,$$
$$p_2 = p_{21} + p_{22} = \frac{\lambda^2}{\mu_1^2\mu_2^2}[(\lambda+\mu_1)(\lambda+\mu_1+\mu_2)+\mu_2(\lambda+\mu_2)] p_0,$$
$$p_3 = p_{31} + p_{32} = \frac{\lambda^3}{\mu_1^3\mu_2^3}[(\lambda\mu_2+\lambda\mu_1+\mu_1^2)(\lambda+\mu_1+\mu_2)$$
$$+\mu_2(\mu_1+\mu_2)(\lambda+\mu_2)] p_0, \tag{11.45.2}$$

where p_0 is defined by the first formula (11.45.1).

The efficiency characteristics of the system can be found in terms of the probabilities p_0, p_1, p_2, p_3 by the formulas

$$P_{ref} = p_3, \quad Q = 1 - p_3, \quad A = \lambda(1 - p_3),$$
$$\bar{z} = 1p_1 + 2p_2 + 3p_3, \quad \bar{r} = 1p_2 + 2p_3,$$
$$\bar{t}_w = \bar{z}/\lambda, \quad t_q = \bar{r}/\lambda. \qquad (11.45.3)$$

Substituting the numerical data $\lambda = 2$, $\mu_1 = 6$, $\mu_2 = 12$ into formulas (11.45.1), we obtain $p_0 = 0.540$, $p_{11} = 0.182$, $p_{12} = 0.090$, $p_{21} = 0.087$, $p_{22} = 0.050$, $p_{31} = 0.029$ and $p_{32} = 0.022$. We now return to the initial (not Markovian) queueing system and obtain $p_0 = 0.540$, $p_1 = 0.272$, $p_2 = 0.137$ and $p_3 = 0.051$.

Furthermore, we have from formulas (11.45.3) $P_{ref} = 0.051$, $Q = 0.949$, $A = 1.89$, $\bar{z} = 0.705$, $\bar{r} = 0.243$, $\bar{t}_w = 0.352$ and $\bar{t}_q = 0.122$.

(2) We calculate the same characteristics for an elementary queueing system with the same $\lambda = 2$ and $\mu = (1/\mu_1 + 1/\mu_2)^{-1} = 4$. From formulas (11.0.12)-(11.0.15) we have: $\rho = 0.5$, $p_0 \approx 0.533$, $\bar{k} = 1 - p_0 \approx 0.467$, $p_1 = \rho p_0 \approx 0.267$, $p_2 = \rho^2 p_0 \approx 0.133$, $p_3 = \rho^3 p_0 \approx 0.067$, $P_{ref} = p_3 \approx 0.067$, $Q = 1 - p_3 \approx 0.933$, $A \approx 1.866$, $\bar{z} \approx 0.733$, $r = 0.267$, $\bar{t}_w \approx 0.367$ and $\bar{t}_q \approx 0.133$.

We see that our nonmarkovian queueing system has some advantages over an elementary queueing system as concerns the capacity for service and differs but little from it (to the better) as concerns the waiting time of a customer and the size of the queue.

11.46*. There is a single-server queueing system with two places in the queue. Customers arrive at it in a Palm process, with an interarrival time T, which has a generalized Erlang distribution with parameters λ_1 and λ_2; the service time is exponential with parameter μ. (1) Applying the method of phases, write the equations for the limiting probabilities of states p_0, p_1, p_2, p_3. Express the following characteristics of the system in terms of these probabilities: P_{ref}, Q, A, $\bar{z}$, $\bar{r}$, $\bar{t}_w$, $\bar{t}_q$. (2) Calculate the limiting probabilities and the efficiency characteristics for the following initial data: $\lambda_1 = 3$, $\lambda_2 = 6$ and $\mu = 4$ and compare them with the characteristics for an elementary system with parameters $\lambda = (1/\lambda_1 + 1/\lambda_2)^{-1} = 2$ and $\mu = 4$, the one considered in Problem 11.45.

Solution. (1) If we consider (as usual) the states of the system numbered in accordance with the number of customers in the system, s_0, s_1, s_2, s_3, then the system will not be Markovian. To make it Markovian, we shall divide the interarrival time T, rather than the service time, into two phases (I and II). $T = T_1 + T_2$, where the random variables T_1 and T_2 have an exponential distribution with parameters λ_1 and λ_2 respectively.

We shall enumerate the states of the system in accordance with the number of customers present in the system and the number of phases between the customers:

s_{01}—there are no customers in the system, the interarrival time is in the first phase;

s_{02}—there are no customers in the system, the interarrival time is in the second phase;

s_{11}—there is one customer in the system (being served), the interarrival time is in the first phase;

s_{12}—there is one customer in the system (being served), the interarrival time is in the second phase;

s_{21}—there are two customers in the system (one is being served and the other is waiting for service), the interarrival time is in the first phase;

s_{22}—there are two customers in the system (one is being served and the other is waiting for service), the interarrival time is in the second phase;

s_{31}—there are three customers in the system (one is being served and two are in the queue), the interarrival time is in the first phase;

s_{32}—there are three customers in the system (one is being served and two are in the queue), the interarrival time is in the second phase.

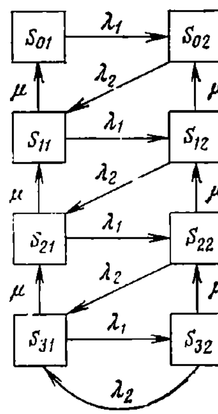


Fig. 11.46

The directed graph of states of the system is shown in Fig. 11.46.

The equations for the limiting probabilities are

$$\lambda_1 p_{01} = \mu p_{11}, \quad \lambda_2 p_{02} = \lambda_1 p_{01} + \mu p_{12}, \quad (\lambda_1 + \mu)\, p_{11} = \lambda_2 p_{02} + \mu p_{21},$$
$$(\lambda_2 + \mu)\, p_{12} = \lambda_1 p_{11} + \mu p_{22}, \quad (\lambda_1 + \mu)\, p_{21} = \lambda_2 p_{12} + \mu p_{31},$$
$$(\lambda_2 + \mu)\, p_{22} = \lambda_1 p_{21} + \mu p_{32}, \quad (\lambda_1 + \mu)\, p_{31} = \lambda_2 p_{22} + \lambda_2 p_{32},$$
$$(\lambda_2 + \mu)\, p_{32} = \lambda_1 p_{31}.$$

The normalizing condition is $p_{01} + p_{02} + p_{11} + p_{12} + p_{21} + p_{22} + p_{31} + p_{32} = 1$.

It is most convenient to solve these equations by expressing the probabilities p_{ij} in terms of the last probability, p_{32}. The expressions for the probabilities p_{ij} ($i = 0, 1, 2, 3$; $j = 1, 2$) have the form

$$p_{32} = \left\{ 1 + \frac{\mu + \lambda_2}{\lambda_1} + \frac{\mu(\mu + \lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} + \frac{\mu(\mu^2 + \lambda_1 \mu + 2\lambda_2 \mu + \lambda_2^2)}{\lambda_1^2 \lambda_2} \right.$$
$$+ \frac{\mu^2(\mu^2 + 2\lambda_1 \mu + 2\lambda_2 \mu + \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)}{\lambda_1^2 \lambda_2^2}$$
$$+ \frac{\mu^2(\mu^3 + 2\lambda_1 \mu^2 + 3\lambda_2 \mu^2 + \lambda_1^2 \mu + 2\lambda_1 \lambda_2 \mu + 3\lambda_2^2 \mu + \lambda_2^3)}{\lambda_1^3 \lambda_2^2}$$
$$+ \frac{\mu^3(\mu^3 + 3\lambda_1 \mu^2 + 3\lambda_2 \mu^2 + 3\lambda_1^2 \mu + 4\lambda_1 \lambda_2 \mu + 3\lambda_2^2 \mu + \lambda_1^3 + \lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2 + \lambda_2^3)}{\lambda_1^3 \lambda_2^3}$$
$$\left. + \frac{\mu^3(\mu^3 + 2\lambda_1 \mu^2 + 3\lambda_2 \mu^2 + \lambda_1^2 \mu + 2\lambda_1 \lambda_2 \mu + 3\lambda_2^2 \mu + \lambda_2^3)}{\lambda_1^4 \lambda_2^3} \right\}^{-1}.$$

$$p_{31}=\frac{\mu+\lambda_2}{\lambda_1}\,p_{32},\qquad p_{22}=\frac{\mu\,(\mu+\lambda_1+\lambda_2)}{\lambda_1\lambda_2}\,p_{32},$$

$$p_{21}=\frac{\mu\,(\mu^2+\lambda_1\mu+2\lambda_2\mu+\lambda_2^2)}{\lambda_1^2\lambda_2}\,p_{32},$$

$$p_{12}=\frac{\mu^2\,(\mu^2+2\lambda_1\mu+2\lambda_2\mu+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)}{\lambda_1^2\lambda_2^2}\,p_{32},$$

$$p_{11}=\frac{\mu^2\,(\mu^3+2\lambda_1\mu^2+3\lambda_2\mu^2+\lambda_1^2\mu+2\lambda_1\lambda_2\mu+3\lambda_2^2\mu+\lambda_2^3)}{\lambda_1^3\lambda_2^2}\,p_{32},$$

$$p_{02}=\frac{\begin{gathered}\mu^3\,(\mu^3+3\lambda_1\mu^2+3\lambda_2\mu^2+\lambda_1^2\mu+4\lambda_1\lambda_2\mu+\\+3\lambda_2^2\mu+\lambda_1^3+\lambda_1^2\lambda_2+\lambda_1\lambda_2^2+\lambda_2^3)\end{gathered}}{\lambda_1^3\lambda_2^2}\,p_{32},$$

$$p_{01}=\frac{\mu^3\,(\mu^3+2\lambda_1\mu^2+3\lambda_2\mu^2+\lambda_1^2\mu+2\lambda_1\lambda_2\mu+3\lambda_2^2\mu+\lambda_2^3)}{\lambda_1^4\lambda_2^2}\,p_{32}.$$

Returning to the initial (nonmarkovian) queueing system, we obtain: $p_0=p_{01}+p_{02}$, $p_1=p_{11}+p_{12}$, $p_2=p_{21}+p_{22}$ and $p_3=p_{31}+p_{32}$.

Furthermore, $P_{\text{ref}}=p_3$, $Q=1-p_3$, $A=Q\lambda$, $\bar{z}=1p_1+2p_2+3p_3$, $\bar{r}=1p_2+2p_3$, $\bar{t}_{\text{w}}=\bar{z}/\lambda$ and $\bar{t}_{\text{q}}=\bar{r}/\lambda$.

The limiting probabilities of states are $p_{01}\approx 0.308$, $p_{02}\approx 0.208$, $p_{11}\approx 0.231$, $p_{12}\approx 0.082$, $p_{21}\approx 0.091$, $p_{22}\approx 0.032$, $p_{31}\approx 0.037$ and $p_{32}\approx 0.011$.

For the initial (nonmarkovian) system $p_0\approx 0.616$, $p_1\approx 0.313$, $p_2\approx 0.123$, $p_3\approx 0.048$, $P_{\text{ref}}\approx 0.048$, $Q\approx 0.952$, $A\approx 1.904$, $\bar{z}\approx 0.703$, $\bar{r}\approx 0.219$, $\bar{t}_{\text{w}}\approx 0.352$ and $\bar{t}_{\text{q}}\approx 0.110$.

Comparing these data with the results of the preceding problem 11.45, we infer that the system considered in Problem 11.46 has insignificant advantages over that considered in Problem 11.45 as concerns all the efficiency characteristics and a little greater advantage over an elementary queueing system with the same λ and μ.

11.47. *An elementary system without a queue and with unlimited assistance between the servers.* Customers arrive at an n-server congestion system in a stationary Poisson process with intensity λ. The servers assist one another, i.e. if some servers are idle when a customer is being served, they all assist the busy server. The intensity of the process is a function $\mu=\varphi(k)$ of the number k of servers who simultaneously serve him. Construct the directed graph of states of the system and find its limiting probabilities of states. Express the following efficiency characteristics of the system in terms of these probabilities: the probability of refusal P_{ref}, the relative capacity for service Q and the average number of busy servers $\bar{k}$. Calculate these characteristics for $n=4$, $\lambda=1$, $\mu(k)=k\mu$, $\mu=0.5$ and compare them with the same characteristics in the case when the servers do not assist one another.

Solution. When the first customer arrives, all n servers begin serving him. This means that all the servers always function as a single server. The n-server system turns into a single-server system with refusals. Its states are: s_0, all the servers are free, s_n, all n servers are busy. The

marked directed graph of states is shown in Fig. 11.47. Using this graph, we get the limiting probabilities of states:

$$p_0 = \left\{1 + \frac{\lambda}{\varphi(n)}\right\}^{-1} = \frac{\varphi(n)}{\varphi(n)+\lambda}, \quad p_1 = \frac{\lambda}{\varphi(n)} p_0 = \frac{\lambda}{\varphi(n)+\lambda}.$$

For $\varphi(n) = n\mu$ we have $p_0 = (n\mu)/(n\mu + \lambda)$, $p_1 = \lambda/(n\mu + \lambda)$.

For $n = 4$, $\lambda = 1$, $\mu = 0.5$ we have $p_0 = 2/3$, $p_1 = 1/3$, $P_{\text{ref}} = p_1 = 1/3$, $Q = 1 - p_1 = 2/3$ and $A = \lambda Q = 2/3 \approx 0.667$.

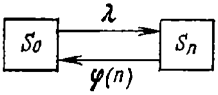


Fig. 11.47

The average number of busy servers $\bar{k} = 4 \cdot 1/3 + 0 \cdot 2/3 = 4/3 \approx 1.333$. We get the same result by dividing A by μ: $\bar{k} = (2/3)/0.5 = 4/3$.

For the sake of comparison we calculate the same efficiency characteristics for a four-server system without a mutual assistance between the servers [see Erlang's formulas (11.0.6) and formula (11.0.7) which follows from them] for $\rho = \lambda/\mu = 2$

$$p_0 = \{1 + \rho + \rho^2/2 + \rho^3/6 + \rho^4/24\}^{-1} = 1/7,$$

$$P_{\text{ref}} = p_4 = (2/3)(1/7) = 2/21, \quad A = \lambda(1 - 2/21) \approx 0.905,$$

$$Q = A, \quad \bar{k} = A/\mu \approx 1.81.$$

Comparing these characteristics with those obtained earlier for a system with assistance between the servers, we infer that in our conditions the assistance is not profitable. This is because for a system with refusals an unlimited assistance (when all the servers "fall upon" one customer while the customers who keep arriving are refused) is never profitable.

11.48. *An elementary system without a queue and with a uniform mutual assistance between the servers.* Customers arrive in a process with intensity λ at an elementary n-server system with refusals. The servers assist one another but not by combining their efforts to form a single-server system, as was in the preceding problem, but by the uniform assistance scheme. Thus if a customer arrives when all n servers are idle, all of them begin serving him; but if another customer arrives when the first customer is still being served, some of the servers begin serving the new arrival. If another customer arrives whilst the first two are being served, some of the servers start serving this new arrival, and so on till all n servers become busy. If they are all busy, the next customer to arrive is refused. The function $\varphi(k) = k\mu$, i.e. k servers serve k times quicker than one server.

Construct a marked directed graph of states of the system and find the limiting probabilities of states and the efficiency characteristics Q, A, $\bar{k}$ for $n = 4$, $\lambda = 1$, $\mu = 0.5$, i.e. on the hypothesis of Problem 11.47. Compare them with results for the absence of assistance*).

*) In this problem it is irrelevant how many servers begin serving a new arrival. It is only relevant that all n servers function all the time and that not a single arriving customer is refused until n customers have arrived in the system and all n servers are serving them individually.

Solution. The states of the system are enumerated in accordance with the number of customers present in the system. The directed graph of states is shown in Fig. 11.48. The graph is the same as that for an elementary one-server system with capacity $\mu^* = n\mu$ and a bounded queue which has $n - 1$ places. To find its characteristics, we use formulas (11.0.16)-(11.0.19). Setting $\rho = \rho^* = \lambda/\mu^* = \lambda/(n\mu) = 0.5$, we have for $m = 3$

$$p_0 = \frac{1-\rho^*}{1-(\rho^*)^5} = \frac{0.5}{1-0.5^5} \approx 0.514, \quad p_4 = (\rho^*)^4 p_0 \approx 0.032,$$

$$A = \lambda(1 - p_4) \approx 0.968, \quad Q = 1 - p_4 \approx 0.968, \quad \bar{k} = A/\mu^* \approx 1.936.$$

Under the same conditions (see Problem 11.47) with no assistance we have $A \approx 0.905$, $Q \approx 0.905$; $\bar{k} \approx 1.81$, i.e. a "uniform" assistance

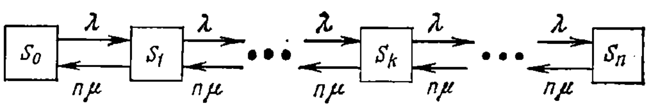


Fig. 11.48

increases somewhat the capacity of the system for service. In this case the increase is insignificant since the load on the system is not very large.

11.49. We are given an elementary three-server system with refusals and parameters $\lambda = 4$ cust/min. The average service time of a customer served by one server $1/\mu = 0.5$ min, the intensity of serving a customer by k servers $\varphi(k) = k\mu$. Find the efficiency characteristics Q, A, $\bar{k}$ for three variants: (1) no assistance; (2) unlimited assistance and (3) uniform assistance.

Answer. (1) $Q \approx 0.79$, $A \approx 3.16$, $\bar{k} \approx 1.58$, (2) $Q = 0.6$, $A = 2.4$, $\bar{k} = 1.2$, (3) $Q \approx 0.887$, $A \approx 3.51$, $\bar{k} \approx 1.76$.

11.50. *An elementary system with an unbounded queue and assistance between the servers.* Customers arrive at an elementary n-server queueing system in a process with intensity λ, the service time of a customer by one server being exponential with parameter μ. The intensity of the service process of a customer by k servers is proportional to k, i.e. $\varphi(k) = k\mu$. The servers are arbitrarily distributed over the customers in the system but if at least one customer is present in the system, then all n servers are busy serving him.

Numbering the states of the system in accordance with the number of customers in it, construct a marked directed graph of states, find the limiting probabilities of states and calculate the efficiency characteristics $\bar{k}$, $\bar{z}$, $\bar{r}$, $\bar{t}_w$, $\bar{t}_q$.

Solution. The directed graph of states of this system coincides with that of an elementary single-server system with an unbounded queue, with the intensity of the arrival process λ and service process $n\mu$ (see 11.0, item 2).

Setting $\rho = \varkappa = \lambda/(n\mu)$ in formulas (11.0.12)-(11.0.15) we get $p_0 = 1 - \varkappa$, $p_k = \varkappa^k(1-\varkappa)$ $(k = 1, 2, \ldots)$; $\bar{z} = \varkappa/(1-\varkappa)$, $\bar{r} = \varkappa^2/(1-\varkappa)$, $\bar{t}_w = \frac{\varkappa}{\lambda(1-\varkappa)}$ and $\bar{t}_q = \frac{\varkappa^2}{\lambda(1-\varkappa)}$.

In this case the efficiency characteristics of the system do not depend on whether all the servers serve one customer or whether their services are distributed uniformly since no customers are refused (note that in this case the mean values of the random variables Z, R, T_w, T_q do not vary but their distributions change).

11.51. *An elementary system with a bounded queue and uniform mutual assistance.* Customers arrive at an elementary queueing system with

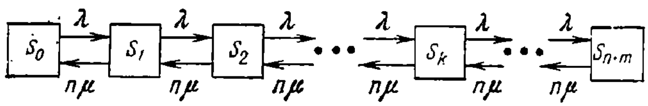


Fig. 11.51

n serves and uniform assistance in a stationary Poisson process with intensity λ. The service process rendered by one server is stationary Poisson's with intensity μ, while the total service process rendered by k servers has intensity $\varphi(k) = k\mu$. The servers are distributed over the customers uniformly in the sense that every new arrival gets service if there is a free server. A customer who arrives when all n servers are busy joins the queue. There are n places in the queue; if they are all occupied, the customer is refused.

Numbering the states of the system in accordance with the number of customers in it, construct a marked graph of states of the system and find the limiting probabilities of states. Find the efficiency characteristics of the system P_{ref}, Q, A, $\bar{z}$, $\bar{r}$, $\bar{t}_w$, $\bar{t}_q$.

Solution. The states of the system are:

s_0—the system is idle;

s_1—one customer is being served by all n servers; . . . ;

s_k—k customers are being served by all n servers $(1 < k < n)$; . . . ;

s_n—n customers are being served by n servers, there is no queue; . . . ;

s_{n+1}—n customers are being served by n servers and one customer is waiting to be served; . . . ;

s_{n+m}—n customers are being served by n servers, m customers are in the queue.

The directed graph of states of the system is shown in Fig. 11.51. This graph coincides with that of an elementary single-server system with m places in the queue, the intensity of the arrival process λ and the intensity of the service process $n\mu$ (see 11.0 item 3). Substituting $\varkappa = \lambda/(n\mu)$ for ρ and $n + m$ for m in formulas (11.0.16)-(11.0.20),

we obtain

$$p_0 = (1 - \varkappa)/(1 - \varkappa^{n+m+2}), \quad p_k = \varkappa^k p_0 \quad (k = 1, \ldots, n + m),$$

$$P_{\text{ref}} = p_{n+m}, \; Q = 1 - p_{n+m}, \quad A = \lambda Q,$$

$$\bar{r} = \frac{\varkappa^2 [1 - \varkappa^{n+m} (n+m+1-(n+m)\varkappa)]}{(1-\varkappa^{n+m+2})(1-\varkappa)},$$

$$\bar{z} = \bar{r} + \bar{k}, \quad \bar{t}_{\text{w}} = \bar{z}/\lambda, \quad \bar{t}_{\text{q}} = \bar{r}/\lambda.$$

11.52. The input of the automated data bank receives $\lambda = 335$ records/h on the average. The first operation when a primary documents arrive is to select the records which must be fed into the data bank. Six sorters select the records, the average productivity of each being $\mu = 60$ records/h. It is known that 61.3 per cent of the primary documents are selected, on the average, from the arrivals. All the processes are stationary Poisson's. Considering the selection system for the primary documents to be a six-server system ($n = 6$) with an unbounded queue, find the efficiency characteristics A, Q, $\bar{k}$, $\bar{z}$, $\bar{r}$, $\bar{t}_{\text{w}}$, $\bar{t}_{\text{q}}$.

Solution. Since the queue is unbounded, $Q = 1$ and $A = \lambda$. The intensity of the arrival process of primary documents to the data bank $\lambda_0 = \lambda p$, where $p \approx 0.613$, i.e. $\lambda_0 = 335 \times 0.613 \approx 205$ records/h.

The intensity of the arrival process of those documents which will not be fed into the data bank $\lambda_{\text{not}} = \lambda (1 - p) \approx 130$ records/h.

The average number of sorters busy with selecting the documents $\bar{k} = \lambda/\mu = \rho = 5.58$ and does not depend on the number of servers (sorters). There is a stationary situation in the system if the condition $\varkappa = \lambda/(n\mu) < 1$, is fulfilled; in our case it is fulfilled ($\varkappa = 0.93$).

The probability that all n sorters in the system will be busy [see formula (11.0.22)] is

$$P_n = P(n, \rho) / \left[R(n, \rho) + P(n, \rho) \frac{\varkappa}{1-\varkappa} \right].$$

Using the tables in Appendices 1 and 2, we obtain

$$p_6 = 0.1584 / \left(0.6703 + 0.1584 \frac{0.93}{0.07} \right) \approx 0.0569.$$

The average number of primary documents in the queue [see (11.0.23)] $\bar{r} = p_n \varkappa/(1 - \varkappa)^2 \approx 10.8$. The average queueing time of a document $\bar{t}_{\text{q}} = \bar{r}/\lambda \approx 1.87$ min. The average number of documents in the system (in the queue and being processed) $\bar{z} = r + \rho \approx 15.38$. The average waiting time of a document $\bar{t}_{\text{w}} = \bar{r}_{\text{q}} + 1/\mu \approx 2.87$ min.

11.53. Demands arrive at the input of a queueing system (Fig. 11.53) in a stationary Poisson process with intensity λ. The service consists of two successive phases performed in system 1 and system 2. A demand is served in system 1 and the quality of the service is checked in system 2. If no faults in the service are detected by system 2, then the demand is considered to have been served by the queueing system; if system 2 detects faults in the service of a demand, the demand is sent

back to system 1 for reservice (see Fig. 11.53). The probability that a demand served in system 1 will be returned to system 1 for reservice is $1 - p$ and does not depend on how many times it has been served in system 1.

Systems 1 and 2 are n_1-channel and n_2-channel systems with unfounded queues and service intensities in each channel of μ_1 and μ_2 respectively. The time needed for reservicing a demand in a channel in system 1 and for the retesting the quality of service by a channel of system 2

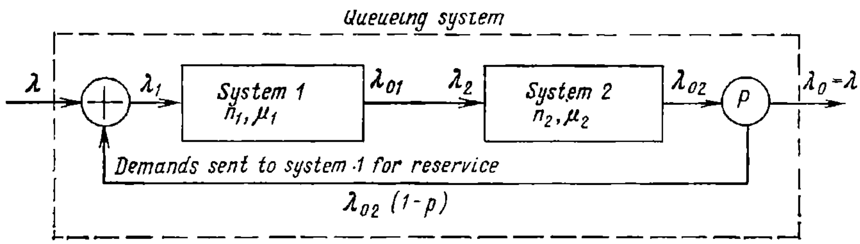


Fig. 11.53

have exponential distributions (the same as when these operations were carried out for the first time) with parameters μ_1 and μ_2 respectively. Find the conditions for a stationary situation in this queueing system, assuming that the demands arrive at system 1 and system 2 in stationary Poisson processes.

Solution. We denote the arrival intensity at the input of system 1 by λ_1. It is evident that $\lambda_1 > \lambda$ since the input of system 1 receives demands arriving at system 1 for the first time (the intensity of the process is λ) plus demands returned for reservicing (see Fig. 11.53). If a stationary situation exists, then the arrival intensity of demands served in system 1, λ_{01}, is equal to the intensity λ_1. The demands served by system 1 arrive in a flow at system 2 and, consequently, the input of system 2 receives demands arriving in a flow with intensity $\lambda_2 = \lambda_1 = \lambda_{01}$. Since there is a stationary situation in system 2, the intensity of the departure process of demands from system 2 (λ_{02}) is also λ_2. Thus

$$\lambda_1 = \lambda_{01} = \lambda_2 = \lambda_{02}. \qquad (11.53.1)$$

Evidently, the intensity λ_0 of the departure process of demands which have been served in a queueing system at steady state is equal to the intensity λ of the arrival process.

The intensity λ_0 with which the demands are served in the queueing system is equal to that at the output of system 2 (λ_{02}) multiplied by the probability p that a demand will not be returned to system 1 for a repeated service:

$$\lambda_{02} p = \lambda_0 = \lambda, \qquad (11.53.2)$$

whence we have

$$\lambda_{02} = \lambda/p. \tag{11.53.3}$$

Thus [see formulas (11.53.1)-(11.53.3)]

$$\lambda_1 = \lambda_2 = \lambda/p. \tag{11.53.4}$$

For the queueing system's operation to be steady state, it is necessary that both system 1 and system 2 should successfully handle all the arriving demands and, consequently, the following two conditions should be fulfilled:

$$\varkappa_1 = \lambda_1/(n_1\mu_1) = \lambda/(pn_1\mu_1) < 1, \tag{11.53.5}$$

$$\varkappa_2 = \lambda_2/(n_2\mu_2) = \lambda/(pn_2\mu_2) < 1. \tag{11.53.6}$$

They follow from the fact that system 1 and system 2 are n_1-channel and n_2-channel systems with channel service intensities μ_1 and μ_2 respectively and with an unbounded queue (see item 4 at the beginning of this chapter).

11.54. For the conditions of the preceding problem find the average waiting time of a demand and the average number of demands present in the system.

Solution. The average time a demand is present in system 1 for the first time $\bar{t}_1$ (see Fig. 11.53) can be found from the condition that demands arrive at the input of the system in a stationary Poisson process with intensity $\lambda_1 = \lambda/p$, the number of channels is n_1, the intensity of service is μ_1, and the number of places in the queue is unlimited. In accordance with (11.0.21)-(11.0.25) we have for these conditions

$$\bar{t}_1 = (\rho_1 + \bar{r}_1)\lambda_1^{-1} = \frac{1}{\mu_1} + \frac{\rho_1^{n_1+1} p_{01}}{n_1 \cdot n_1!\,(1-\varkappa_1)^2 \lambda_1}, \tag{11.54.1}$$

where

$$\rho_1 = \frac{\lambda}{p\mu_1}, \quad \varkappa_1 = \frac{\rho_1}{n_1}, \quad \lambda_1 = \frac{\lambda}{p},$$

$$p_{01} = \left[1 + \frac{\rho_1}{1!} + \frac{\rho_1^2}{2!} + \dots + \frac{\rho_1^{n_1}}{n_1!} + \frac{\rho_1^{n_1+1}}{n_1 \cdot n_1!}\,\frac{1}{1-\varkappa_1}\right]^{-1}.$$

By analogy we calculate the variable $\bar{t}_2$, which is the average time a demand is present in system 2 for the first time for the conditions $\lambda_2 = \lambda/p$, n_2 and μ_2:

$$\bar{t}_2 = (\rho_2 + \bar{r}_2)\lambda_2^{-1} = \frac{1}{\mu_2} + \frac{\rho_2^{n_2+1} p_{02}}{n_2 \cdot n_2!\,(1-\varkappa_2)^2 \lambda_2}, \tag{11.54.2}$$

where

$$\rho_2 = \frac{\lambda}{p\mu_2}, \quad \varkappa_2 = \frac{\rho_2}{n_2}, \quad \lambda_2 = \frac{\lambda}{p},$$

$$p_{02} = \left[1 + \frac{\rho_2}{1!} + \frac{\rho_2^2}{2!} + \dots + \frac{\rho_2^{n_2}}{n_2!} + \frac{\rho_2^{n_2+1}}{n_2 \cdot n_2!}\,\frac{1}{1-\varkappa}\right]^{-1}$$

Consequently, the average time needed for a once through service of a demand by system 1 and system 2

$$\bar{\tau}_{12} = \bar{t}_1 + \bar{t}_2. \tag{11.54.3}$$

It follows from problem 11.53 that a random variable X, defined as the number of times a demand is served by system 1 and system 2, has a geometric distribution beginning with unity with parameter p:

$$X: \left| \begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & \dots & k & \dots \\ \hline p & pq & pq^2 & \dots & pq^{k-1} & \dots \end{array} \right. . \tag{11.54.4}$$

We denote the time of the first, second, . . ., the kth cycle of processing a demand in system 1 and system 2 by $T_1, T_2, \dots, T_k, \dots$. By the hypothesis the random variables $T_1, \dots, T_k, \dots$ are mutually independent, have the same distribution and the mean value $\bar{\tau}_{12}$.

We can write an expression for the time a demand remains in the queueing system (with due regard for the possibility that a demand is returned for reservice) in the form $T = \sum_{k=1}^{X} T_k$, i.e. the sum of a random number of random terms, where the number of terms X does not depend on the random variables $T_1, T_2, T_3, \dots$. In accordance with the solution of Problem 7.64 we find the mean value of the variable T:

$$\bar{t} = \mathrm{M}[T] = \mathrm{M}[T_k]\,\mathrm{M}[X] = \bar{\tau}_{12}/p. \tag{11.54.5}$$

Little's formula is also valid for the queueing system we have considered, and, therefore, the average number of demands present in the system can be found from the formula

$$\bar{z} = \bar{t}\lambda. \tag{11.54.6}$$

11.55. The conditions of Problem 11.53 are changed so that in both system 1 and system 2 a demand is served for the first time and the quality of service is checked. If a demand does not pass the check in system 2, it is sent to system 3 for a reservice and to system 4 for rechecking (Fig. 11.55). System 3 and system 4 are n_3-channel and n_4-channel systems with exponential distributions of service time of demands in the channels and with parameters μ_3 and μ_4 respectively. The probability that a demand processed in system 3 will be returned to system 3 after the check in system 4 is $1 - \pi$. Find the conditions for steady-state operation of this queueing system, assuming that the demands arrive at system 1, system 2, system 3 and system 4 in processes which are stationary Poisson's.

Solution. When there is a stationary situation in the queueing system, the following relationship is obvious (see Fig. 11.55):

$$\lambda = \lambda_1 = \lambda_{01} = \lambda_2 = \lambda_{02}. \tag{11.55.1}$$

Consequently, the intensity of the departure process of demands which have been served for the first time and checked in systems 1 and 2 is λp.

The intensity of the process in which the demands that have not passed the check in system 2 arrive for reserving at system 3 and system 4 is $\lambda(1-p)$. The operation of the reservice system (system 3 and system 4) (from point 1 to point 2 in Fig. 11.55) does not differ in principle from the operation of the system considered in Problem 11.53.

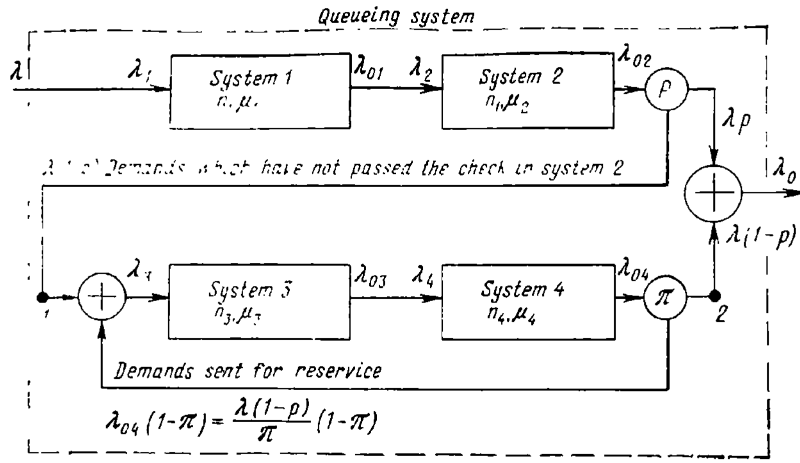


Fig. 11.55

Thus the following conditions must be fulfilled simultaneously for the stationary mode of operation of the system considered in this problem:

$$\varkappa_1=\frac{\lambda}{n_1\mu_1}<1,\quad \varkappa_2=\frac{\lambda}{n_2\mu_2}<1,\quad \varkappa_3=\frac{\lambda(1-p)}{\pi n_3\mu_3}<1,$$

$$\varkappa_4=\frac{\lambda(1-p)}{\pi n\mu_4}<1. \qquad (11.55.2)$$

In addition, the following equalities will hold true (see Fig. 11.55):

$$\lambda_3=\lambda_{03}=\lambda_4=\lambda_{04}=\lambda(1-p)\pi^{-1}. \qquad (11.55.3)$$

11.56. For the conditions of the preceding problem find the average waiting time $\bar{t}$ of a demand and the average number of demands $\bar{z}$ present in the system.

Solution. The average waiting time of a demand can be found from the condition that demands arrive at the input of the system in a stationary Poisson process with intensity $\lambda_1=\lambda$ [see Fig. 11.55 and formula (11.55.1)], the number of channels is n_1, the intensity of service in a channel is μ_1, and the number of places in the queue is unlimited. In accordance with (11.0.21)-(11.0.25), we have for these conditions

$$\bar{t}_1=(\rho_1+\bar{r}_1)\lambda^{-1}=\frac{1}{\mu_1}+\frac{\rho_1^{n_1+1}p_{01}}{n_1 n_1!(1-\varkappa_1)^2\lambda}. \qquad (11.56.1)$$

where

$$\rho_1 = \frac{\lambda}{\mu_1}, \quad \varkappa_1 = \frac{\rho_1}{n_1}.$$

$$p_{01} = \left[1 + \frac{\rho_1}{1!} + \frac{\rho_2^2}{2} + \ldots + \frac{\rho_1^{n_1}}{n_1!} + \frac{\rho_1^{n_1+1}}{n_1 n_1!} \frac{1}{1-\varkappa_1}\right]^{-1}.$$

By analogy we calculate the variable $\bar{t}_2$ for system 2 at $\lambda_2 = \lambda$, n_2 and μ_2, i.e.

$$\bar{t}_2 = (\rho_2 + \bar{r}_2)\lambda^{-1} = \frac{1}{\mu_2} + \frac{\rho_2^{n_2+1} p_{02}}{n_2 n_2! (1-\varkappa_2)^2 \lambda}, \tag{11.56.2}$$

where

$$\rho_2 = \frac{\lambda}{\mu_2}, \quad \varkappa_2 = \frac{\rho_2}{n_2},$$

$$p_{02} = \left[1 + \frac{\rho_2}{1!} + \frac{\rho_2^2}{2!} + \ldots + \frac{\rho_2^{n_2}}{n_2!} + \frac{\rho_2^{n_2+1}}{n_2 n_2!} \frac{1}{1-\varkappa_2}\right]^{-1}.$$

The average waiting time of a demand in system 1 and system 2

$$\bar{\tau}_{12} = \bar{t}_1 + \bar{t}_2. \tag{11.56.3}$$

The average time $\bar{t}_3$ a demand is present in system 3 for a once through service can be found by assuming that demands arrive at the input of this system in a stationary Poisson process with intensity $\lambda_3 = \lambda(1-p)/\pi$ [see Fig. 11.55 and formula (11.55.3)], the number of channels is n_3, the intensity of service in a channel is μ_3, and the number of places in the queue is unlimited. Then, in accordance with (11.0.21)-(11.0.25), we have

$$\bar{t}_3 = (\rho_3 + \bar{r}_3)\lambda_3^{-1} = 1/\mu_3 + \rho_3^{n_3+1} p_{03}/[n_3 n_3! (1-\varkappa_3)^2 \lambda_3], \tag{11.56.4}$$

where

$$\rho_3 = \frac{\lambda_3}{\mu_3} = \frac{\lambda(1-p)}{\pi\mu_3}, \quad \varkappa_3 = \frac{\rho_3}{n_3}$$

$$p_{03} = \left[1 + \frac{\rho_3}{1!} + \frac{\rho_3^2}{2!} + \ldots + \frac{\rho^{n_3}}{n_3!} + \frac{\rho_3^{n_3+1}}{n_3 n_3!} \frac{1}{1-\varkappa_3}\right]^{-1}.$$

By analogy we calculate the variable $\bar{t}_4$ for system 4 at $\lambda_4 = \lambda(1-p)/\pi$, n_4, μ_4;

$$\bar{t}_4 = (\rho_4 + \bar{r}_4)\lambda_4^{-1} = 1/\mu_4 + \rho_4^{n_4+1} p_{04}/[n_4 n_4! (1-\varkappa_4)^2 \lambda_4], \tag{11.56.5}$$

where

$$\rho_4 = \frac{\lambda_4}{\mu_4} = \frac{\lambda(1-p)}{\pi\mu_3}, \quad \varkappa_4 = \frac{\rho_4}{n_4},$$

$$p_{04} = \left[1 + \frac{\rho_4}{1!} + \frac{\rho_4^2}{2!} + \ldots + \frac{\rho_4^{n_4}}{n_4!} + \frac{\rho_4^{n_4+1}}{n_4 n_4!} \frac{1}{1-\varkappa_4}\right]^{-1}.$$

Consequently, the average time per demand processed once in system 3 and system 4

$$\bar{\tau}_{34}^{(1)} = \bar{t}_3 + \bar{t}_4. \tag{11.56.6}$$

In accordance with formula (11.54.6), the expectation of the waiting time of a demand in system 3 and system 4 (see Fig. 11.56), with due regard for possible reprocessing, is

$$\bar{\tau}_{34} = \bar{\tau}_{34}^{(1)}/\pi. \tag{11.56.7}$$

To find the average waiting time of a demand in the queueing system, we consider two hypotheses: (1) H_1, the demand was only processed once; $\mathrm{P}(H_1) = p$ and (2) H_2, the demand was processed many times, $\mathrm{P}(H_2) = 1 - p$.

On the hypothesis H_1 the expectation of the waiting time of a demand in the queueing system can be found from formula (11.56.3), on the hypothesis H_2 the expectation can be found from formulas (11.56.7) and (11.56.3). The complete expectation of the waiting time of a demand in the system

$$\bar{t} = \bar{\tau}_{12} + \bar{\tau}_{34}^{(1)}(1 - p)/\pi. \tag{11.56.8}$$

The average number of demands present in the system can be found by Little's formula:

$$\bar{z} = \bar{t}\lambda. \tag{11.56.9}$$

11.57. Let us consider the formulation of Problem 11.53, as applied to the multi-fold processing of information in two phases. To check that an input card has been correctly punched, the punching is performed twice, on a punch unit and on a verifier, which can detect an error. The punch-unit operator makes one error per 1000 symbols on the average. Each punch-card contains an average of 80 symbols. When at least one error is detected on a punch-card, the card is returned for a repunching. It is necessary to process 50 000 documents in a year, each of which contains an average of 400 symbols.

Find the minimum possible number of punch-unit and verifier operators needed if one punch-unit operator punches 4.2×10^6 symbols a year, and find the characteristics of the system.

Solution. It follows from the conditions of the problem that an average of $50\,000 \times 400/80 = 250\,000$ cards must be punched and verified per year. Consequently, $\lambda = 250\,000$ cards/year $= 125$ cards/h $= 2.083$ cards/min. The productivity of a punch-unit or verifier operator is $\mu_1 = \mu_2 = 4.2 \times 10^6/80 = 52 \times 5000$ cards/year $= 26.25$ cards/h $= 0.438$ cards/min. The probability that no errors will be found on a punch-card is $p = 0.999^{80} \approx 0.9$. In this case we neglect the probability that both the punch-unit and the verifier operators will err at the same symbol. This probability is much smaller than $(0.001)^2 = 10^{-6}$ since the keyboards of the punch unit and the verifier has about 50 symbols. If we assume that an error can be introduced by pressing any one

of the 50 keys, then we find that the probability of the same error is $10^{-6}/2500$. If we assume that an error can only be introduced by pressing a key neighbouring the correct symbol (of which there are from five to eight), then the probability is $10^{-6}/25$-$10^{-6}/64$.

For stationary operation of the punch-unit operators [see (11.53.5), (11.53.6)]

$$\lambda/(n_1\mu_1 p) = \varkappa_1 < 1, \quad \lambda/(n_2\mu_2 p) = \varkappa_2 < 1,$$

whence $n_1 > \lambda/(\mu_1 p) = 250\,000/(52 \times 500 \times 0.9) = 5.29$ and $n_2 > 5.29$.

Thus we must have six punch-unit operators and six verifier operators.

The characteristics of the operation for the punch-unit operators: $\lambda_{\text{in }1} = \lambda/p = 250\,000/0.9 = 278\,000$ cards/year, $n_1 = 6$, $\mu_1 = 52 \times 500$ cards/year, $\varkappa_1 = \lambda/(n_1 p\mu) = 0.882$ and $\rho_1 = \lambda/(p\mu_1) = 5{,}29$.

We use formula (11.0.22) to find the probability that all the six punch-unit operators are busy and there is no queue of documents which must be punched:

$$p_{n_1} = \frac{\mathrm{P}(n_1, \rho_1)}{R(n_1, \rho_1) + \mathrm{P}(n_1, \rho_1)\dfrac{\varkappa_1}{1-\varkappa_1}} = \frac{0.152}{0.282 + 0.152\dfrac{0.882}{1-0.882}} = 0.107.$$

The average number of cards waiting to be punched $\bar{r}_1 = p_{n_1}\varkappa_1/(1-\varkappa_1)^2 = 6.78$.

The average queueing time of a card $\bar{t}_q = \bar{r}_1 p\lambda^{-1} = 2{,}92$ min and $\bar{t}_1 = \bar{t}_{q\,1} + 1/\mu_1 = 5.21$ min.

The total number of cards in the first phase $\bar{z}_1 = \bar{r}_1 + \bar{k}_1 = \bar{r}_1 + \rho_1 = 12.07$.

Since the characteristics of the second phase (verification) are the same as those of the first phase, we can write the general characteristics for the operation of the system with due regard for the possibility that cards may be returned for repunching. The overall average number of cards present in the system $\bar{z} = 2\bar{z}_1 = 24.14$ and the number of these cards in the queue $\bar{r} = 2\bar{r}_1 = 13.56$. The overall average processing time per card, with due account of the possible return of a card for reprocessing, $\bar{t} = \bar{\tau}_{12}/p = (\bar{t}_1 + \bar{t}_2)/p = 2\bar{t}_1/p = 11.57$ min.

Appendices

APPENDIX 1

Poisson's Distribution $P(m, a) = \frac{a^m}{m!} e^{-a}$

m \ a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
0	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	
1	0.0905	0.1638	0.2222	0.2681	0.3033	0.3293	0.3476	0.3595	0.3659	
2	0.0045	0.0164	0.0333	0.0536	0.0758	0.0988	0.1217	0.1438	0.1647	
3	0.0002	0.0019	0.0033	0.0072	0.0126	0.0198	0.0284	0.0383	0.0494	
4		0.0001	0.0002	0.0007	0.0016	0.0030	0.0050	0.0077	0.0111	
5				0.0001	0.0002	0.0004	0.0007	0.0012	0.0020	
6							0.0001	0.0002	0.0003	

m \ a	1	2	3	4	5	6	7	8	9	10
0	0.3679	0.1353	0.0498	0.0183	0.0067	0.0025	0.0009	0.0003	0.0001	0.0000
1	0.3679	0.2707	0.1494	0.0733	0.0337	0.0149	0.0064	0.0027	0.0011	0.0005
2	0.1839	0.2707	0.2240	0.1465	0.0842	0.0446	0.0223	0.0107	0.0050	0.0023
3	0.0613	0.1804	0.2240	0.1954	0.1404	0.0892	0.0521	0.0286	0.0150	0.0076
4	0.0153	0.0902	0.1680	0.1954	0.1755	0.1339	0.0912	0.0572	0.0337	0.0189
5	0.0031	0.0361	0.1008	0.1563	0.1755	0.1606	0.1277	0.0916	0.0607	0.0378
6	0.0005	0.0120	0.0504	0.1042	0.1462	0.1606	0.1490	0.1221	0.0911	0.0631
7	0.0001	0.0037	0.0216	0.0595	0.1044	0.1377	0.1490	0.1396	0.1171	0.0901
8		0.0009	0.0081	0.0298	0.0653	0.1033	0.1304	0.1396	0.1318	0.1126
9		0.0002	0.0027	0.0132	0.0363	0.0688	0.1014	0.1241	0.1318	0.1251
10			0.0008	0.0053	0.0181	0.0413	0.0710	0.0993	0.1186	0.1251
11			0.0002	0.0019	0.0082	0.0225	0.0452	0.0722	0.0970	0.1137
12			0.0001	0.0006	0.0034	0.0126	0.0263	0.0481	0.0728	0.0948
13				0.0002	0.0013	0.0052	0.0142	0.0296	0.0504	0.0729
14				0.0001	0.0005	0.0022	0.0071	0.0169	0.0324	0.0521
15					0.0002	0.0009	0.0033	0.0090	0.0194	0.0347
16						0.0003	0.0014	0.0045	0.0109	0.0217
17						0.0001	0.0006	0.0021	0.0058	0.0128
18							0.0002	0.0009	0.0029	0.0071
19							0.0001	0.0004	0.0014	0.2237
20								0.0002	0.0006	0.0019
21								0.0001	0.0003	0.0009
22									0.0001	0.0004
23										0.0002
24										0.0001

APPENDIX 2

$$\textit{Probabilities}^{*)}\ \overline{R}(m, a) = 1 - R(m, a) = 1 - \sum_{k=0}^{m} \frac{a^k}{k!} e^{-a}$$

m	$a = 0.1$	$a = 0.2$	$a = 0.3$	$a = 0.4$	$a = 0.5$
0	9.5163^{-2}	1.8127^{-1}	2.5918^{-1}	3.2968^{-1}	8.9347^{-1}
1	4.6788^{-3}	1.7523^{-2}	3.6936^{-2}	6.1552^{-2}	9.0204^{-2}
2	1.5465^{-4}	1.1485^{-3}	3.5995^{-3}	7.9263^{-3}	1.4388
3	3.8468^{-6}	5.6840^{-5}	2.6581^{-4}	7.7625^{-4}	1.7516^{-3}
4		2.2592^{-6}	1.5785^{-5}	6.1243^{-5}	1.7212^{-4}
5				4.0427^{-6}	1.4165^{-5}
6					1.0024^{-6}

m	$a = 0.6$	$a = 0.7$	$a = 0.8$	$a = 0.9$	
0	4.5119^{-1}	5.0341^{-1}	5.5067^{-1}	5.9343^{-1}	
1	1.2190	1.5580	1.9121	2.2752	
2	2.3115^{-2}	3.4142^{-2}	4.7423^{-2}	6.2857^{-2}	
3	3.3581^{-3}	5.7535^{-3}	9.0799^{-3}	1.3459	
4	3.9449^{-4}	7.8554^{-4}	1.4113	2.3441^{-3}	
5	3.8856^{-5}	9.0026^{-5}	1.8434^{-4}	3.4349^{-4}	
6	3.2931^{-6}	8.8836^{-6}	2.0747^{-5}	4.3401^{-5}	
7			2.0502^{-6}	4.8172^{-6}	

m	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$
0	6.3212^{-1}	8.6466^{-1}	9.5021^{-1}	9.8168^{-1}	9.9326^{-1}
1	2.6424	5.9399	8.0085	9.0842	9.5957
2	8.0301^{-2}	3.2332	5.7681	7.6190	8.7535
3	1.8988	1.4288	3.5277	5.6653	7.3497
4	3.6598^{-3}	5.2653^{-2}	1.8474	3.7116	5.5951
5	5.9418^{-4}	1.6564	8.3918^{-2}	2.1487	3.8404
6	8.3241^{-5}	4.5338^{-3}	3.3509	1.1067	2.3782
7	1.0219	1.0967	1.1905	5.1134^{-2}	1.3337
8	1.1252^{-6}	2.3745^{-4}	3.8030^{-3}	2.1363	6.8094^{-2}

*) $P(m, a) = \frac{a^m}{m!} e^{-a}$ can be found in terms of $\overline{R}(m, a)$ as follows: $P(m, a) = \overline{R}(m-1, a) - \overline{R}(m, a)$ $(m > 0)$, $P(0, a) = 1 - \overline{R}(0, a)$.

Continued

m	$a=1$	$a=2$	$a=3$	$a=4$	$a=5$
9		4.6498^{-5}	1.1025	8.1322^{-3}	3.1828
10		8.3082^{-6}	2.9234^{-4}	2.8398	1.3695
11		1.3646	7.1387^{-5}	9.1523^{-4}	5.4531^{-3}
12			1.6149	2.7372	2.0189
13			3.4019^{-6}	7.6328^{-5}	6.9799^{-4}
14				1.9932	2.2625
15				4.8926^{-6}	6.9008^{-5}
16				1.1328	1.9869
17					5.4163^{-6}
18					1.4017

m	$a=6$	$a=7$	$a=8$	$a=9$	$a=10$
0	9.9752^{-1}	9.9909^{-1}	9.9966^{-1}	9.9988^{-1}	9.9995^{-1}
1	9.8265	9.9270	9.9698	9.9877	9.9950
2	9.3803	9.7036	9.8625	9.9377	9.9723
3	8.4880	9.1823	9.5762	9.7877	9.8966
4	7.1494	8.2701	9.0037	9.4504	9.7075
5	5.5432	6.9929	8.0376	8.8431	9.3291
6	3.9370	5.5029	6.8663	7.9322	8.6986
7	2.5602	4.0129	5.4704	6.7610	7.7978
8	1.5276	2.7091	4.0745	5.4435	6.6718
9	8.3924^{-2}	1.6950^{-1}	2.8338^{-1}	4.1259^{-1}	5.4207^{-1}
10	4.2621	9.8521^{-2}	1.8411	2.9401	4.1696
11	2.0092	5.3350	1.1192	1.9699	3.0322
12	8.8275^{-3}	2.7000	6.3797^{-2}	1.2423	2.0344
13	3.6285	1.2811	3.4181	7.3851^{-2}	1.3554
14	1.4004	5.7172^{-3}	1.7257	4.1466	8.3458^{-2}
15	5.0910^{-4}	2.4066	8.2310^{-3}	2.2036	4.8740
16	1.7488	9.5818^{-4}	3.7180	1.1106	2.7042
17	5.6917^{-5}	3.6178	1.5943	5.3196^{-3}	1.4278
18	1.7597	1.2985	6.5037^{-4}	2.4264	7.1865^{-3}

m	$a=6$	$a=7$	$a=8$	$a=9$	$a=10$
19	5.1802^{-5}	4.4402^{-5}	2.5294	1.0560	3.4543
20	1.4551	1.4495	9.3968^{-5}	4.3925^{-4}	1.5883
21		4.5263^{-6}	3.3407	1.7495	6.9965^{-4}
22		1.3543	1.1385	6.6328^{-5}	2.9574
23			3.7255^{-6}	2.4519	1.2012
24			1.1722	8.6531^{-6}	4.6949^{-5}
25				2.9414	1.7680
26					6.4229^{-6}
27					2.2535

Example. We must find the probability that an event A will occur no more than twice if $a = 7$.

We have

$R(2, 7) = 1 - \bar{R}(2, 7) = 1 - 9.7036^{-1} = 1 - 0.97036 = 0.02964.$

Remark. 1. If a number in the table has no exponent, then the exponent of the previous number in the column is the exponent. For example, $\bar{R}(33, 19) = 1.2067 \times 10^{-3}$.

2. For $a > 20$ the probability $R(m, a)$ can be approximated by

$$R(m, a) \approx \Phi\left(\frac{m+0.5-a}{\sqrt{a}}\right)+0.5,$$

where $\Phi(x)$ is the error function (Appendix 5).

m	$a=11$	$a=12$	$a=13$	$a=14$	$a=15$
0	9.9998^{-1}	9.9999^{-1}			
1	9.9980	9.9992	9.9997^{-1}	9.9999^{-1}	
2	9.9879	9.9948	9.9978	9.9991	9.9996^{-1}
3	9.9508	9.9771	9.9895	9.9953	9.9979
4	9.8490	9.9240	9.9626	9.9819	9.9914
5	9.6248	9.7966	9.8927	9.9447	9.9721
6	9.2139	9.5418	9.7411	9.8577	9.9237
7	8.5681	9.1050	9.4597	9.6838	9.8200
8	7.6801	8.4497	9.0024	9.3794	9.6255
9	6.5949	7.5761	8.3419	8.9060	9.3015
10	5.4011	6.5277	7.4832	8.2432	8.8154
11	4.2073	5.3840	6.4684	7.3996	8.1525
12	3.1130	4.2403	5.3690	6.4154	7.3239
13	2.1871	3.1846	4.2696	5.3555	6.3678
14	1.4596	2.2798	3.2487	4.2956	5.3435
15	9.2604^{-2}	1.5558	2.3639	3.3064	4.3191
16	5.5924	1.0129	1.6451	2.4408	3.3588
17	3.2191	6.2966^{-2}	1.0954	1.7280	2.5114
18	1.7687	3.7416	6.9833^{-2}	1.1736	1.8053
19	9.2895^{-3}	2.1280	4.2669	7.6505^{-2}	1.2478
20	4.6711	1.1598	2.5012	4.7908	8.2972^{-3}
21	2.2519	6.0651^{-3}	1.4081	2.8844	5.3106
22	1.0423	3.0474	7.6225^{-3}	1.6712	3.2744
23	4.6386^{-4}	1.4729	3.9718	9.3276^{-3}	1.9465
24	1.9871	6.8563^{-4}	1.9943	5.0199	1.1165
25	8.2050^{-5}	3.0776	9.6603^{-4}	2.6076	6.1849^{-3}
26	3.2693	1.3335	4.5190	1.3087	3.3119
27	1.2584	5.5836^{-5}	2.0435	6.3513^{-4}	1.7158
28	4.6847^{-6}	2.2616	8.9416^{-5}	2.9837	8.6072^{-4}

Continued

m	$a = 11$	$a = 12$	$a = 13$	$a = 14$	$a = 15$
29	1.6882	8.8701^{-6}	3.7894	1.3580	4.1843
30		3.3716	1.5568	5.9928^{-5}	1.9731
31		1.2432	6.2052^{-6}	2.5665	9.0312^{-5}
32			2.4017	1.0675	4.0155
33				4.3154^{-6}	1.7356
34				1.6968	7.2978^{-6}
35					2.9871
36					1.1910

m	$a = 16$	$a = 17$	$a = 18$	$a = 19$	$a = 20$
0					
1					
2	9.9998^{-1}	9.9999^{-1}			
3	9.9991	9.9996	9.9998^{-1}	9.9999^{-1}	
4	9.9960	9.9982	9.9992	9.9996	9.9998^{-1}
5	9.9862	9.9933	9.9968	9.9985	9.9993
6	9.9599	9.9794	9.9896	9.9948	9.9974
7	9.9000	9.9457	9.9711	9.9849	9.9922
8	9.7801	9.8740	9.9294	9.9613	9.9791
9	9.5670	9.7388	9.8462	9.9114	9.9500
10	9.2260	9.5088	9.6963	9.8168	9.8919
11	8.7301	9.1533	9.4511	9.6533	9.7861
12	8.0688	8.6498	9.0833	9.3944	9.6099
13	7.2545	7.9913	8.5740	9.0160	9.3387
14	6.3247	7.1917	7.9192	8.5025	8.9514
15	5.3326	6.2855	7.1335	7.8521	8.4349
16	4.3404	5.3226	6.2495	7.0797	7.7893
17	3.4066	4.3598	5.3135	6.2164	7.0297
18	2.5765	3.4504	4.3776	5.3052	6.1858
19	1.8775	2.6368	3.4908	4.3939	5.2974
20	1.3183	1.9452	2.6928	3.5283	4.4091
21	8.9227^{-2}	1.3853	2.0088	2.7450	3.5630
22	5.8241	9.5272^{-2}	1.4491	2.0687	2.7939
23	3.6686	6.3296	1.0111	1.5098	2.1251
24	2.2315	4.0646	6.8260^{-2}	1.0675	1.5677
25	1.3119	2.5245	4.4608	7.3126^{-2}	1.1218
26	7.4589^{-3}	1.5174	2.8234	4.8557	7.7887^{-2}
27	4.1051	8.8335^{-3}	1.7318	3.1268	5.2481
28	2.1886	4.9838	1.0300	1.9536	3.4334

Continued

m	$a = 16$	$a = 17$	$a = 18$	$a = 19$	$a = 20$
29	1.1312	2.7272	5.9443^{-3}	1.1850	2.1818
30	5.6726^{-4}	1.4484	3.3308	6.9819^{-3}	1.3475
31	2.7620	7.4708^{-4}	1.8133	3.9982	8.0918^{-3}
32	1.3067	3.7453	9.5975^{-4}	2.2267	4.7274
33	6.0108^{-5}	1.8260	4.9416	1.2067	2.6884
34	2.6903	8.6644^{-5}	2.4767	6.3674^{-4}	1.4890
35	1.1724	4.0035	1.2090	3.2732	8.0366^{-4}
36	4.9772^{-6}	1.8025	5.7519^{-5}	1.6401	4.2290
37	2.0599	7.9123^{-6}	2.6684	8.0154^{-5}	2.1708
38		3.3882	1.2078	3.8224	1.0875
39		1.4162	5.3365^{-6}	1.7797	5.3202^{-5}
40			2.3030	8.0940^{-6}	2.5426
41				3.5975	1.1877[illegible]
42				1.5634	5.4252^{-6}
43					2.4243
44					1.0603

APPENDIX 3

The Values of the Function e^{-x}

x	e^{-x}	Δ	x	e^{-x}	Δ	x	e^{-x}	Δ	x	e^{-x}	Δ
0.00	1.000	10	0.40	0.670	7	0.80	0.449	4	3.00	0.050	5
0.01	0.990	10	0.41	0.664	7	0.81	0.445	5	3.10	0.045	4
02	980	10	42	657	7	82	440	4	3.20	41	4
03	970	9	43	650	6	83	436	4	3.30	37	4
04	961	10	44	644	6	84	432	5	3.40	33	3
05	951	9	45	638	7	85	427	4	3.50	30	3
06	942	10	46	631	6	86	423	4	3.60	27	2
07	932	9	47	625	6	87	419	4	3.70	25	3
08	923	9	48	619	6	88	415	4	3.80	22	2
09	914	9	49	613	7	89	411	4	3.90	20	2
0.10	0.905	9	0.50	0.606	6	0.90	0.407	4	4.00	0.0183	17
11	896	9	51	600	5	91	403	4	4.10	166	16
12	887	9	52	595	6	92	399	4	4.20	150	14
13	878	9	53	589	6	93	395	4	4.30	136	13
14	869	8	54	583	6	94	391	4	4.40	123	12
15	861	9	55	577	6	95	387	4	4.50	111	10
16	852	8	56	571	6	96	383	4	4.60	101	10
17	844	9	57	565	5	97	379	4	4.70	0.0091	9
18	835	8	58	560	6	98	375	3	4.80	82	8
19	827	8	59	554	5	99	372	4	4.90	74	7
0.20	0.819	8	0.60	0.549	6	1.00	0.368	35	5.00	0.0067	6
21	811	8	61	543	5	1.10	333	31	5.10	61	6
22	803	8	62	538	5	1.20	302	29	5.20	55	5
23	795	8	63	533	6	1.30	273	26	5.30	50	5
24	787	8	64	527	5	1.40	247	24	5.40	45	4
25	779	8	65	522	5	1.50	223	21	5.50	41	4
26	771	8	66	517	5	1.60	202	19	5.60	37	4
27	763	7	67	512	5	1.70	183	18	5.70	33	3
28	756	8	68	507	5	1.80	165	15	5.80	30	3
29	748	7	69	502	5	1.90	150	15	5.90	27	2
0.30	0.741	8	0.70	0.497	5	2.00	0.135	13	6.00	0.0025	3
31	733	7	71	492	5	2.10	122	11	6.10	22	2
32	726	7	72	487	5	2.20	111	11	6.20	20	2
33	719	7	73	482	5	2.30	100	9	6.30	18	1
34	712	7	74	477	5	2.40	0.091	9	6.40	17	2
35	705	7	75	472	4	2.50	82	8	6.50	15	1
36	698	7	76	468	5	2.60	74	7	6.60	14	2
37	691	7	77	463	5	2.70	67	6	6.70	12	1
38	684	7	78	458	4	2.80	61	6	6.80	11	1
39	677	7	79	454	5	2.90	55	5	6.90	10	1
0.40	0.670		0.80	0.449		3.00	0.050		7.00	0.0009	

APPENDIX 4

The Density of the Normal Distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.3989	3989	3989	3988	3986	3984	3982	3980	3977	3973
0.1	3970	3965	3961	3956	3951	3945	3939	3932	3925	3918
0.2	3910	3902	3894	3885	3876	3867	3857	3847	3836	3825
0.3	3814	3802	3790	3778	3765	3752	3739	3726	3712	3697
0.4	3683	3668	3653	3637	3621	3605	3589	3572	3555	3538
0.5	3521	3503	3485	3467	3448	3429	3410	3391	3372	3352
0.6	3332	3312	3292	3271	3251	3230	3209	3187	3166	3144
0.7	3123	3101	3079	3056	3034	3011	2989	2966	2943	2920
0.8	2897	2874	2850	2827	2803	2780	2756	2732	2709	2685
0.9	2661	2637	2613	2589	2565	2541	2516	2492	2468	2444
1.0	0.2420	2396	2371	2347	2323	2299	2275	2251	2227	2203
1.1	2179	2155	2131	2107	2083	2059	2036	2012	1989	1965
1.2	1942	1919	1895	1872	1849	1826	1804	1781	1758	1736
1.3	1714	1691	1669	1647	1626	1604	1582	1561	1539	1518
1.4	1497	1476	1456	1435	1415	1394	1374	1354	1334	1315
1.5	1295	1276	1257	1238	1219	1200	1182	1163	1145	1127
1.6	1109	1092	1074	1057	1040	1023	1006	0989	0973	0957
1.7	0940	0925	0909	0893	0878	0863	0848	0833	0818	0804
1.8	0790	0775	0761	0748	0734	0721	0707	0694	0681	0669
1.9	0656	0644	0632	0620	0608	0596	0584	0573	0562	0551
2.0	0.0540	0529	0519	0508	0498	0488	0478	0468	0459	0449
2.1	0440	0431	0422	0413	0404	0396	0387	0379	0371	0363
2.2	0355	0347	0339	0332	0325	0317	0310	0303	0297	0290
2.3	0283	0277	0270	0264	0258	0252	0246	0241	0235	0229
2.4	0224	0219	0213	0208	0203	0198	0194	0189	0184	0180
2.5	0175	0171	0167	0163	0158	0154	0151	0147	0143	0139
2.6	0136	0132	0129	0126	0122	0119	0116	0113	0110	0107
2.7	0104	0101	0099	0096	0093	0091	0088	0086	0084	0081
2.8	0079	0077	0075	0073	0071	0069	0067	0065	0063	0061
2.9	0060	0058	0056	0055	0053	0051	0050	0048	0047	0046
3.0	0.0044	0043	0042	0040	0039	0038	0037	0036	0035	0034
3.1	0033	0032	0031	0030	0029	0028	0027	0026	0025	0025
3.2	0024	0023	0022	0022	0021	0020	0020	0019	0018	0018
3.3	0017	0017	0016	0016	0015	0015	0014	0014	0013	0013
3.4	0012	0012	0012	0011	0011	0010	0010	0010	0009	0009
3.5	0009	0008	0008	0008	0008	0007	0007	0007	0007	0006
3.6	0006	0006	0006	0005	0005	0005	0005	0005	0005	0004
3.7	0004	0004	0004	0004	0004	0004	0003	0003	0003	0003
3.8	0003	0003	0003	0003	0003	0002	0002	0002	0002	0002
3.9	0002	0002	0002	0002	0002	0002	0002	0002	0001	0001

APPENDIX 5

The Values of the Error Function $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{z^2}{2}}\,dz$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	00399	00798	01197	01595	01994	02392	02790	03188	03586
0.1	03983	04380	04776	05172	05567	05962	06356	06749	07142	07535
0.2	07926	08317	08706	09095	09483	09871	10257	10642	11026	11409
0.3	11791	12172	12552	12930	13307	13683	14058	14431	14803	15173
0.4	15542	15910	16276	16640	17003	17364	17724	18082	18439	18793
0.5	19146	19497	19847	20194	20540	20884	21226	21566	21904	22240
0.6	22575	22907	23237	23565	23891	24215	24537	24857	25175	25490
0.7	25804	26115	26424	26730	27035	27337	27637	27935	28230	28524
0.8	28814	29103	29389	29673	29955	30234	30511	30785	31057	31327
0.9	31594	31859	32121	32381	32639	32894	33147	33398	33646	33891
1.0	34134	34375	34614	34850	35083	35314	35543	35769	35993	36214
1.1	36433	36650	36864	37076	37286	37493	37698	37900	38100	38298
1.2	38493	38686	38877	39065	39251	39435	39617	39796	39973	40147
1.3	40320	40490	40658	40824	40988	41149	41309	41466	41621	41774
1.4	41924	42073	42220	42364	42507	42647	42786	42922	43056	43189
1.5	43319	43448	43574	43699	43822	43943	44062	44179	44295	44408
1.6	44520	44630	44738	44845	44950	45053	45154	45254	45352	45449
1.7	45543	45637	45728	45818	45907	45994	46080	46164	46246	46327
1.8	46407	46485	46562	46638	46712	46784	46856	46926	46995	47062
1.9	47128	47193	47257	47320	47381	47441	47500	47558	47615	47670
2.0	47725	47778	47831	47882	47932	47982	48030	48077	48124	48169
2.1	48214	48257	48300	48341	48382	48422	48461	48500	48537	48574
2.2	48610	48645	48679	48713	48745	48778	48809	48840	48870	48899
2.3	48928	48956	48983	49010	49036	49061	49086	49111	49134	49158
2.4	49180	49202	49224	49245	49266	49286	49305	49324	49343	49361
2.5	49379	49396	49413	49430	49446	49461	49477	49492	49506	49520
2.6	49534	49547	49560	49573	49585	49598	49609	49621	49632	49643
2.7	49653	49664	49674	49683	49693	49702	49711	49720	49728	49736
2.8	49744	49752	49760	49767	49774	49781	49788	49795	49801	49807
2.9	49813	49819	49825	49831	49836	49841	49846	49851	49856	49861

3.0	0.49865	3.1	49903	3.2	49931	3.3	49952	3.4	49966
3.5	49977	3.6	49984	3.7	49989	3.8	49993	3.9	49995
4.0	499968								
4.5	499997								
5.0	49999997								

Appendix 6

Some of the Properties of Generalized Functions

When solving problems dealing with random functions, it is often convenient to use transformations involving various step functions as well as generalized functions like the delta function.

Here are the definitions and main properties of these functions of a real argument τ.

1. $|\tau|$, a modulus (absolute value):

$$|\tau| = \begin{cases} \tau & \text{for} \quad \tau \geqslant 0, \\ -\tau & \text{for} \quad \tau < 0. \end{cases}$$

2. $1(\tau)$, **a unit function (a unit jump):**

$$1(\tau) = \begin{cases} 1 & \text{for} \quad \tau > 0, \\ \frac{1}{2} & \text{for} \quad \tau = 0, \\ 0 & \text{for} \quad \tau < 0. \end{cases}$$

3. sign τ, **the sign of the value of** τ **(the signum):**

$$\operatorname{sign} \tau = \begin{cases} 1 & \text{for} \quad \tau > 0 \\ 0 & \text{for} \quad \tau = 0 \\ -1 & \text{for} \quad \tau < 0 \end{cases}$$

4. $\delta(\tau)$, **the delta function:**

$$\delta(\tau) = \frac{d}{d\tau} 1(\tau).$$

The delta function is an even function of τ. The main properties of the delta function are

(a) $\tau\delta(\tau) \equiv 0$ and, in general, $\varphi(\tau)\,\delta(\tau) \equiv 0$ if $\varphi(\tau)$ is an odd function continuous at $\tau = 0$;

(b) $\int\limits_{0-\varepsilon}^{0+\varepsilon} \psi(\tau)\,\delta(\tau)\,d\tau = \psi(0)$ if the function $\psi(t)$ is continuous at the point $\tau = 0$ $(\varepsilon > 0)$;

$$\int\limits_{0-\varepsilon}^{0} \psi(\tau)\,\delta(\tau)\,d\tau = \int\limits_{0}^{0+\varepsilon} \psi(\tau)\,\delta(\tau)\,d\tau = \frac{1}{2}\psi(0)$$

if the function $\psi(\tau)$ is continuous at the point $\tau = 0$.

These definitions yield the following properties which hold for any real τ and any odd function $\varphi(\tau)$:

$$(1)\ |\tau| = \tau \operatorname{sign} \tau, \quad (2)\ \tau = |\tau| \operatorname{sign} \tau,$$

$$(3)\ \varphi(\tau) = \varphi(|\tau|) \operatorname{sign} \tau, \quad (4)\ \varphi(|\tau|) = \varphi(\tau) \operatorname{sign} \tau,$$

$$(5)\ \varphi^2(|\tau|) = \varphi^2(\tau), \quad (6)\ \operatorname{sign} \tau = 2\cdot 1(\tau) - 1,$$

(7) $1(\tau)=\dfrac{\operatorname{sign}\tau+1}{2}$,

(8) $|\tau|=\tau\,[2\cdot 1(\tau)-1]$,

(9) $\dfrac{d\,|\tau|}{d\tau}=\operatorname{sign}\tau$,

(10) $\dfrac{d^2\,|\tau|}{d\tau^2}=\dfrac{d\operatorname{sign}\tau}{d\tau}=2\delta(\tau)$,

(11) $1(\tau)=\displaystyle\int_{-\infty}^{\tau}\delta(\tau)\,d\tau=\frac{1}{2}\int_{-\infty}^{\tau}d(\operatorname{sign}\tau)$.

APPENDIX 7

A Table of the Relationships Between Various Correlation Functions $R_x(\tau)$ and Spectral Densities $S_x^*(\omega)$

$R_x(\tau)$	$S_x^*(\omega)$
1. Var $\delta(\tau)$, where $\delta(t)$ is a delta function	$\mathrm{Var}/(2\pi)$
2. Var	$\mathrm{Var}\ \delta(\omega)$
3. Var $\cos\beta\tau$	$(\mathrm{Var}/2)\,[\delta(\omega+\beta)+\delta(\omega-\beta)]$
4. $\sum\limits_{i=1}^{n} \mathrm{Var}_i \cos\beta_i\tau$	$\frac{1}{2}\sum\limits_{i=1}^{n} \mathrm{Var}_i\,[\delta(\omega+\beta_i)+\delta(\omega-\beta_i)]$
5. Var $e^{-\alpha\|\tau\|}$	$(\mathrm{Var}\ \alpha/\pi)(\alpha^2+\omega^2)^{-1}$
6. $\sum\limits_{i=1}^{n} \mathrm{Var}_i\, e^{-\alpha_i\|\tau\|}$	$\frac{1}{\pi}\sum\limits_{i=1}^{n} \frac{\mathrm{Var}_i\alpha_i}{\alpha_i^2+\omega^2}$
7. Var $e^{-\alpha\|\tau\|}\cos\beta\tau$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{\alpha^2+\beta^2+\omega^2}{[\alpha^2+(\beta-\omega)^2]\,[\alpha^2+(\beta+\omega)^2]}$
8. Var $e^{-\alpha\|\tau\|}\left(\cos\beta\tau+\frac{\alpha}{\beta}\sin\beta\|\tau\|\right)$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{2(\alpha^2+\beta^2)}{(\omega^2+\alpha^2-\beta^2)^2+4\alpha^2\beta^2}$
9. Var $e^{-\alpha\|\tau\|}\left(\cos\beta\tau-\frac{\alpha}{\beta}\sin\beta\|\tau\|\right)$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{2\omega^2}{(\omega^2+\alpha^2+\beta^2)^2-4\beta^2\omega^2}$
10. Var $e^{-\alpha\|\tau\|}\left(\cosh\beta\tau+\frac{\alpha}{\beta}\sinh\beta\|\tau\|\right)$ $(\alpha\geqslant\beta)$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{2(\alpha^2-\beta^2)}{[(\alpha-\beta)^2+\omega^2][(\alpha+\beta)^2+\omega^2]}$
11. Var $(1-\|\tau\|)\,1(1-\|\tau\|)$, where $1(x)$ is a unit function	$\frac{\mathrm{Var}}{2\pi}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^2$
12. Var $e^{-\alpha\|\tau\|}(1+\alpha\|\tau\|)$	$(\mathrm{Var}\ \alpha/\pi)\ 2\alpha^3/(\omega^2+\alpha^2)^2$
13. Var $e^{-\alpha\|\tau\|}\left(1+\alpha\|\tau\|+\frac{1}{3}\alpha^2\tau^2\right)$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{\alpha^4}{3(\omega^2+\alpha^2)^3}$
14. Var $e^{-\alpha\|\tau\|}\left(1+\alpha\|\tau\|-2\alpha^2\tau^2+\frac{1}{3}\alpha^3\|\tau\|^3\right)$	$\frac{\mathrm{Var}\,\alpha}{\pi}\,\frac{16\alpha^3\omega^4}{(\omega^2+\alpha^2)^4}$
15. $2\alpha\sin\beta\tau/\tau$	$\alpha 1(1-\|\omega\|/\beta)$
16. $2\alpha^2(2\cos\beta\tau-1)\frac{\sin\beta\tau}{\tau}$	$\begin{cases} 0 \text{ for } 0\leqslant\|\omega\|\leqslant\beta, \\ \alpha^2 \text{ for } \beta<\|\omega\|\leqslant 2\beta, \\ 0 \text{ for } 2\beta<\|\omega\| \end{cases}$
17. Var $e^{-(\alpha\tau)^2}$	$\frac{\mathrm{Var}}{2\alpha\sqrt{\pi}}\exp\left[-\left(\frac{\omega}{2\alpha}\right)^2\right]$
18. Var $e^{-\alpha\|\tau\|}\,[2\delta(\tau)-\alpha(\mathrm{sign}\,\tau)^2]$	$(\mathrm{Var}\ \alpha/\pi)[\omega^2/(\alpha^2+\omega^2)]$

BIBLIOGRAPHY

1. Feller, W. *An Introduction to Probability Theory and Its Applications*, John Wiley and Sons, New York, 1962, 1966.
2. Gnedenko, B. V. *The Theory of Probability*, Mir Publishers, Moscow, 1982.
3. Khinchine, A. *Works on the Theory of Queues*, John Wiley and Sons, New York, 1961.
4. Ovcharov, L. A. *Applied Problems in the Queueing Theory*, Mashinostroenie Publishers, Moscow, 1969 (in Russian).
5. Pugachev, V. S. *Probability Theory and Mathematical Statistics*, Nauka, Moscow, 1979.
6. Pugachev, V. S. *Probability Theory and Mathematical Statistics for Engineers*, Pergamon Press, Oxford, 1984.
7. Takacs, K. *An Introduction to the Theory of Queues*. Oxford University Press, New York, 1962.
8. Tarakanov, K. V., Ovcharov, L. A., Tyryshkin, A. N. *Analytical Methods of System Analysis*, Sovetskoe Radio Publishers, Moscow, 1974 (in Russian).
9. Wentzel, E. S. *Probability Theory*, Nauka, Moscow, 1969 (in Russian).
10. Wentzel, E. S., Ovcharov, L. A. *Probability Theory*, Nauka, Moscow, 1973 (in Russian).

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